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STABILIZATION OF NONHOLONOMIC SYSTEMS USING HOMOGENEOUS FINITE-TIME CONTROL TECHNIQUE

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Abstract: This research focuses on the problem of finite-time stabilization of chained-form nonholonomic systems using discontinuous homogeneous feedback and introduces the discontinuous homogeneous (with dilation) controller. The proposed controller has no singular point, stabilizes the system in finite-time, and is not complicated. Moreover, we demonstrate an exponentially stable controller.

Keywords: discontinuous control, nonlinear systems, stabilization, state feedback

1. INTRODUCTION

In this paper, we consider the problems of stabilizing nonholonomic systems (for example, wheeled mobile vehicles, and space robots) These systems are controllable but cannot be stabilized by any smooth time-invariant state feedback control lows (Brockett, 1983).

For these systems, various control methods have been proposed. We can divide these methods into basically two approaches: the discontinuous time-invariant state feedback and smooth time-varying state feedback approach.

However, these control methods have problems. Controllers based on the discontinuous approach lack sophisticated control strategy and those based on the time-varying approach suffer from slow convergence. Constructing controllers using back-stepping (Xu and Huo, 2000) improved the former problem, but another problem was raised, namely, that there was a set that the input cannot be defined. For the latter problem, M'Closkey and Murray (M'Closkey and Murray, 1998) introduced a time-varying controller based on homogeneity with dilation, but this controller was discontinuous at origin and smoothness was lost.

We propose a time-invariant discontinuous homogeneous (with dilation) controller for solving these problems. In the proposed method, the system has no singular point, is finite-time stable and the controller is not complicated. Moreover, the convergence speed is selectable; we can design the system so that it is either finite-time or exponentially stable.

2. CONTROL STRATEGY

We consider the following chained-form systems.

$$
\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_2 \\
\dot{x}_3 &= x_2 u_1 \\
&\vdots \\
\dot{x}_n &= x_{n-1} u_1 .
\end{aligned}
\tag{1}
$$

For the system, we stabilize the following two steps.

Step 1
Move states $x_2, \cdots, x_n$ to a region where they are settled at zero at time $T_s < |x_1|$.

Step 2
Let $u_1 = -\operatorname{sgn} x_1$ and choose u_2 to make $x_2, \cdots, x_n$ stabilize in finite-time, all states $x_1, \cdots, x_n$ converge to origin.

In prior research on discontinuous approach, *Step 1* and *Step 2* are dealt with independently. However, in this research, we construct a controller that can automatically switch using one equation. For system (1), we choose input u_1 as

$$u_1 = \operatorname{sgn}[T_s(x_2, \cdots, x_n) - |x_1|] \operatorname{sgn} x_1, \qquad (2)$$

where the function $T_s : \mathbb{R}^n \to \mathbb{R}$ is called a guaranteed settling time function that assures that the states $x_2, \cdots, x_n$ converge to the origin until T_s, as defined below.

Definition 1. (guaranteed settling function). Consider system (1), and assume $u_1 = -\operatorname{sgn} x_1(t_0)$ and u_2 to make $x_2, \cdots, x_n$ stabilize in finite-time. T_s is defined as a guaranteed settling time function if a function $T_s : \mathbb{R}^n \to \mathbb{R}$ satisfies the following conditions.

(1) T_s is continuous and $T_s(\operatorname{sgn}(x_1(t_0)), x_2, \cdots, x_n) > 0 \ \forall (\operatorname{sgn}(x_1(t_0)), x_2, \cdots, x_n)$.
(2) $\exists \delta \in \mathbb{R}^+$ s.t. $T_s(\operatorname{sgn}(x_1(t_0)), x_2, \cdots, x_n) < \delta$, $\forall \varepsilon \in \mathbb{R}^+$ s.t. $\|x_2, \cdots, x_n\| < \varepsilon$.
(3) If at any t_0, $T_s(\operatorname{sgn}(x_1(t_0)), x_2(t_0), \cdots, x_n(t_0)) = T_{s0}$, then $(x_2(t_0+T_{s0}), \cdots, x_n(t_0+T_{s0})) = 0$.
(4) In the region $T_s(x_1(t_0), x_2, \cdots, x_n) < |x_1|$, $\dot{T}_s < -1$.

