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$\mathcal{H}_\infty$ LOOP SHAPING FOR MIXED CONTINUOUS/DISCRETE-TIME TIME-VARYING AND PERIODIC SYSTEMS

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Abstract: The loop shaping scheme is applied to $\mathcal{H}_\infty$-control of time-varying and periodic systems. The solution is given in a unified mixed continuous/discrete-time compact form. This implies that pure continuous-time and discrete-time applications simply appear as two different interpretations of the given results. Among the applications are also hybrid systems with time-driven discrete events. For periodic continuous-time systems, a lifted (or discretised) system model is used to catch the behaviour during the period. The results can be considered as a generalisation of the Glover-McFarlane loop shaping procedure.

Keywords: H-infinity control, Continuous-time systems, Discrete-time systems, Time-varying systems, Sampled-data systems, Loop shaping, Periodic systems

1. INTRODUCTION

$\mathcal{H}_\infty$-design is by now well investigated, see e.g. the classical book (Green and Limebeer 1995) where, as is mostly the case, the continuous-time and discrete-time solutions are presented one at a time. A general and unified $\mathcal{H}_\infty$-design framework that applies to mixed continuous/discrete-time systems was presented in (Christiansson *et al.* 1999) and (Christiansson *et al.* 2000) as a generalisation of many known works, e.g. (Green and Limebeer 1995, Zhou *et al.* 1996, Ravi *et al.* 1991). This paper shows how the unified framework easily is adopted to the loop shaping scheme.

The loop shaping design procedure for $\mathcal{H}_\infty$-design presented in (McFarlane and Glover 1992) is nowadays widely adopted. One great advantage is that there is no need for the so called γ-iteration. The reason for this is the simple system setup that is obtained when weight functions are placed *in* the loop, as opposed to the general scheme, where weight functions can be placed anywhere. Loop shaping is mostly presented when the plant is normalised coprime factored, see e.g. (Glover and McFarlane 1989) in continuous time, or (Walker 1990) in discrete time. The unified mixed $\mathcal{H}_\infty$-solution mentioned above is in this paper applied to the loop shaping scheme without the need for such normalised coprime factorisation. It is also shown how multiplicative and additive uncertainties at plant input and output can be interpreted.

The contributions of this paper is mainly the unified mixed continuous/discrete-time solution that also applies to time-varying and periodic systems. Typical applications can be pure continuous-time and discrete-time systems as well as systems with mixed continuous/discrete-time measurements. It also applies to hybrid systems in the meaning time-driven discrete-event systems including continuous dynamics. It gives a solution that indirectly implies a test on detectability and stabilisability, which is normally not a trivial task. The results can be considered as generalisations of (Glover and McFarlane 1989, Walker 1990). Other related works in this field are e.g. (Aripirala and Syrmos 1997, Xie and Syrmos 1997, Iglesias

2000). For continuous-time periodic systems, the periodic behaviour is achieved from a lifted (or discretised) equivalent discrete-time system. Such discretisation is often discussed in the sampled-data case, cf. (Toivonen and Sågfors 1997). The results are derived for proper, but not strictly proper, system models, i.e. the plant model can have a direct through term $D \neq 0$.

Some useful mixed notations are introduced in Section 2 in order to be able to give a compact solution. In Section 3 the $\mathcal{H}_\infty$-problem is presented together with the loop shaping scheme. The general design method is then applied, and the mixed static feedback-, filter- and output feedback solutions are given. Finally, for periodic systems, an attractive discretisation method is presented, which considers the continuous-time integration over the period. The results are summarised in Section 4.

2. MIXED NOTATIONS

First some mixed continuous/discrete-time notations are introduced in order to be able to give a unified result. Consider a general mixed system with μ_c/μ_d as inputs and η_c/η_d as outputs, with $\mathcal{G}_c$ for $t \neq t_k$ and $\mathcal{G}_d$ for $t = t_k$:

$$\mathcal{G}_c : \begin{cases} \dot{x}(t) = A_c(t)x(t) + B_c(t)\mu_c(t) \\ \eta_c(t) = C_c(t)x(t) + D_c(t)\mu_c(t) \end{cases} \tag{1a}$$

$$\mathcal{G}_d : \begin{cases} x(t_k^+) = A_d(t_k)x(t_k^-) + B_d(t_k)\mu_d(t_k) \\ \eta_d(t_k) = C_d(t_k)x(t_k^-) + D_d(t_k)\mu_d(t_k) \end{cases} \tag{1b}$$

