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GLOBAL CONTROLLER OPTIMIZATION USING HOROWITZ BOUNDS

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Abstract: A procedure for global optimization of PID type controller parameters for SISO plants with model uncertainty is presented. Robustness to the uncertainties is guaranteed by the use of Horowitz bounds, which are used as constraints when low frequency performance is optimized. The basic idea of both the optimization and the parameter tuning is to formulate separate criteria for low, mid and high frequency closed loop properties. The trade-off between stability margins, high frequency robustness and low frequency performance is then elucidated and, hence, the final choice of parameters is facilitated. The optimization problems are non-convex and ill-conditioned and we use a combination of new global and standard local optimization algorithms available in the TOMLAB optimization environment to solve the problem. The method does not rely on a good initial guess and converges fast and robustly. It is applied to a controller structure comparison for a plant with an uncertain mechanical resonance. For a given control activity and stability margin as well as identical tuning parameters it is shown that a PID controller achieves slightly improved low frequency performance compared to an $\mathcal{H}_\infty$ controller based on loop-shaping. The reason for this somewhat surprising result is the roll-off in the $\mathcal{H}_\infty$ controller, which adds additional high frequency robustness compared to the PID controller. Computationally, a factor of $10-20$ has been gained compared to an earlier, less general, version of the procedure.

Keywords: $\mathcal{H}_\infty$, PID, global optimization, Horowitz bounds, performance, robustness

1. INTRODUCTION

In many controller design techniques a mixed, possibly weighted, performance criterion is used to ensure that the closed loop achieves desirable behavior. This criterion includes multiple closed loop objectives and is minimized to obtain an optimal solution with respect to the performance objectives (as in, e.g., the mixed sensitivity optimization). However, it is important to have separate criteria for the closed loop properties at different frequency regions such that the trade-off between performance and robustness can easily be evaluated, especially in case of a change of closed loop specifications. Such a procedure has been used to evaluate PI and PID controllers, as well as for tuning the controller parameters by optimization (Lennartson and Kristiansson, 1997; Kristiansson and Lennartson, 1999).

For plants with uncertainties, the mid frequency (MF) robustness properties are crucial, whereas the low frequency (LF) performance and robustness are less affected by the uncertainties due to high gain (integral action). For high frequencies (HF) it is simply a question of having a small loop gain. In case explicit descriptions of the plant uncertainties have been formulated, the Quantitative Feedback Theory (QFT) by Horowitz (1993) can be used to design controllers such that specified bounds on, e.g., the sensitivity function $S(j\omega)$ and the control sensitivity function $KS(j\omega)$ are satisfied in spite of the uncertainties. The basis for this method is a translation of the bounds on S and KS to so-called Horowitz bounds on the nominal open loop. For this purpose a toolbox QSYN (Gutman, 2001) running on MATLAB (Mathworks, 2000) can be used.

The aim of the traditional QFT method is a minimization of the HF open loop gain (Bryant and Halikias, 1995; Gera and Horowitz, 1980). It has no fixed structure of the controller and, hence, in its general form it gives an unlimited number of tuning parameters. Zolotas and Halikias (1999) used the QFT approach with bounds on the complementary sensitivity but fixed the structure to an ideal PID controller and minimized the derivative gain. Neither of these design methods considers the trade-off between the LF, MF and HF properties of the closed loop. This trade-off is important since, e.g., the process disturbance rejection can often be significantly improved at only a marginal reduction of the HF robustness (Kristiansson and Lennartson, 1999). To pursue this, Fransson *et al.* developed a constrained optimization procedure where PID and PID weighted $\mathcal{H}_\infty$ loop shaping controllers were designed based on an optimization of the LF performance subject to specified bounds on the maximum sensitivity $\|S(\Delta)\|_\infty$ and the maximum nominal control sensitivity $\|KS\|_\infty$ (Fransson *et al.*, 2000; Fransson *et al.*, 2001*b*). Robustness to plant model uncertainties (for S) was guaranteed by Horowitz bounds. The procedure, however, suffered from the fact that local optimization methods were used to solve highly nonlinear problems with potential discontinuities in the parameter space. As a result, difficulties with initial guesses, convergence and local optima were frequently experienced.