A function sgn is defined as

$$\operatorname{sgn}(x) = \begin{cases} -1 & (x < 0) \\ 1 & (x \geq 0). \end{cases} \qquad (3)$$

To define the sgn() function, the input $u_1 \neq 0$ in region $x_1 = 0$ and $T_s \neq 0$. This u_1 switches when x_1 reaches the region where $x_2, \cdots, x_n$ converge to the origin in time T_s.

Lemma 1. Assume system (1) and input (2). If $T_s(\operatorname{sgn}(x_1(t_0)), x_2(t), \cdots, x_n(t))$ is bounded for $t \geq 0$ in *Step 1*, then the u_1 switches one time at most.

Proof . Since $|x_1|$ is a strictly increasing function in $T_s \geq |x_1|$ and $T_s(\operatorname{sgn}(x_1(t_0)), x_2(t), \cdots, x_n(t))$ is bounded $\forall t \geq t_0$, x_1 cannot fail to become $|x_1| > T_s$ when enough long time have elapsed. Therefore, u_1 has one switch. On the other hand, when $T_s < |x_1|$, by

$$\dot{x}_1 = -\operatorname{sgn} x_1 \qquad (4)$$

and $\dot{T}_s < |\dot{x}_1|$ in *Step 2* by assumption, u_1 has no switch. Therefore, u_1 switches at most only one time.

Therefore, the system does not return from *Step 2* to *Step 1*.

Remark . T_s may include $\operatorname{sgn} x_1(t_0)$. Since $\operatorname{sgn} x_1(t) = \operatorname{sgn} x_1(t_0) \forall t > t_0$ with lemma 1 and its proof, we refer to '$\operatorname{sgn} x_1(t_0)$' simply as '$\operatorname{sgn} x_1$'.

In *Step 1*, system (1) is shown as

$$\begin{aligned} \dot{x}_1 &= \operatorname{sgn} x_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_2 \operatorname{sgn} x_1 \\ &\vdots \\ \dot{x}_n &= x_{n-1} \operatorname{sgn} x_1, \end{aligned} \qquad (5)$$

and in *Step2*, as

$$\begin{aligned} \dot{x}_1 &= -\operatorname{sgn} x_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= -x_2 \operatorname{sgn} x_1 \\ &\vdots \\ \dot{x}_n &= -x_{n-1} \operatorname{sgn} x_1. \end{aligned} \qquad (6)$$

The control aim of *Step1* is that $x_2, \cdots, x_n$ are held bounded, and that of *Step 2* is that they are stabilized in finite-time. To deal with these two systems together, let the new variables ξ be

$$\begin{aligned} \xi_1 &= x_1 \\ \xi_i &= x_i [\operatorname{sgn} u_1]^{n-i} \qquad (2 \leq i \leq n-1) \\ \xi_n &= x_n \\ v_2 &= [\operatorname{sgn} u_1]^{n-2} u_2. \end{aligned} \qquad (7)$$

The two systems then are shown in single form such that

$$\begin{aligned} \dot{\xi}_1 &= \operatorname{sgn}[T_s(\xi_2, \cdots, \xi_n) - |\xi_1|] \operatorname{sgn} \xi_1 \\ \dot{\xi}_2 &= v_2 \\ \dot{\xi}_3 &= \xi_2 \\ &\vdots \\ \dot{\xi}_n &= \xi_{n-1}. \end{aligned} \qquad (8)$$

Therefore, if v_2 stabilizes $\xi_2, \cdots, \xi_n$ in finite-time, both aims of *Step 1* and *Step 2* are satisfied and the two problems in both steps are reduced to one problem. For these variables ξ, we do not consider the moment when the system switches from *Step 1* to *Step 2*, but the system displays no unexpected behavior at the switching moment with lemma 1. However, the fact that the transformation of ξ to x is discontinuous may cause some problems when constructing T_s.

3. FINITE-TIME CONTROL

There are a number of approaches for finite-time control. For non-singularity and guaranteed settling time functions, we use the homogeneous (with dilation) finite-time control introduced by Bhat and Bernstein (Bhat and Bernstein, 1997).

Preparing for finite-time control, we state homogeneity with dilation.