When no index $_c$ or $_d$ is used, the signals and matrices are assumed to be mixed, i.e. continuous-time when $t \neq t_k$ and discrete-time when $t = t_k$. Let the input signal μ be composed of the disturbance signal w and the control signal u as $\mu = [w' \quad u']'$. Similarly, let the output signal η be composed of the performance signal z and the measured output y as $\eta = [z' \quad y']'$. The system is now in the standard form for $\mathcal{H}_\infty$-applications. A mixed signal is composed of both discrete-time and continuous-time contributions, and its size is expressed as (for z)

$$\|z\|^2_{[0,T_f]} = \int_0^{T_f} z_c'(t)z_c(t)dt + \sum_{t_k \in [0,T_f]} z_d'(t_k)z_d(t_k) \tag{2}$$

Note that z_c in (2) is not formally defined at times t_k. However, the time limits exist, and the integral can be considered as a sum of integrals. Variables, that are solutions to a mix of differential- and difference equations, appear as piece-wise continuous variables with jumps. To describe differential- and difference-equations for such variables similarly, a "forward" notation is introduced:

$$x^+ = \dot{x}(t), \quad t \neq t_k \tag{3a}$$

$$x^+ = \lim_{\epsilon \to 0} x(t_k + \epsilon) = x(t_k^+) \tag{3b}$$

and similarly for "backward" (x^-), $\epsilon \in \mathbb{R}^+$. The systems in (1) can then be considered as the mixed system

$$\mathcal{G} : \begin{bmatrix} x^+ \\ z \\ y \end{bmatrix} = \begin{bmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix} \tag{4}$$

where the B-, C- and D-matrices in (1) are partitioned accordingly. Note that the absence of time notations in systems and signals is just for simpler notation; the system may be time-varying throughout this paper.

The state-transition matrix $\Pi_A(t,s)$ for the mixed system $x^+ = Ax$ from time s to t, is defined by

$$\dot{\Pi}_A(t,s) = A_c(t)\Pi_A(t,s), \quad t \neq t_k \tag{5a}$$

$$\Pi_A(t_k^+, s) = A_d(t_k)\Pi_A(t_k^-, s) \tag{5b}$$

The mixed system is exponentially stable, see e.g. (Sun *et al.* 1993), if there exist positive real numbers c_1, c_2 such that $\|\Pi_A(t,s)\| \leq c_1 e^{-c_2(t-s)}$, $0 \leq s \leq t$. This is also denoted as the system A being stable. The mixed system $\mathcal{G}$ in (4) is said to be stabilisable (detectable), if there exist a bounded mixed matrix L (K), such that the system $A - BL$ ($A - KC$) is stable. When the system is periodic with period T_p, the system matrices are repeated periodically, e.g. $A(t + \ell T_p) = A(t)$, l being an arbitrary integer.

To present the mixed solution in a compact form, define a mixed matrix notation $\diamond$ suitable for Lyapunov- and Riccati equations according to

$$A \diamond P \equiv A_c P + P A_c', \quad t \neq t_k \tag{6a}$$

$$A \diamond P \equiv A_d P A_d', \quad t = t_k \tag{6b}$$

Also introduce the notations δ_{t_k}, A_{t_k} that allow us to express some continuous- and discrete-time matrices in a unified way:

$$\delta_{t_k} = \begin{cases} 0, & t \neq t_k \\ 1, & t = t_k \end{cases} ; \quad A_{t_k} = \delta_{t_k} A + (1 - \delta_{t_k}) I \tag{7}$$

See (Christiansson *et al.* 2000) for more details on the mixed notation.

3. THE $\mathcal{H}_\infty$ LOOP SHAPING PROBLEM

The general mixed $\mathcal{H}_\infty$-control problem can be expressed for the system (4): *Find an output feedback controller $u = \mathcal{K}(y)$, such that the bound on the induced norm of the mixed closed loop system from w to z*

$$\|\mathcal{G}_{zw}\|_{[0,T_f]} = \sup_{\|w\|_{[0,T_f]} \neq 0} \frac{\|z\|_{[0,T_f]}}{\|w\|_{[0,T_f]}} < \gamma \tag{8}$$

holds for a specified constant $\gamma > 0$. The supremum is taken over all w in $\mathcal{L}_{2[0,T_f]} \oplus l_{2[1,N]}$ such that $\|w\|_{[0,T_f]} \neq 0$. In infinite-time horizon, i.e.