In (Fransson *et al.*, 2001*a*), the method was extended to MIMO systems, resulting in two major differences compared to the earlier work. Firstly, the QFT techniques were replaced with μ analysis because of the mathematical complexity of QFT for MIMO systems. Secondly, to remedy the numerical problems a combination of state of the art global (Jones, 2001; Jones *et al.*, 1993) and local (Murtagh and Saunders, 1998) optimization algorithms was used to solve the following non-convex design problem:

$$K^* = \arg\min_K J_v \tag{1}$$

$$\|S(\Delta)\|_\infty \leq c_S, \quad \|KS(\Delta)\|_\infty \leq c_{KS} \quad \forall \Delta \in \mathbf{\Delta}, \tag{2}$$

where K^* is the controller obtained by minimizing the LF performance measure J_v subject to user defined specifications c_S and c_{KS} on $\|S\|_\infty$ and $\|KS\|_\infty$ respectively. Δ defines a deviation from the nominal plant model and $\mathbf{\Delta}$ is the set of all such possible plant uncertainties. K is taken as either a PID controller or a PID weighted $\mathcal{H}_\infty$ controller, synthesized according to the method by McFarlane and Glover (1992). Implementations of the algorithms in (Jones, 2001; Jones *et al.*, 1993) and (Murtagh and Saunders, 1998) are available in the optimization environment TOMLAB (Björkman and Holmström, 1999; Holmström, 1999; Holmström, 2001), which runs on top of MATLAB (Mathworks, 2000).

In this paper, the design method in (Fransson *et al.*, 2000; Fransson *et al.*, 2001*b*) is improved in the following ways:

- A reliable test is given that determines if L_{nom} is (pointwise) inside or outside the Horowitz bounds in the Nichols chart.
- Robustness to plant model uncertainties for $\|KS\|_\infty$ is guaranteed.
- The numerical procedure in (Fransson *et al.*, 2001*a*) is used, resulting in fast convergence towards a global optimum of (1)–(2).

2. PLANT UNCERTAINTY AND HOROWITZ BOUNDS

A plant having parametric uncertainty can be defined as

$$G(s) \in \{G(s, \Delta)\},$$

where $\Delta \in \mathbf{\Delta} \subset \mathbb{R}^p$ is a vector of uncertain parameters. Approximate the set of transfer functions obtained by the uncertainty representation with $\{G_i(s)\}_{i=1}^N$. For each frequency, $\omega_k \in \Omega$, the set $\{G_i(j\omega_k)\}_{i=1}^N$ in the complex plane is then called a *template* (Horowitz, 1993). The template should enclose all possible frequency responses of the plant at the frequency ω_k.

To design a controller that satisfies the specifications for all plant variations within the uncertainty set, QFT can be used (Horowitz, 1993). It is required that the plant uncertainty be represented by a set of templates and that the design specifications be in the form of bounds on the magnitudes of some frequency response functions. The frequency response specifications, in turn, result in constraints on the nominal open loop $L_{nom}(j\omega) = G_{nom}(j\omega)K(j\omega)$. These constraints

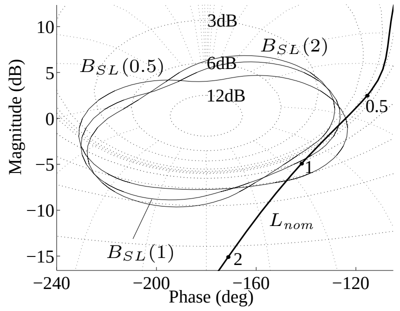

Fig. 1. Nichols chart with Horowitz bounds and a nominal open loop for $\omega_k = 0.5$, 1 and 2.