Definition 2. (dilation). Dilation Δ_ϵ^r is a mapping, depending on positive dilation coefficients $r = (r_1, r_2, \cdots, r_n) \in (\mathbb{R}^+)^n$, which assigns to every $\varepsilon \in \mathbb{R}^+$ a global diffeomorphism

$$\Delta_\varepsilon^r(x) = (\varepsilon^{r_1} x_1, \cdots, \varepsilon^{r_n} x_n), \qquad \varepsilon \in \mathbb{R}^+ \tag{9}$$

where $x_1, \cdots, x_n$ are suitable coordinates on $\mathbb{R}^n$.

Definition 3. (homogeneous function). A function $V(x) : \mathbb{R}^n \to \mathbb{R}$ is called homogeneous of degree $q \in \mathbb{R}$ with respect to the dilation Δ_ε^r, if there exists $q \in \mathbb{R}$ such that

$$V(\Delta_\varepsilon^r(x)) = \varepsilon^q V(x). \tag{10}$$

Definition 4. (homogeneous vector field). A vector field $f(x) = (f_1(x), \cdots, f_n(x)) : \mathbb{R}^n \to \mathbb{R}^n$ is called homogeneous of degree $k \in \mathbb{R}$ with respect to the dilation Δ_ε^r, if there exists $k \in \mathbb{R}$ such that

$$f_i(\Delta_\varepsilon^r(x)) = \varepsilon^{k+r_i} f_i(x), \quad i = 1, \cdots, n. \tag{11}$$

A system, $\dot{x} = f(x)$ is called homogeneous if its vector field $f(x)$ is homogeneous.

Definition 5. (homogeneous norm). A continuous map $\rho : \mathbb{R}^n \to \mathbb{R}$ is called a homogeneous norm with respect to the dilation Δ_ϵ^r, if it is positive definite function homogeneous of degree 1 with respect to the dilation Δ_ϵ^r.

In this research, homogeneous norm is taken in the form of

$$||x||_{hom} = (|x_1|^{\frac{c}{r_1}} + \cdots + |x_n|^{\frac{c}{r_n}})^{\frac{1}{c}}, \tag{12}$$

where $c > \max\{r_i, i = 1, \cdots, n\}$.

Finite-time control for homogeneous systems is shown below results(Bhat and Bernstein, 1997),(Hong *et al.*, 1999).

Lemma 2. Assume system

$$\dot{x} = f(x), \quad f(0) = 0, \quad x \in \mathbb{R}, \quad x(0) = x_0 \tag{13}$$

is homogeneous of degree k with respect to Δ_ε^r, $x = 0$ is its asymptotically stable equilibrium. Let $V(x)$ be the homogeneous Lyapunov function of degree l. Then the equilibrium of the origin of system is finite-time stable if $k < 0$. Moreover settling time $T(x_0)$ at initial values x_0 is shown

$$T(x_0) \le -\frac{l}{k} \min_{||e||_{hom}=1} \left(\frac{V(e)^{\frac{k+l}{l}}}{\dot{V}(e)} \right) V(x_0)^{-\frac{k}{l}}. \tag{14}$$

Proof is easy for using same argument as (Hong *et al.*, 1999).

We show some examples of such homogeneous finite-time controller. For system

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= w, \end{aligned} \tag{15}$$

controller

$$w = -k_1 |y_1|^{2r-1} \operatorname{sgn} y_1 - k_2 |y_2|^{\frac{2r-1}{r}} \operatorname{sgn} y_2 \tag{16}$$

$$\left(\frac{1}{2} < r < 1 \right) \tag{17}$$

or

$$\begin{aligned} w &= -|y_2|^\alpha \operatorname{sgn} y_2 - \phi_\alpha^{\frac{\alpha}{2-\alpha}} \\ \phi_\alpha &= y_1 + \frac{1}{2-\alpha} |y_2|^{2-\alpha}, \qquad 0 < \alpha < 1, \end{aligned} \tag{18}$$

is a finite-time controller(Bhat and Bernstein, 1997) (Bhat and Bernstein, 1998). For more general system

$$\begin{aligned} \dot{y}_1 &= y_2^{m_1} \\ &\vdots \\ \dot{y}_{n-1} &= y_n^{m_{n-1}} \\ \dot{y}_n &= w, \end{aligned} \tag{19}$$

controller $w = u_n(w), u_0(w) = 0$,

$$u_{i+1}(y) = -l_{i+1} \left[y_{i+1}^{m_i \beta_i} - u_i(y)^{\beta_i} \right]^{\frac{r_{i+1}+k}{r_{i+1}^{m_i \beta_i}}} \tag{20}$$

makes the system finite-time(Hong, 2001). Thus we can design v in system (8) using these controllers.