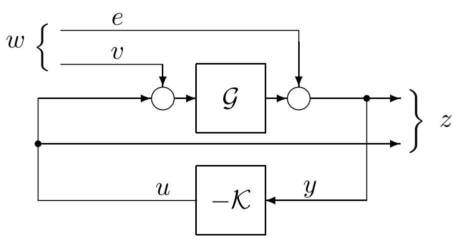

Fig. 1. A general performance $\mathcal{H}_\infty$-problem setup for loop shaping.

when $T_f \to \infty$, the controller shall give a stable closed loop system. The system (4) may be time-varying. The problem statement implies that z and w can contain both continuous- and discrete-time signals. The continuous-time periodic case is specially treated when $T_f \to \infty$. When choosing the system setup according to Fig. 1, the solution will be "γ-free" in the meaning that no γ-iteration is needed. Let the performance measure be $z = [y' \;\; u']'$, and the disturbances be $w = [e' \;\; v']'$. Note that B_u is used for both u and w input signals and C_y both for z and y output signals! This is in fact the reason for the simplifications that imply that no γ-iteration is needed. Let us for simpler notations introduce $B = B_u, C = C_y, D = D_{yu}$. The shorthands in (4) to use are then

$$B_w = \begin{bmatrix} 0 & B \end{bmatrix},\; D_{zw} = \begin{bmatrix} I & D \\ 0 & 0 \end{bmatrix}, D_{zu} = \begin{bmatrix} D \\ I \end{bmatrix}$$
$$C_z = \begin{bmatrix} C \\ 0 \end{bmatrix},\; C_w = \begin{bmatrix} C_z \\ C \end{bmatrix}, D_{yw} = \begin{bmatrix} I & D \end{bmatrix} \quad (9)$$

Furthermore introduce $D_z = \begin{bmatrix} D_{zw} & D_{zu} \end{bmatrix}, D_w = \begin{bmatrix} D_{zw} \\ D_{yw} \end{bmatrix}$. The $\mathcal{H}_\infty$-problem will be solved as a special case of the "three Riccati equation" mixed general method, i.e. first the static feedback case, then the filter for estimating $\hat{z}$, and finally the output feedback case. The presentation will be held in the mixed framework, such that continuous-time and discrete-time results are simply two different interpretations of the general results when subscripts $_c$ or $_d$ respectively are introduced in the matrix notations.

Fig. 2 and 3 show how the performance problem in Fig. 1 can be considered as a robustness problem, i.e. considering uncertainties. This paper considers the situation in Fig. 1 and Fig. 2 , while the Glover-McFarlane plant structure can be considered as in Fig. 3. They all lead to the same result.

3.1 *Static feedback*

Follow the general mixed full information static feedback solution as in (Christiansson 2000), with the aggregated input signal $\mu = \begin{bmatrix} [e' \;\; v'] & u' \end{bmatrix}'$. The general mixed static feedback Riccati equation can then be expressed as

$$S^- = A' \diamond S + C_z' C_z - L' Q_\mu L \qquad (10)$$

where

$$L = Q_\mu^{-1} Q_{\mu x},\; Q_{\mu x} = D_z' C_z + B' S A_{t_k}$$
$$Q_\mu = D_z' D_z - \gamma^2 Q_\gamma + \delta_{t_k} B_w' S B_w$$

The matrix $Q_\gamma = \text{diag}(I_{n_w}, 0_{n_u})$ implying that $w'w = \mu' Q_\gamma \mu$. This general solution is applied to the simplified system (9). The matrix Q_μ can be factored as $Q_\mu = U' \text{diag}(-\hat{Q}_w, \hat{Q}_u) U$, with U unitary. Furthermore, introduce the notations

$$\nu = 1 - \gamma^{-2}, Q_\nu = I + D'D + \delta_{t_k} B' S_\nu B \quad (11a)$$
$$S_\nu = \nu S, \qquad L_\nu = Q_\nu^{-1}(D'C + B' S_\nu A_{t_k}) \quad (11b)$$

