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PIECEWISE QUADRATIC ESTIMATES OF DOMAINS OF ATTRACTION FOR LINEAR SYSTEMS WITH SATURATION

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Abstract: We show how piecewise quadratic Lyapunov functions can be used to estimate regions of attraction for linear systems with saturation. The central issues of how to restrict the analysis region and how to optimize the size of the domain of attraction are addressed, and the approach is demonstrated on several examples. We observe that the piecewise quadratic Lyapunov functions yield significant improvements over recently proposed methods based on the circle and Popov criteria.

Keywords: Stability, saturation, convex optimization

1. INTRODUCTION

Linear control systems closed by saturated feedback loops occur frequently in practice. Such system exhibit nonlinear behavior such as local stability, finite disturbance rejection and performance degradation when operating in saturation. In many applications it therefore becomes essential to determine the domain of attraction of the system and desirable to estimate the performance degradation when the system operates in saturation.

The problem of determining the stability domains for nonlinear systems has received considerable attention in the literature. Most techniques use a Lyapunov function to estimate the domain of attraction, see the survey by Genesio *et al.* (1985). Traditionally, such Lyapunov functions were found by trial-and-error, or by graphical or analytical procedures, see (Weissenberger, 1968; Margolis and Vogt, 1963). Since the late 70's, various computer-generated Lyapunov functions have been used for estimating the domain of attraction, see, *e.g.*, (Ohta *et al.*, 1993; Blanchini, 1995) and the references therein. More recently, it has been shown how both stability domains and local performance measures based on quadratic and Luré-type Lyapunov functions can be computed using semi-definite programming (Pittet *et al.*, 1997; Hindi and Boyd, 1998). In particular, the paper (Hindi and Boyd, 1998) shows that the search for the quadratic or Luré-type Lyapunov function that guarantees the largest domain of attraction can be cast as a convex optimization problem.

Unfortunately, current analysis tools tend to either provide conservative estimates or be computationally demanding. Conservativeness typically comes from limiting the analysis region to the unsaturated region (Kamenetskiy, 1996) or from treating the saturation as an uncertain (sector-bounded) element (Pittet *et al.*, 1997; Hindi and Boyd, 1998). Procedures that exploit the exact description of the saturation nonlinearity, such as (Romachuk, 1996), are computationally demanding and not easily extended to system of higher order or systems with multiple saturation loops.

In this paper, which is of methodological rather than theoretical nature, we show how piecewise quadratic estimates of the domain of attraction can be computed via convex optimization. The approach uses a piecewise quadratic Lyapunov function and a piecewise linear description of the closed-loop system. The central issues of how to restrict the analysis region, and

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how to optimize the "size" of the domain of attraction are addressed, and the approaches are compared on several examples from the literature.

This paper is organized as follows. Section 2 describes the set-up and discusses the models of the closed-loop system that arise when the saturation nonlinearity is considered as an uncertain element or described using piecewise linear techniques. Section 3 details an approach for estimating regions of attraction based on the circle criterion, while Section 4 describes the novel approach based on piecewise quadratic Lyapunov functions. We compare the computational techniques on a number of examples from the literature in Section 5, and conclude the paper in Section 6.

2. LINEAR SYSTEMS WITH SATURATION

2.1 *System Model and Problem Formulation*

Consider a linear system under saturated feedback

$$\begin{cases} \dot{x}(t) = Ax(t) + B_p p(t) & x(0) = x_0 \\ q(t) = C_q x(t) & p(t) = \operatorname{sat}(q(t)) \end{cases} \qquad (1)$$

where $\operatorname{sat}(\cdot)$ denotes the unit saturation

$$\operatorname{sat}(q) = \begin{cases} 1 & \text{if} \quad q \geq 1 \\ q & \text{if} \quad |q| \leq 1 \\ -1 & \text{if} \quad q \leq -1 \end{cases}$$

For simplicity, we will focus on the case of a single saturation nonlinearity; extensions to multiple saturations are discussed in Section 6. Hence, we assume that $x \in \mathbf{R}^n$, $A \in \mathbf{R}^{n\times n}$, $B_p \in \mathbf{R}^n$ and $C_q \in \mathbf{R}^n$.

Let $x(t, x_0)$ be the solution of (1) with initial condition $x(0) = x_0$. The set

$$\mathscr{D} = \left\{ x_0 \mid x(t, x_0) \to 0 \text{ as } t \to \infty \right\}$$

is called the *domain of attraction* of the system (1).