Consider system (1) and $u_2 = [\operatorname{sgn} u_1]^{n-2} v_2$ such that v_2 stabilize $\xi_2, \cdots, \xi_n$ in finite-time. To demonstrate the relation between eq. (14) and the guaranteed settling time function, by eq. (14), we define a constant c as

$$c > -\frac{l}{k} \min_{||e||_{hom}=1} \left(\frac{V(e)^{\frac{k+l}{l}}}{\dot{V}(e)} \right) > 0, \tag{21}$$

then a function

$$T_e = cV(x)^{-\frac{k}{l}}, \tag{22}$$

satisfies $\dot{T}_e < 1$. Since T_e may be discontinuous at the switching point, it generally does not satisfy the guaranteed settling-time function definition. However, when we always consider T_e in system (6), namely

$$T_e = cV(x_1, \cdots, x_i[-\operatorname{sgn} x_1]^{n-i}, \cdots, x_n)^{-\frac{k}{l}}, \tag{23}$$

it satisfies definition of the guaranteed settling time function, and this is identified with T_s. Thus, T_s is homogeneous of degree $-k$ with respect to $\Delta_\varepsilon^{(r_2, \cdots, r_n)}$.

Therefore, these homogeneous finite-time controllers v for ξ of degree k with respect to

$\Delta_\varepsilon^{(r_2,\cdots,r_n)}$ make system (1) homogeneous of degree k with respect to $\Delta_\epsilon^{(-k,r_2,\cdots,r_n)}$. Since we choose Lyapunov candidate function

$$V_a = T_s + |x_1|, \tag{24}$$

then its derivative $\dot{V}_a$ is negative-definite in *Step 2*, and system (8) is finite-time by lemma 2.

4. CONVERGENCE SPEED DESIGN

Discontinuity of the proposed controller in the previous section causes chattering at the origin and unknown motion after the convergence. In this section, we design the convergence speed of system (1) based on the homogeneity introduced in the previous section as a solution to these problems and for other benefits.

Finite-time stability is regarded as a part of exponential stability, particularly local stability. However, in this research we refer to 'exponentially stable without finite-time stability' simply as 'exponentially stable' and distinguish it from finite-time stability.

Generally, the next theorem exists for driftless-systems.

Theorem 1. Consider the system

$$\dot{y} = Y_1(y)w_1 + \cdots + Y_m(y)u_m, \qquad y \in \mathbb{R}^n, \tag{25}$$

and inputs

$$w_i = f_i(y) \tag{26}$$

for the system. If the closed-loop system

$$\dot{y} = \sum_{i=1}^{m} Y_i(y) f_i(y) \tag{27}$$

is asymptotically stable, then if input

$$w_i = \rho(y) f_i(y) \tag{28}$$

is chosen using positive definite function $\rho : \mathbb{R}^n \to \mathbb{R}$, where $\rho : \mathbb{R}^n \to \mathbb{R}$ is a positive-definite function, then the system is asymptotically stable.

Summary of proof If a Lyapunov function for system (25) exists, refer to the arguments of (M'Closkey and Murray, 1998). If not, the same conclusion is reached by the time-scale transformation

Using the argument set forth in the previous section, we can design a homogeneous controller for system (1). Since stability is maintained when the input is multiplied by a positive-definite function, the following theorem is obtained for homogeneous systems.

Theorem 2. Consider that sref2:1 under the inputs u_1, u_2 designed in the previous section is homogeneous of degree k with respect to dilation $\Delta_\varepsilon^{(-k,r_2,\cdots,r_n)}$. If we choose new input

$$\begin{aligned} u_1' &= \|x\|_{hom}^h u_1 \\ u_2' &= \|x\|_{hom}^h u_2, \end{aligned} \tag{29}$$

then

(1) if $0 \le h \le -k$,
the new closed-loop system is finite-time stable.
(2) if $h = -k$,
the new closed-loop system is exponentially stable.