For a solution S in (10) to exist, the matrix Q_μ must be invertible, and thus also $\hat{Q}_w$ and $\hat{Q}_u$. These matrices are different in the the full information (FI) and state feedback (SF) cases, see more details in (Christiansson *et al.* 2001). For the system (9) they are

$$FI: \begin{cases} \hat{Q}_w = (\gamma^2 - 1)I + \begin{bmatrix} -D \\ I \end{bmatrix} \hat{Q}_u^{-1} \begin{bmatrix} -D' & I \end{bmatrix} \\ \hat{Q}_u = I + D'D + \nu^{-1} \delta_{t_k} B' S_\nu B \end{cases} \quad (12a)$$

$$SF: \begin{cases} \hat{Q}_w = \begin{bmatrix} (\gamma^2 - 1)I & -D \\ -D' & \gamma^2 I - (Q_u - I) \end{bmatrix} \\ \hat{Q}_u = (\gamma^2 - 1) Q_\nu (\gamma^2 I - Q_\nu)^{-1} \end{cases} \quad (12b)$$

For the full information case a sufficient demand is $\gamma > 1$. In the following, only the FI case is shown. The static feedback gain matrix L can be partitioned as $L = [-L_w' \quad \nu^{-1} L_\nu']'$, where

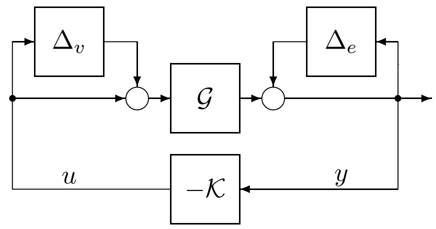

Fig. 2. Multiplicative uncertainties at input and output; $y = (I - \Delta_e)^{-1} \mathcal{G} (I + \Delta_v) u$

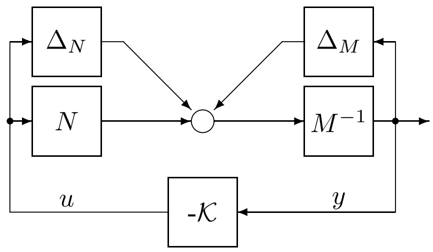

Fig. 3. Additive uncertainties in the coprime factors, $\mathcal{G} = M^{-1}N$; $y = (M - \Delta_M)^{-1}(N + \Delta_N)u$

$L_w = \frac{1}{\gamma^2-1}\left[\, C' - L_\nu' D' \;\; L_\nu' \,\right]'$. The mixed static feedback Riccati equation (10) can now be expressed in S_ν instead of S as

$$S_\nu^- = A' \diamond S_\nu + C'C - L_\nu' Q_\nu L_\nu \tag{13}$$

Note that all matrices in the Riccati equation (13) are independent of γ, and thus no γ-iteration is needed when solving this equation, however a lowest possible $\gamma > 1$ as was discussed above. One suboptimal static feedback *full information* controller that achieves the norm bound (8) is

$$\begin{aligned} u^* = -Q_u^{-1}(D'C + \nu^{-1}B'S_\nu A_{t_k})x - \\ -Q_u^{-1}D'e - Q_u^{-1}(D'D + \nu^{-1}\delta_{t_k}B'S_\nu B)v \end{aligned} \tag{14}$$

When the disturbances e, v are the worst ones, this optimal control is expressed as

$$u^* = -L_\nu x \tag{15}$$

Remark: This is the same controller as is achieved for the SF-case, i.e. when the states can be fed back directly, however the sub-optimal γ-value might differ, see discussion after (12b).

3.2 *Filter for estimating $\hat{z}$*

Estimation of $\hat{z}$ is performed with the aggregated output signal $\eta = [z' \;\; y']'$, and initially without the control signal u. The filter solution is, as is more or less standard, solved as an adjoint full information static feedback solution. The general mixed filter Riccati equation is, cf. (10)

$$P^+ = A \diamond P + B_w B_w' - K R_\eta K' \tag{16}$$

where

$$\begin{aligned} K = R_{x\eta}R_\eta^{-1},\; R_{x\eta} &= B_w D_w' + A_{t_k} P C_w' \\ R_\eta &= D_w D_w' - \gamma^2 R_\gamma + \delta_{t_k} C_w P C_w' \end{aligned}$$