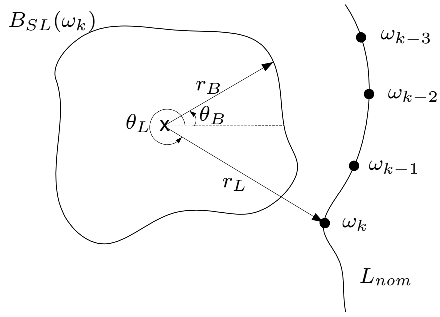

Fig. 2. A nominal open loop and the corresponding Horowitz bound for ω_k transformed to polar coordinates.

are called *Horowitz bounds* and reflect the interaction between the plant uncertainty and the closed loop specifications. We define

$$S_i(s) = \frac{1}{1 + G_i(s)K(s)}, \qquad i = 1, 2, \ldots, N,$$

where G_i is the uncertain plant, and further impose an upper bound c_S on the maximum frequency response of this sensitivity function for all plants in $\{G_i\}_{i=1}^N$. To satisfy this bound, the following must hold:

$$M_S \equiv \max_i \left| \frac{1}{1 + G_i(j\omega_k)K(j\omega_k)} \right| \le c_S \ \forall k. \quad (3)$$

Thus, the controller must be chosen such that (3) is fulfilled for the complex numbers $K(j\omega_k)$. Clearly, for each G_i there is a domain in the complex plane of $K(j\omega_k)$ values for which (3) does not hold. The union of all such domains gives a domain, or possibly several domains, that contains the unacceptable values of $K(j\omega_k)$. The boundary of this domain is called the Horowitz bound for K with respect to S and ω_k and is denoted $B_{SK}(\omega_k)$. Multiplying by the nominal plant yields $B_{SL}(\omega_k)$. From the plant templates and the specifications, the Horowitz bounds can be computed with QSYN and placed in a Nichols chart together with L_{nom}. To ensure that (3) is satisfied for all plants within the uncertainty set, L_{nom} must be shaped such that, at each frequency ω_k, it lies outside the bound for that frequency (see Figure 1). Analogously to (3) we also define

$$M_{KS} \equiv \max_{i,k} \left| \frac{K(j\omega_k)}{1 + G_i(j\omega_k)K(j\omega_k)} \right| \le c_{KS}. \quad (4)$$

In an automated procedure one would like to have a test that determines if $L(j\omega_k)$ lies outside, e.g., $B_{SL}(\omega_k)$ $\forall$ k or not. Now consider a single k and transform the origin in the Nichols chart to the interior of the considered B (where the subindex $_{SL}$ and the argument ω have been dropped). If B is transformed to polar coordinates we can write $r_B = B(\theta_B)$ and if L is transformed to the same coordinates it can be represented by (r_L, θ_L). The test for L being outside B is then simply (see Figure 2)

$$r_L > B(\theta_L). \quad (5)$$

Note that the test can fail if $B(\theta_L)$ has more than one solution (i.e. if the non-convexity of B is very high). This is unlikely, however, if the origin is chosen carefully.

3. CONTROLLER STRUCTURES

Both PID controllers and $\mathcal{H}_\infty$ synthesis by loop shaping (McFarlane and Glover (1992)) are considered when solving (1)–(2). By choosing a PID structure for the weight function in the loop shaping procedure and using its parameters for tuning, we make the optimization procedure more or less the same as for a standard PID controller. Further, it has the advantage of not including a γ-iteration. Instead, sub-optimality is introduced in terms of a stability margin ϵ.

The first step in the method is to shape G with a weight W to give an open loop that meets some nominal performance specifications. For the shaped system $\tilde{G} = GW$, a controller K_∞ is then obtained by solving a set of Riccati equations. The final feedback controller for the plant G is then $K = WK_\infty$ (see Figure 3). For a given weight W the controller derived in this way will, in some sense, have optimal robustness. The degree of optimality is determined by $\epsilon = \epsilon_{max}\alpha^{-1}$, where ϵ_{max} is called the maximum stability margin and $\alpha > 1$ is a scaling factor. In this paper W is parameterized as a PID controller,

$$K_{PID}(s) = k_I \frac{1 + 2\zeta\tau s + (\tau s)^2}{s(1 + s\frac{\tau}{\beta})}. \quad (6)$$

These PID parameters are then subject to tuning (by an optimization procedure) such that a desired closed loop behavior is obtained.