Summary of proof Finite-time stability is obvious with lemma 2. With respect to exponential stability, refer to the arguments of (M'Closkey and Murray, 1998).

In the case of $h > -k$, the system has a high-order convergence property and its convergence is slow in the neighborhood of the origin (away from the origin, convergence is fast).

We can now make the system exponentially stable to multiply the input by a homogeneous norm. Thus, problems such as chattering are avoided.

5. CONTROLLER FOR THIRD ORDER CHAINED-FORM SYSTEMS

In this section, we apply the proposed method to a third-order chained-form system. The target system is shown as

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_2 u_1. \end{aligned} \tag{30}$$

For this system, we design u_1, u_2 with the procedure set forth in Section 2 and 3. First, let input

$$u_1 = \operatorname{sgn}[T_s(\operatorname{sgn} x_1, x_2, x_3) - |x_1|] \operatorname{sgn} x_1. \tag{31}$$

Next, apply the following variable-transformations.

$$\begin{aligned} \xi_1 &= x_1 \\ \xi_2 &= x_2[\operatorname{sgn} u_1] \\ \xi_3 &= x_3 \\ v_2 &= [\operatorname{sgn} u_1] u_2. \end{aligned} \tag{32}$$

the system becomes

$$\begin{aligned} \dot{\xi}_1 &= \operatorname{sgn}[T_s(\xi_2, \xi_3) - |\xi_1|] \operatorname{sgn} \xi_1 \\ \dot{\xi}_2 &= v_2 \\ \dot{\xi}_3 &= \xi_2. \end{aligned} \tag{33}$$

Assume an input with the form $v_2 = -k_1|\xi_3|^{2r-1} \operatorname{sgn} \xi_3 - k_2|\xi_2|^{\frac{2r-1}{r}} \operatorname{sgn} \xi_2$ for the finite-time controller. Then the system

$$\begin{aligned} \dot{\xi}_2 &= -k_1|\xi_3|^{2r-1}\operatorname{sgn}\xi_3 - k_2|\xi_2|^{\frac{2r-1}{r}}\operatorname{sgn}\xi_2 \\ \dot{\xi}_3 &= \xi_2 \end{aligned} \quad (34)$$

is homogeneous of degree $r-1$ with respect to $\Delta_\epsilon^{(r,1)}$ and, in the case $\frac{1}{2} < r < 1$, the system is finite-time stable if the system is asymptotically stable by lemma 2.

Next, we look for a Lyapunov function. Let candidate Lyapunov function be

$$V = \frac{k_1}{r}|\xi_3|^{2r-1} + b\xi_3|\xi_2|^{\frac{2r-1}{r}}\operatorname{sgn}\xi_2 + |\xi_2|^2. \quad (35)$$

Then the candidate Lyapunov derivative is

$$\dot{V} = |\xi_2|^{\frac{r-1}{r}}\left[\frac{2r-1}{r}bk_1|\xi_3|^{2r} - \frac{2r-1}{r}k_2 b\xi_3|\xi_2|^{\frac{2r-1}{r}}\operatorname{sgn}\xi_2 - (2k2-b)|\xi_2|^2\right]. \quad (36)$$

Here, if $k_1, k_2 > 0$,

$$b < \left[\left(\frac{k_2^{2r}}{k_1}\right)^{\frac{1}{2r-1}}\left(\frac{1}{2r}\right)^{\frac{2r}{2r-1}}\frac{(2r-1)^2}{r} + 1\right]^{-1} 2k_2 \quad (37)$$

and

$$b < \left(\frac{2r}{2r-1}\right)^{\frac{2r-1}{2r}}(2k_1)^{\frac{1}{2r}}, \quad (38)$$

then we obtain $V > 0, \dot{V} < 0$ by thorem3, and V is the Lyapunov function for system (34).