The matrix $R_\gamma = \mathrm{diag}(I_{n_z}, 0_{n_y})$. The filter Riccati equation for the system (9) can be further simplified with the notations

$$R_y = I + DD' + \delta_{t_k} CPC' \tag{17a}$$

$$K_y = (BD' + A_{t_k} PC')R_y^{-1} \tag{17b}$$

Then the mixed filter Riccati equation (16) can be rewritten

$$P^+ = A \diamond P + BB' - K_y R_y K_y' \tag{18}$$

As (13), this Riccati equation does not depend on γ, and thus no γ-iteration is needed for this solution either. In fact, from a filter point of view, γ can be arbitrary small, however positive. One filter that achieves the bound $\|\mathcal{G}_{(z-\hat{z})w}\|_{[0,T_f]} < \gamma$, when also the control signal is included, is

$$\hat{x}^+ = A\hat{x} + Bu + K_y(y - \hat{y}) \tag{19}$$

$$\hat{z} = \begin{bmatrix} y \\ u \end{bmatrix}, \qquad \hat{y} = C\hat{x} + Du$$

Note that this filter $\hat{z}$ only holds the measured output y and the known controller output u.

3.3 *The output feedback controller*

The output feedback solution is achieved from studying a "new" transformed system with input and output signals as weighted deviations of the "optimal" ones, here denoted $\bar{e}, \bar{v}$ and $\bar{u}$ respectively, see the general mixed solution in (Christiansson 2000). The transformed system, without the input signal u, is

$$\begin{bmatrix} x \\ \bar{u} \\ y \end{bmatrix} = \begin{bmatrix} \bar{A} & \bar{B}_{\bar{w}} \\ \bar{C}_{\bar{u}} & \bar{D}_{\bar{u}\bar{w}} \\ \bar{C}_y & \bar{D}_{y\bar{w}} \end{bmatrix} \begin{bmatrix} x \\ \begin{bmatrix} \bar{e} \\ \bar{v} \end{bmatrix} \end{bmatrix} \tag{20}$$

where

$$\bar{A} = A + \gamma^{-2}\nu^{-1}BL_\nu, \quad \bar{B}_{\bar{w}} = \left[\, 0 \quad \gamma B \,\right] \hat{Q}_w^{-\frac{1}{2}}$$

$$\bar{C}_{\bar{u}} = \nu^{-1}Q_u^{\frac{1}{2}}L_\nu, \quad \bar{C}_y = \nu^{-1}C$$

$$\bar{D}_{\bar{u}\bar{w}} = \gamma Q_u^{-\frac{1}{2}} \left[\, D' \quad Q_u - I \,\right] \hat{Q}_w^{-\frac{1}{2}}$$

$$\bar{D}_{y\bar{w}} = \gamma \left[\, I \quad D \,\right] \hat{Q}_w^{-\frac{1}{2}}$$

The output feedback solution is achieved from a filter solution for the transformed system (20). The corresponding Riccati equation has a solution $\bar{P}$, which can be obtained from the static feedback and filter solutions S_ν and P. Also introduce $\bar{P}_\nu$ and the following holds

$$\bar{P}_\nu = \nu^{-1}\bar{P} = P(I - \gamma^{-2}(I + S_\nu P))^{-1} \tag{21}$$

For simpler controller formulation, introduce the matrices

$$\begin{aligned} K_\nu &= (BD' + A_{t_k}\bar{P}_\nu C')(I + DD' + \delta_{t_k}C\bar{P}_\nu C')^{-1} \\ K_{\nu y} &= (D' + \delta_{t_k}L_\nu \bar{P}_\nu C')(I + DD' + \delta_{t_k}C\bar{P}_\nu C')^{-1} \end{aligned} \tag{22}$$

The results are now summarised in a theorem.

Theorem 1. Consider the mixed time-varying system (4) with (9) on $t \in [0, T_f]$. Then

- there exists a mixed controller $u = \mathcal{K}(y)$ on $t \in [0, T_f]$, which achieves the bound $\|\mathcal{G}_{zw}\|_{[0,T_f]} < \gamma$

if and only if ($\Leftrightarrow$)

- there exist mixed piece-wise continuous matrix functions $S_\nu \geq 0$, $P \geq 0$ satisfying the mixed Riccati equations (13) and (18) on $t \in [0, T_f]$.
- the spectral radius $\rho(S_\nu P) < \gamma^2 - 1$ on $t \in [0, T_f]$.