The aim of this paper is to derive a computationally efficient approach for obtaining good estimates of the domain of attraction for linear systems with saturation. In particular, we will develop an approach that

(1) provides good estimates of the domain of attraction,
(2) is simple to implement and computationally efficient to execute,
(3) applies to systems of arbitrary order and with multiple saturation loops.

2.2 *Uncertain Linear Models of Closed Loop System*

The classical approach for dealing with saturation nonlinearities is based on absolute stability theory. Although absolute stability analysis is often carried out globally, extensions to systems with finite domains of attractions were made over 30 years ago, see, *e.g.*, the papers by Walker and McClamroch (1967) and Weissenberger (1968). These authors propose to model the saturation nonlinearity as a locally sector-bounded element

$$q/r \leq \operatorname{sat}(q) \leq q \qquad \text{if } |q| \leq r$$

and consider the resulting polytopic model of the closed-loop system

$$\dot{x}(t) \in \overline{\operatorname{co}} \left\{ (A + B_p C_q) x(t), (A + r^{-1} B_p C_q) x(t) \right\}$$

valid for all trajectories $x(t)$ of (1) that satisfy

$$|C_q x(t)| \leq r \qquad \forall t \geq 0.$$

Recently, it has been shown how domains of attractions for systems described by local sector conditions can be computed via semi-definite programming (Pittet *et al.*, 1997; Hindi and Boyd, 1998). In particular, the paper (Hindi and Boyd, 1998) shows how the size of the domain of attraction can be optimized in this framework. We will return to these results in Section 3.

2.3 *Piecewise Linear Models of Closed Loop System*

An alternative approach is to exploit the fact that the dynamics of the closed loop system is piecewise linear

$$\dot{x} = A_i x + a_i \qquad \text{for } x \in X_i, \quad i \in I \qquad (2)$$

Here, $X = \cup_{i\in I} X_i$ is a partition of the state space into polyhedral cells, and I is the index set for the cells. Specifically, the saturation nonlinearity partitions the state-space of the system (1) into three regions

$$\begin{aligned} X_N &= \{x \mid C_q x \leq -1\} \\ X_L &= \{x \mid -1 \leq C_q x \leq 1\} \\ X_P &= \{x \mid C_q x \geq 1\} \end{aligned}$$

corresponding to negative saturation, linear operation, and positive saturation, respectively. The dynamics of (1) can be written on the form (2) since

$$\dot{x} = \begin{cases} Ax - B_p & \text{for } x \in X_N \\ (A + B_p C_q)x & \text{for } x \in X_L \\ Ax + B_p & \text{for } x \in X_P \end{cases}$$

We will return to this model formulation in Section 4 and see how it can be used to obtain piecewise quadratic estimates of the domain of attraction of the saturated system.

3. ESTIMATES FROM CIRCLE CRITERION

In this section, we will show how quadratic Lyapunov functions

$$V(x) = x^T P_c x \qquad (3)$$

can be used together with the local polytopic model from Section 2.2 to estimate regions of attraction for the system (1). The main idea is to require that (3) is a simultaneous Lyapunov function for the extreme system of the polytopic model. Since the polytopic model is only valid within the "slab"

$$\mathscr{S}_A = \{x \mid -r \leq C_q x \leq r\}$$

this analysis only guarantees that level sets of (3) fully contained in the analysis region $\mathscr{S}_A$ are regions of attraction for the system (1). The following result shows how to find the simultaneous Lyapunov function with largest level sets (in terms of trace) fully contained in the analysis region.

Proposition 1. Let $P_c = P_c^T$ be a solution to the convex optimization problem

$$\begin{array}{ll} \text{minimize} & \mathbf{Tr} P_c \\ \text{subject to} & 0 > (A+B_pC_q)^T P_c + P_c(A+B_pC_q) \\ & 0 > (A+r^{-1}B_pC_q)^T P_c + P_c(A+r^{-1}B_pC_q) \\ & 0 < \begin{bmatrix} r^2 & C_q \\ C_q^T & P_c \end{bmatrix} \end{array}$$

Then,

$$\mathscr{D}_c = \left\{x \mid x^T P_c x \leq 1\right\}$$

is a region of attraction for the system (1).