We choose guaranteed settling time function T_s,

$$T_s = c\left[\frac{k_1}{r}|x_3|^{2r-1} + bx_3|x_2|^{\frac{2r-1}{r}}\operatorname{sgn}(-x_2\operatorname{sgn}x_1) + |x_2|^2\right]^{\frac{1-r}{2r-1}}, \quad (39)$$

in consideration of the fact that the Lyapunov function V obtained above is discontinuous at the moment when the system changes from *Step 1* to *Step 2*, where b satisfies equations (37) and (38) and c satisfies eq. (21).

The controller thus designed is shown as

$$\begin{aligned} u_1 &= \operatorname{sgn}[T_s(\operatorname{sgn}x_1, x_2, x_3) - |x_1|]\operatorname{sgn}x_1 \\ u_2 &= -k_1|x_3|^{2r-1}\operatorname{sgn}x_3\operatorname{sgn}u_1 - k_2|x_2|^{\frac{2r-1}{r}}\operatorname{sgn}x_2. \end{aligned} \quad (40)$$

Equation (40) confirms that controllers u_1, u_2 have no singular point.

We have constructed a discontinuous finite-time stable controller, but this controller has some problems, as described in Section 4. Therefore we have also designed the exponentially stable controller shown in Section 4.

The closed loop system substituting controller eq. (40) into system (30) is

$$\begin{aligned} \dot{x}_1 &= \operatorname{sgn}[T_s(\operatorname{sgn}x_1, x_2, x_3) - |x_1|]\operatorname{sgn}x_1 \\ \dot{x}_2 &= -k_1|x_3|^{2r-1}\operatorname{sgn}x_3\operatorname{sgn}u_1 - k_2|x_2|^{\frac{2r-1}{r}}\operatorname{sgn}x_2 \\ \dot{x}_3 &= x_2\operatorname{sgn}[T_s(\operatorname{sgn}x_1, x_2, x_3) - |x_1|]\operatorname{sgn}x_1. \end{aligned} \quad (41)$$

Therefore, the total system is homogeneous of degree $r-1$ with respect to $\Delta_\epsilon^{(1-r,r,1)}$. Hence, we can design a new input

$$\begin{aligned} u_1 &= \|x\|_{hom}^{1-r}\operatorname{sgn}[T_s(\operatorname{sgn}x_1, x_2, x_3) - |x_1|]\operatorname{sgn}x_1 \\ u_2 &= \|x\|_{hom}^{1-r}\left[-k_1|x_3|^{2r-1}\operatorname{sgn}x_3\operatorname{sgn}u_1 - k_2|x_2|^{\frac{2r-1}{r}}\operatorname{sgn}x_2\right] \end{aligned} \quad (42)$$

which makes the system exponentially stable.

6. COMPUTER SIMULATION

In this section, we evaluate the controller for the third-order chained-form system shown in the previous section by computer simulation. Each parameter of the input is chosen as $r = \frac{11}{20}, k_1 = 1, k_2 = 1, b = 1, c = 10$ and the homogeneous norm is defined as

$$\|x\|_{hom} = \left(|x_1|^{\frac{2}{1-r}} + |x_2|^{\frac{2}{r}} + |x_3|^2\right)^{\frac{1}{2}}. \quad (43)$$

The simulation result for the finite-time controller is shown in Fig. 1 and the case of the exponentially stable controller is shown in Fig. 2. Both controllers stabilize all states to the origin. It appears that the exponentially stable controller converges faster in the region away from origin and slower in the neighborhood of origin.

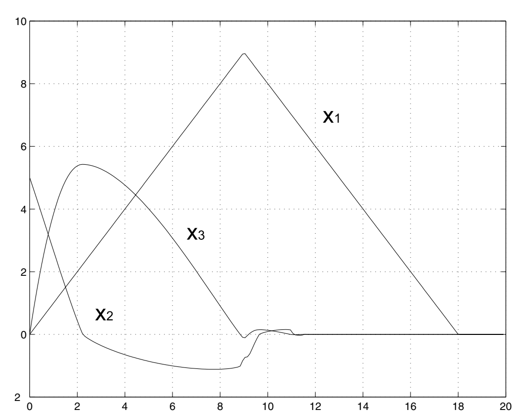

Fig. 1. Simulation Results: Finite-time Controller

7. CONCLUSION

In this research, we proposed a discontinuous homogeneous controller for chained-form systems that is controllable but cannot stabilize with smooth time-invariant state feedback control.