One controller, that achieves the bound is

$$\hat{x}_\nu^+ = A\hat{x}_\nu + Bu + K_\nu(y - \hat{y}) \tag{23a}$$

$$\hat{y} = C\hat{x}_\nu + Du \tag{23b}$$

$$u = -L_\nu \hat{x}_\nu - K_{\nu y}(y - \hat{y}) \tag{23c}$$

where $K_\nu, K_{\nu y}$ are defined in (22), L_ν in (11b) and $\bar{P}_\nu$ in (21). $\square$

Proofs for the general case are found in (Christiansson 2000), and applied to the system (9) also in (Christiansson *et al.* 2001). This special application is attractive since there is no need for γ-iteration when solving the Riccati equations, however the spectral radius demand must be fulfilled, i.e.

$$\gamma > \sqrt{1 + \rho(S_\nu P)} \tag{24}$$

which is > 1, as was a sufficient demand for the S_ν-solution to exist. Pure continuous-time and discrete-time solutions are just different interpretations of this mixed result. Note that the matrix D can be non-zero, and that there was no need for any normalised coprime factorisation of the plant. In continuous time, the controller is strictly proper when $D = 0$, however not when $D \neq 0$. In discrete-time, the controller is never strictly proper. Strictly proper discrete-time controllers are discussed in e.g. (Mirkin 1997, Iglesias 2000).

The situation in Theorem 1 can be extended to infinite time horizon, if additional detectability and stabilisability conditions for the system are added, see e.g. (Christiansson *et al.* 2000).

Theorem 2. Consider the situation in Theorem 1, when $T_f \to \infty$. Assume further that the system (A, B, C) is stabilisable and detectable. Then the controller (23) is stabilising. □

Proofs are found in (Christiansson *et al.* 2001). In cases where it is not obvious how to check the detectability and stabilisability criteria, such as for periodic and mixed systems, these tests can preferably be replaced by a check afterwards that the systems $A - BL_\nu$ and $A - K_y C$ are stable, see discussion in next subsection.

3.4 *Periodic continuous time systems*

For periodic systems, the continuous-time static feedback and filter solutions between time instants t and $t + T_p$, where T_p is the period, can be achieved from a lifting (or discretisation) procedure as in the sampled-data case in e.g. (Toivonen and Sågfors 1997). The γ-free continuous-time Riccati equations (13) and (18) can be reorganised as

$$-\dot{S}_\nu = A'_{\nu_c} S_\nu + S_\nu A_{\nu_c} + C'_{\nu_c} C_{\nu_c} - S_\nu B_{\nu_c} B'_{\nu_c} S_\nu \tag{25a}$$

$$\dot{P} = A_{\nu_c} P + P A'_{\nu_c} + B_{\nu_c} B'_{\nu_c} - P C'_{\nu_c} C_{\nu_c} P \tag{25b}$$

where

$$A_{\nu_c} = A_c - B_c(I + D'_c D_c)^{-1} D'_c C_c$$
$$B_{\nu_c} = B_c(I + D'_c D_c)^{-1} \begin{bmatrix} -D'_c & I \end{bmatrix}$$
$$C_{\nu_c} = \begin{bmatrix} I \\ -D'_c \end{bmatrix} (I + D_c D'_c)^{-1} C_c$$

The Hamiltonian matrices associated with the static feedback and filter Riccati equations (25) are respectively

$$\mathcal{H}_{S_\nu} = \begin{bmatrix} A_{\nu_c} & -B_{\nu_c} B'_{\nu_c} \\ -C'_{\nu_c} C_{\nu_c} & -A'_{\nu_c} \end{bmatrix} \tag{26a}$$

$$\mathcal{H}_P = \begin{bmatrix} -A'_{\nu_c} & C'_{\nu_c} C_{\nu_c} \\ B_{\nu_c} B'_{\nu_c} & A_{\nu_c} \end{bmatrix} \tag{26b}$$

These are related as $\mathcal{H}_P = T^{-1} \mathcal{H}_{S_\nu} T$ with $T = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. The transition matrix from time s to t associated with $\mathcal{H}_{S_\nu}$ is denoted $\Pi(t, s)$. Let Π be partitioned as $\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix}$. Now there are enough preliminaries to give a discretisation theorem reflecting the situation over the period.