A more elegant solution that treats the multiple saturation-case more efficiently is given in (Hindi and Boyd, 1998). Furthermore, the papers (Pittet *et al.*, 1997; Hindi and Boyd, 1998) show how (often superior) estimates based on the Popov criterion can also be computed via optimization over linear matrix inequalities.

4. PIECEWISE QUADRATIC ESTIMATES

A general procedure for analysis of piecewise linear systems using piecewise quadratic Lyapunov functions was developed in (Johansson and Rantzer, 1998; Rantzer and Johansson, 2000). A particular feature of the approach is that the search for Lyapunov, storage, and value functions for piecewise linear systems can be performed via convex optimization over linear matrix inequalities. However, for systems with finite stability regions the analysis conditions given in (Johansson and Rantzer, 1998; Rantzer and Johansson, 2000) cannot be verified globally and the techniques cannot be immediately applied. In this section, we will describe how the original results can be extended to estimation of the domain of attraction. The novel developments include an approach for restricting the domain of analysis, and a method for optimizing the size of the domain of attraction. Both developments are critical for obtaining good estimates.

Piecewise linear systems We consider piecewise linear systems on the form

$$\dot{x} = A_i x + a_i \qquad \text{for } x \in X_i, \quad i \in I$$

Here, $X = \cup_{i\in I} X_i \subseteq \mathbf{R}^n$ is a partition of the state space into a number of closed (possibly unbounded) polyhedral cells, and I is the index set of the cells. We let $I_0 \subseteq I$ be the set of indices that contain the origin, and I_1 be the indices of the cells that do not contain the origin. For linear systems with saturation, I_0 will contain the index of the region of linear operation, while I_1 will contain the indices of all other cells. For convenient notation, we define

$$\bar{A}_i = \begin{bmatrix} A_i & a_i \\ 0 & 0 \end{bmatrix} \qquad \bar{x} = \begin{bmatrix} x \\ 1 \end{bmatrix}$$

and write the system dynamics as

$$\dot{\bar{x}} = \bar{A}_i \bar{x} \qquad \text{for } x \in X_i, \quad i \in I \tag{4}$$

Since the cells are polyhedra, they can be represented as the intersection of a finite number of halfspaces. In other words, for each X_i, there exist matrices

$$\bar{G}_i = \begin{bmatrix} G_i & g_i \end{bmatrix}$$

such that

$$X_i = \{x \mid \bar{G}_i \bar{x} \succeq 0\} \qquad i \in I. \tag{5}$$

Piecewise quadratic Lyapunov functions We will use Lyapunov functions candidates that are continuous and piecewise quadratic

$$V(x) = \begin{cases} x^T P_i x & x \in X_i,\ i \in I_0 \\ \bar{x}^T \bar{P}_i \bar{x} = x^T P_i x + 2q_i^T x + r_i & x \in X_i,\ i \in I_1 \end{cases}$$

To enforce continuity, we construct matrices

$$\bar{F}_i = \begin{bmatrix} F_i & f_i \end{bmatrix}$$

with $f_i = 0$ for $i \in I_0$ that satisfy

$$\bar{F}_i \bar{x} = \bar{F}_j \bar{x} \qquad x \in X_i \cap X_j \qquad i, j \in I. \tag{6}$$

This is always possible, since the cells are polyhedra. Now, by parameterizing the matrices P_i and $\bar{P}_i$ as

$$\begin{array}{ll} P_i = F_i^T T F_i & i \in I_0 \\ \bar{P}_i = \bar{F}_i^T T \bar{F}_i & i \in I_1 \end{array}$$

the Lyapunov function candidate is continuous and piecewise quadratic. The free parameters (over which the Lyapunov function candidate is optimized) are collected in the symmetric matrix T.

In our computations, we will also need matrices

$$\bar{E}_i = \begin{bmatrix} E_i & e_i \end{bmatrix}$$

with $e_i = 0$ for $i \in I_0$ such that

$$\bar{E}_i \bar{x} \succeq 0 \qquad \text{for } x \in X_i$$

These matrices can be constructed directly from the cell identifiers $\bar{G}_i$ without introducing any conservatism in the analysis, see (Johansson, 1999). For linear systems with saturation, the general procedure in (Johansson, 1999) reduces to

$$\begin{array}{ll} \bar{E}_i = 0 & i \in I_0 \\ \bar{E}_i = \begin{bmatrix} G_i & g_i \\ 0 & 1 \end{bmatrix} & i \in I_1 \end{array}$$

Describing the saturated system Procedures for determining matrices $\bar{G}_i$ and $\bar{F}_i$ for polyhedral partitions were given in (Johansson, 1999). For sake of completeness, however, we will show explicitly how the saturated system (1) can be described in the notation above, and how cell identifiers $\bar{G}_i$ and continuity matrices $\bar{F}_i$ can be determined.