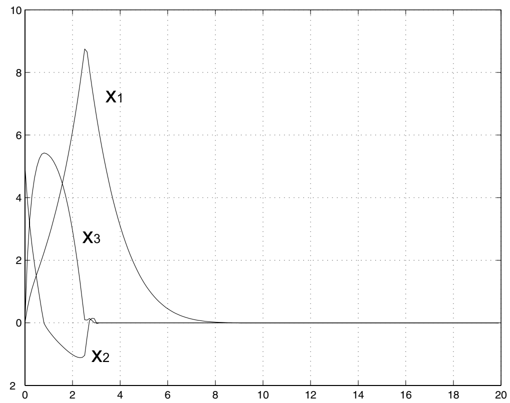

Fig. 2. Simulation Results: Exponentially Stable Controller

Using the proposed controller, we demonstrated that the controllers do not have a singular point and that a finite-time or exponentially stable convergence speed can be selected. Moreover, the computer simulation confirms the availability of the proposed method.

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Appendix A. POSITIVE OR NEGATIVE DEFINITY OF HOMOGENEOUS FUNCTIONS

Lemma 3. Assume a function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = a|x|^\alpha + bx|y|^{\frac{\alpha-1}{r}} \operatorname{sgn} y + c|y|^{\frac{\alpha}{r}} \quad (A.1)$$

is homogeneous with respect to $\Delta_\varepsilon^{(1,r)}$, where $\alpha > 1$ and $a, b, c, r \in R$. If

$$a|c|^{\alpha-1} \operatorname{sgn} c \cdot \alpha^\alpha > (\alpha-1)^{\alpha-1}|b|^\alpha, \quad (A.2)$$

then

$$f(x,y) > 0 (a > 0), \quad f(x,y) < 0 (a < 0), \quad (A.3)$$

except the origin $(x, y) = (0, 0)$.

Proof . Since $\frac{\partial^2 f}{\partial x^2} = \alpha(\alpha-1)a|x|^{\alpha-2}$, it is always the same sign. Therefore, the function f is convex with respect to x when $a > 0$ and concave when $a < 0$. Then, x has only one minimal (if $a > 0$) or maximal (if $a < 0$) point. For avoidance of complexity, assume $a > 0$. Next, we find for x, which makes f minimal. When $\frac{\partial f}{\partial x} = 0$,

$$x = \left(\frac{|b|}{\alpha|a|}\right)^{\frac{1}{\alpha-1}} |y|^{\frac{1}{r}} \operatorname{sgn}(-by). \quad (A.4)$$

The minimal value function $g(y)$ can be defined when the x obtained above is substituted into eq. (A.1). It is shown as

$$g(y) = \left\{ \left(\frac{1-\alpha}{\alpha^{\frac{\alpha}{\alpha-1}}}\right) \frac{|b|^{\frac{\alpha}{\alpha-1}}}{|a|^{\frac{1}{\alpha-1}}} + c \right\} |y|^{\frac{\alpha}{r}}. \quad (A.5)$$

Therefore if

$$a|c|^{\alpha-1} \operatorname{sgn} c \cdot \alpha^\alpha > (\alpha-1)^{\alpha-1}|b|^\alpha \quad (A.6)$$

then g is a positive-definite. Arguing the same in the case of $a < 0$, we obtain lemma 3.

By lemma 3 and the fact that if $\beta > 0$, then $x \mapsto x^\beta$ is a homeomorphism, the following theorem is held.

Theorem 3. Assume a function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = a|x|^{\alpha\beta} + b|x|^\beta \operatorname{sgn} x \cdot |y|^{\frac{\alpha-1}{r}} \operatorname{sgn} y + c|y|^{\frac{\alpha}{r}} \quad (A.7)$$

is homogeneous with respect to $\Delta_\varepsilon^{(1,r)}$, where $\alpha > 1, \beta > 0$ and $a, b, c, r \in R$. If

$$a|c|^{\alpha-1} \operatorname{sgn} c \cdot \alpha^\alpha > (\alpha-1)^{\alpha-1}|b|^\alpha, \quad (A.8)$$

then

$$f(x,y) > 0 (a > 0), \quad f(x,y) < 0 (a < 0), \quad (A.9)$$

except the origin$(x, y) = (0, 0)$.