Theorem 3. Consider the periodic system (9) in continuous time over the period T_p, from time instant t to $t + T_p$. The discretised system model

$$\begin{bmatrix} x(t + T_p) \\ \tilde{z}(t, T_p) \end{bmatrix} = \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \tilde{u}(t, T_p) \end{bmatrix} \tag{27}$$

where $\tilde{A}(t, T_p) = \Pi_{11}^{-1}(t, t+T_p)$

$$\tilde{B}(t, T_p)\tilde{B}'(t, T_p) = \Pi_{11}^{-1}(t, t+T_p)\Pi_{12}(t, t+T_p)$$
$$\tilde{C}'(t, T_p)\tilde{C}(t, T_p) = \Pi_{21}(t, t+T_p)\Pi_{11}^{-1}(t, t+T_p)$$

generates discrete-time static feedback and filter Riccati equations with the same solutions as the continuous-time system does at time instants t and $t + T_p$ respectively. These Riccati equations are

$$S_\nu(t) = \tilde{A}'(I + S_\nu(t+T_p)\tilde{B}\tilde{B}')^{-1} S_\nu(t+T_p)\tilde{A} + \tilde{C}'\tilde{C}$$
$$P(t+T_p) = \tilde{A}P(t)(I + \tilde{C}'\tilde{C}P(t))^{-1}\tilde{A}' + \tilde{B}\tilde{B}' \quad \square$$

The state transition matrices for $A - BL_\nu$ (static feedback) and $A - K_y C$ (filter) respectively from times t to $t+T_p$ can be derived for the discretised system (27) and are respectively

$$\Pi_{A-BL}(t+T_p, t) = (I + \tilde{B}\tilde{B}' S_\nu(t+T_p))^{-1}\tilde{A}$$
$$\Gamma_{A-KC}(t+T_p, t) = \tilde{A}(I + P(t_k^+)\tilde{C}'\tilde{C})^{-1}$$

The discretised system model (27) can thus be used for periodic continuous-time systems to achieve the solution integrated over the period. In steady state, when final time $T_f \to \infty$, the solutions, if they exist, are periodic such that $S_\nu = S_\nu(t) = S_\nu(t + T_p), P = P(t) = P(t + T_p)$, and algebraic Riccati equations are obtained.

The corresponding discrete-time state updates at times t_k are achieved from $A_d - B_d L_{\nu_d}$ and $A_d - K_{y_d} C_d$ respectively, and the state transition including both the continuous and discrete parts between two arbitrary sampling instants t_k^- and t_{k+1}^- can be calculated as

$$\Pi_{A-BL}(t_{k+1}^-, t_k^-) = \Pi_{A-BL}(t_{k+1}^-, t_k^+)(A_d - B_d L_d)$$
$$\Gamma_{A-KC}(t_{k+1}^-, t_k^-) = \Gamma_{A-KC}(t_{k+1}^-, t_k^+)(A_d - K_{y_d} C_{y_d})$$

When there are a number, r, of discrete updates during the period the combined state transitions for $A - BL_\nu$ over the period are considered as

$$\Pi_{A-BL}(t_{k+r}^-, t_k^-) = \Pi_{A-BL}(t_{k+r}^-, t_{k+r-1}^-) \cdots \cdots \Pi_{A-BL}(t_{k+1}^-, t_k^-) \qquad (30)$$

and similarly for the filter. Then the closed system is stable, if the eigenvalues of $\Pi_{A-BL}(t_{k+r}^-, t_k^-)$ all are inside the open unit disk and similarly for the filter. If $A - BL_\nu$ and $A - K_y C$ so are determined to be stable over the period, the system is obviously stabilisable and detectable. This is a constructive way to check stabilisability/detectability of the system (A, B, C), especially when this is difficult to do in advance. On the other hand, if it is easily tested that the system (A, B, C) is detectable and stabilisable there is no need to do the test afterwards.

4. SUMMARY OF MIXED $\mathcal{H}_\infty$LOOP SHAPING DESIGN

There are a number of advantages to point out about the $\mathcal{H}_\infty$-design procedure presented in this paper. First of all the system setup (9) does not require any γ-iteration, as is the case in a more general $\mathcal{H}_\infty$-setup. Nor is there a need for any normalised coprime factorisation of the plant, as is the case in traditional loop shaping solutions. The mixed approach is attractive since it shows a close relation between the continuous-time and discrete-time solutions, such that these are simply obtained as two different interpretations of the unified mixed result. Simple discrete-time results are given for continuous-time periodic systems, reflecting the behaviour over the period. The results are applicable for a number of situations, e.g. time-varying systems and hybrid systems in the meaning time-driven discrete-event systems including continuous dynamics. When $\gamma \to \infty$, this loop shaping solution is identical to the well known LQG-solution. The paper also gives a constructive way to test whether a time-varying system is stabilisable and detectable.

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