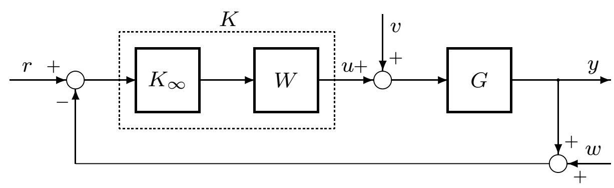

Fig. 3. Closed loop system with an $\mathcal{H}_\infty$ loop shaping controller K, and a plant G.

4. PERFORMANCE MEASURES

In (Lennartson and Kristiansson, 1997) the following LF performance measure was suggested:

$$J_v = \|s^{-1}SG\|_\infty, \tag{7}$$

The disturbance sensitivity function SG is the transfer function from (LF) load disturbance v to output y, which implies that J_v is a measure of load disturbance rejection (see Figure 3). KS is the transfer function from measurement noise w to the control signal u and, thus, M_{KS} is a measure of the control activity caused by (HF) measurement noise, c.f. (4). $1/\|S\|_\infty$ is the shortest distance from the open loop (GK) to the instability point $(-1, 0)$ in the Nyquist diagram and, hence, M_S is a natural robustness measure, c.f. (3). We next introduce a general expression for controllers including integral action as

$$K(s) = \frac{\kappa_I}{s} + \tilde{K}(s), \tag{8}$$

where $|\tilde{K}(j0)|$ is bounded and $|K(j\infty)| = \kappa_\infty$. It can be readily shown that $|(j\omega)^{-1}SG(j\omega)| \to \kappa_I^{-1}$ as $\omega \to 0$ (provided $|G(j0)| \neq 0$) (Fransson *et al.*, 2000). For PID controllers, $\kappa_I = k_I$ and in fact, k_I^{-1} serves as a close approximation to J_v. It means that this LF performance measure will be relatively independent of the plant transfer function and, hence, also the uncertainty.

Measures and criteria for each frequency range (LF, MF and HF) have now been presented and these can be used to determine a controller that gives an acceptable behavior of the closed loop system for all frequencies, in spite of the uncertainties. The constrained optimization problem to be solved is

$$\min_{k_I,\zeta,\tau,\beta} J_v \quad s.t. \quad M_S \leq c_S, \quad M_{KS} \leq c_{KS}. \tag{9}$$

If we choose the loop shaping weight $W = K_{PID}$, the difference between optimizing PID controllers and $\mathcal{H}_\infty$ controllers with (9) is that the optimization for $\mathcal{H}_\infty$ design consists of two steps: an outer loop for tuning the weight W, and an inner loop for the standard $\mathcal{H}_\infty$ optimization.

5. GLOBAL OPTIMIZATION

The optimization problem (9) is non-convex, nonsmooth and ill-conditioned. Using a standard sequential quadratic programming (SQP) method (see e.g., Bertsekas (1995)) is troublesome because it is only guaranteed to find a local optimum, and relies heavily on a good initial guess. Furthermore, analytic gradients are not available and numerical differences must be used. However, recent development of algorithms for non-convex global optimization makes it possible to overcome these problems.

Jones *et al.* (1993) have developed an algorithm DIRECT for finding the global minimum of a multi-variate function subject to simple bounds, using no derivative information. The algorithm is a modification of the standard Lipschitzian approach that eliminates the need to specify a Lipschitz constant. The idea is to carry out simultaneous searches using all possible constants from zero to infinity. In (Jones *et al.*, 1993) the Lipschitz constant is viewed as a weighting parameter that indicates how much emphasis to place on global versus local search. In standard Lipschitzian methods, this constant is usually large because it must be equal to or exceed the maximum rate of change of the objective function. As a result, these methods place a high emphasis on global search, which leads to slow convergence. In contrast, the DIRECT algorithm carries out simultaneous searches using all possible constants, and therefore operates on both the global and local level. It is guaranteed to converge to the global optimal function value if the objective function is continuous in the neighborhood of a global optimum (this could be guaranteed because as the number of iterations goes to infinity, the set of points sampled by DIRECT forms a dense subset of the unit hypercube).