Let $I = \{N, L, P\}$ be the indices for the regions corresponding to negative saturation, linear operation, and positive saturation, respectively. Hence, $I_0 = \{L\}$, and $I_1 = \{N, P\}$. The dynamics of (1) can be written on the form (4) with

$$\bar{A}_N = \begin{bmatrix} A & -B_p \\ 0 & 0 \end{bmatrix} \quad A_L = A + B_p C_q \quad \bar{A}_P = \begin{bmatrix} A & B_p \\ 0 & 0 \end{bmatrix}.$$

The cells can be described as in (5) with

$$\bar{G}_N = -\begin{bmatrix} C_q & 1 \end{bmatrix} \quad \bar{G}_L = \begin{bmatrix} C_q & 1 \\ -C_q & 1 \end{bmatrix} \quad \bar{G}_P = \begin{bmatrix} C_q & -1 \end{bmatrix}$$

while the matrices

$$\bar{F}_N = \begin{bmatrix} C_q & 1 \\ 0 & 0 \\ I & 0 \end{bmatrix} \quad \bar{F}_L = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ I & 0 \end{bmatrix} \quad \bar{F}_P = \begin{bmatrix} 0 & 0 \\ C_q & -1 \\ I & 0 \end{bmatrix}$$

satisfy the condition (6).

Local Analysis Since the systems that we consider have a limited domain of attraction, any proper Lyapunov function candidate can only be decreasing in a limited region around the origin. Hence, to extend the methods from (Johansson and Rantzer, 1998) to this setting we have to restrict the analysis region accordingly. A simple approach is to verify the Lyapunov inequalities on some ellipsoid $\mathcal{E}_A \subset \cup_{i \in I} X_i$, and a natural choice of ellipsoid is to use an inflated version of the domain of attraction estimated by the circle criterion. To this end, consider the analysis ellipsoid

$$\mathcal{E}_A(t) = \{x \mid x^T P_c x \le t\} = \{x \mid \bar{x}^T \bar{S}_A(t) \bar{x} \ge 0\}$$

with

$$\bar{S}_A(t) = \begin{bmatrix} -P_c & 0 \\ 0 & t \end{bmatrix} \tag{7}$$

In this formulation, the analysis region can be inflated by increasing the value of the parameter t.

When we restrict the analysis to the ellipsoid $x \in \mathcal{E}_A(t)$, we can also refine the descriptions of the saturated regions. In particular, if $x \in \mathcal{E}_A(t)$, then $|C_q x| \le k_{\max}$ where

$$k_{\max} = \sqrt{t C_q P_c^{-1} C_q^T}.$$

Hence, when we consider $x \in \mathcal{E}_A(t)$, we can describe the relevant parts of the saturated regions by

$$\bar{G}_N = \begin{bmatrix} -C_q & -1 \\ C_q & k_{\max} \end{bmatrix} \qquad \bar{G}_P = \begin{bmatrix} C_q & -1 \\ -C_q & k_{\max} \end{bmatrix}$$

Analysis Conditions using Piecewise Quadratics We are now ready to sum up the developments.

Theorem 1. Consider positive scalars w_i, and symmetric matrices T and U_i such that U_i has non-negative entries, while

$$\begin{aligned} P_i &= F_i^T T F_i & i \in I_0 \\ \bar{P}_i &= \bar{F}_i^T T \bar{F}_i & i \in I_1 \end{aligned}$$

satisfy

$$\begin{aligned} 0 &> A_i^T P_i + P_i A_i & i \in I_0 \\ 0 &> \bar{A}_i^T \bar{P}_i + \bar{P}_i \bar{A}_i + \bar{E}_i^T U_i \bar{E}_i + w_i \bar{S}_A(t) & i \in I_1 \end{aligned}$$

If $P_i > 0$ for $i \in I_0$, then the system (1) is locally exponentially stable and every level set

$$\mathcal{D}_{\text{pwq}}(\alpha) = \{x \mid \bar{x}^T \bar{P}_i \bar{x} \le \alpha \quad x \in X_i,\, i \in I\}$$

such that $\mathcal{D}_{\text{pwq}}(\alpha) \subseteq \mathcal{E}_A(t)$ is a domain of attraction for the system (1).