The most recently developed DIRECT algorithm (Jones, 2001) handles nonlinear and integer constraints, whereas the original algorithm does not. Both of them have been implemented in the optimization environment TOMLAB as the routines `glcFast`, and `glbFast` (Björkman and Holmström, 1999) and have been successfully used in train design optimization (Björkman and Holmström, 2000) and for the design of trading algorithms in computational finance (Hellström and Holmström, 1999). TOMLAB runs on top of MATLAB and includes a large set of solvers, e.g. the NPSOL solver, implementing an SQP algorithm (`npsol`) by Gill *et al.* (1998).

Since `glcFast` can handle nonlinear constraints, it is suitable for solving (9). In this algorithm one has to specify lower and upper bounds for the independent variables and also the maximum number of function evaluations N. The DIRECT algorithms converges to the global optimum in the limit $N \to \infty$. However to achieve results in finite time, N must be finite and a global optimum can no longer be guaranteed. To increase the performance of the optimization procedure, and allow for smaller N, we switch to NPSOL when the global algorithm has reached the convergence region for `npsol` (which is estimated prior to the optimization). With this approach the probability of finding the global minimizer K^* to (9) increases significantly compared to using SQP-like methods alone. To summarize, the procedure is described in the following algorithm:

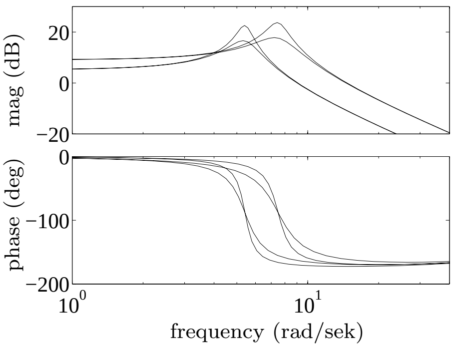

Fig. 4. Bode plot of G at the ends of the parameter intervals.

Algorithm 1:

(1) Define the plant model and the uncertainty set $\mathbf{\Delta}$ along with the grid for each uncertain parameter.
(2) Specify c_S, c_{KS} and Ω along with its grid, then generate $B_{SL}(\omega_k)$ and $B_{KSL}(\omega_k)$ with QSYN.
(3) Specify upper and lower bounds for the PID parameters and the number of function evaluations N for the optimization. Also specify when the switch from the global optimization algorithm to the local is to occur.
(4) Run the optimization.
(5) Go to 2 and repeat with new specifications.
(6) Evaluate with respect to the proposed LF, MF, and HF measures as well as other system properties, especially in the time domain.
(7) If needed go to 2 and repeat.

6. EXAMPLE

Consider the following plant transfer function:

$$G(s) = \frac{4s + 800}{Js^2 + d(1 + 0.08J)s + 16J + 200},$$

where the uncertainty intervals are assumed to be $J \in [5\ 15]$ and $d \in [5\ 10]$ (see Figure 4) and the nominal plant is described by $J = 10$ and $d = 7.5$. The design parameters were chosen as $c_S = 1.7$ and $c_{KS} = 3$, for which Horowitz bounds for 25 frequencies were computed with a grid of 16 points for each of the uncertain parameters. Algorithm 1 was then performed both for an $\mathcal{H}_\infty$ controller (choosing the scaling factor $\alpha = 1.01$) and a PID controller. c_{KS} was gradually increased up to 14 to reflect less strict constraints on the control signal. The entire procedure was repeated for $c_S = 1.5$. All calculations were done with MATLAB on a 1GHz Pentium III processor.

For a given c_S and c_{KS} the optimization required roughly 300 function evaluations for `glcFast` (45 sec) and 50 for `npsol` (5 sec). This was more than adequate and can be compared to $10-20$ minutes in (Fransson *et al.*, 2000; Fransson *et al.*, 2001*b*) for solving a less general problem than here. Figure 5 shows the results of the optimization in terms of obtained J_v versus specified c_{KS} and c_S. We note the general trade-off between performance and control activity for all controllers, i.e., increased performance (reduced J_v) can be achieved without reducing the stability margin, but at a cost of higher control signals (increased c_{KS}). It is seen that for a given bound on the control activity (fixed c_{KS}), PID control can achieve slightly better results in terms of LF performance.

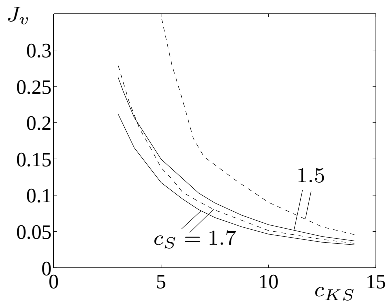

Fig. 5. J_v versus c_{KS} as obtained from Algorithm 1. The upper two curves are for $c_S = 1.5$ and the lower two for $c_S = 1.7$. Solid: PID control. Dashed: PID weighted $\mathcal{H}_\infty$ control.

We also choose to study the step responses from process disturbance $v(t)$ to output $y(t)$, from reference signal $r(t)$ to control signal $u(t)$ and from reference to output for $c_S = 1.7$ and $c_{KS} = 10$. In Figure 6 the step responses are shown for fixed values of the uncertain parameters and for each of the controllers. The difference between the two controller structures is seen to be marginal. However, the general advantage that $\mathcal{H}_\infty$ control implies a roll off in the controller should be taken into consideration. This means that unmodeled HF dynamics will not be amplified to the same extent as for PID control. Optimizing also over α will benefit the $\mathcal{H}_\infty$ design somewhat, in terms of the criteria, but not enough to beat the PID controller. Finally we note that the $\mathcal{H}_\infty$ controller is of order 6 compared to 2 for the PID controller.

7. CONCLUSIONS

An existing optimization method for the design of robust PID and $\mathcal{H}_\infty$ loop shaping controllers has been extended. The new method includes a combination of global and local optimization algorithms and results in a fast algorithm with robust convergence towards a global optimum. A reliable test has been given that can determine if the nominal open loop is (pointwise) inside or outside the Horowitz bounds in the Nichols chart.

The criteria used take important aspects for achieving robust performance into account, including a guaranteed robustness to explicit plant

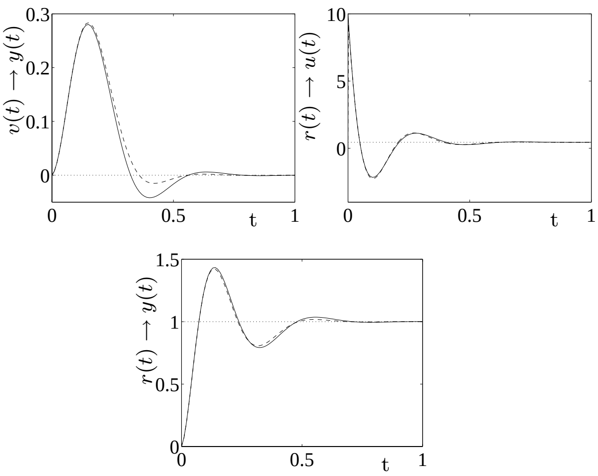

Fig. 6. Step responses for $G(s)$, $J_2 = 15$, $d = 5$ and $c_{KS} = 10$. Solid line: PID control. Dashed line: $\mathcal{H}_\infty$ control.

uncertainties by use of Horowitz bounds. The trade-off is elucidated in the same way as for plants with no uncertainties. Having separate criteria for the closed loop properties in the different frequency regions allowed the trade-off between robustness and performance to be studied easily and clarified the consequences of a change of specifications.

The design method has been applied to an example showing that a PID controller achieves slightly improved low frequency performance compared to an $\mathcal{H}_\infty$ controller based on loop-shaping. Computationally, a factor of $10 - 20$ has been gained compared to an earlier, less general, version of the design procedure.

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