Proof: Similarly to (Johansson and Rantzer, 1998).

Remark 1. Note that the matrices P_i used to define the quadratic term of the Lyapunov function in each region do not need to be positive definite. It is straightforward to enforce positive definiteness in the LMI conditions, but this might lead to unnecessarily conservative estimates (compare Example 1).

Remark 2. Since the choice of analysis region is important, it is natural to apply Theorem 1 iteratively. Initially, we restrict the analysis region to an inflated version of the domain of attraction estimated by the circle criterion. Once a piecewise quadratic estimate $V(x) = \bar{x}^T \bar{P}_i^{(0)} \bar{x}$ has been found, we can try to verify the conditions of Theorem 1 on an inflated version of this domain. We then replace $\bar{S}_A(t)$ by

$$\begin{bmatrix} 0 & 0 \\ 0 & t \end{bmatrix} - \bar{P}_i^{(0)}$$

in the analysis conditions, and sweep t to find the largest value for which the analysis conditions admit a solution (compare Example 3).

Optimizing the Size Since there might be several solutions to the inequalities in Theorem 1, it is natural to try to maximize the “size” of the computed level sets. However, even determining the volume of a *given* convex body (even if it is restricted to be a convex polyhedron) is computationally hard in general (Gritzmann and Klee, 1994) and a formula that allows direct optimization of the volume of $\mathcal{D}_{\text{pwq}}(\alpha)$ appears to be out of reach. Inspired by the developments for quadratic Lyapunov functions, we suggest to minimize the sum of traces of the matrices P_i that describe the Lyapunov function in each region. Hence, we propose to solve the convex optimization problem

$$\text{minimize} \sum_{i \in I} \operatorname{Tr} P_i \tag{8}$$

subject to the conditions in Theorem 1. This is a heuristic criterion that, as we will see in Section 5, appears to work very well in practice.

Finding Large Level Sets within Analysis Region To extract the best estimate of the domain of attraction provided by Theorem 1, we need to find the largest level set of $V(x)$ contained in the analysis region. A good estimate can often be obtained using the following result.

Proposition 2. For every $i \in I$, let $\alpha_i^\star$ be the largest α_i such that

$$\bar{S}_A(t) > w_i \left(\begin{bmatrix} 0 & 0 \\ 0 & \alpha_i \end{bmatrix} - \bar{P}_i \right) + \bar{G}_i^T W_i \bar{G}_i$$

has a feasible solution $W_i \succeq 0$, $w_i \geq 0$. Then,

$$\mathscr{D}_{\text{pwq}}(\alpha) \subseteq \mathscr{E}_A(t)$$

for all $\alpha < \min_{i \in I} \alpha_i^\star$.

Proof: For any $x \in \mathscr{D}_{\text{pwq}}(\alpha)$, the inequality implies $\bar{x}^T \bar{S}_A(t) \bar{x} \geq 0$, and hence $x \in \mathscr{E}_A(t)$.

4.1 *Algorithm in Summary*

Our methodology for finding a piecewise quadratic estimate of the domain of attraction for the system (1) can be summarized as follows:

Algorithm 1

(1) *Compute an initial estimate of the domain of attraction using the circle criterion described in Proposition 1.*

(2) *Use this quadratic estimate to define an initial analysis ellipsoid as in (7), and redefine the cell identifiers for the saturated regions.*

(3) *Compute a piecewise quadratic estimate by minimizing the objective function (8) subject to the conditions in Theorem 1. Extract the estimated region of attraction using Proposition 2.*

(4) *If desired, use the piecewise quadratic estimate to redefine the analysis region, and goto step 3.*

5. EXAMPLES

In this section, we will compare the approaches on several examples from the literature. The methods that we will compare is analysis methods based on the circle and Popov-criteria from (Pittet *et al.*, 1997; Hindi and Boyd, 1998) and the piecewise quadratic approach described in this paper. The piecewise quadratic estimates are computed using Algorithm 1 without any manual intervention. As stressed in (Hindi and Boyd, 1998), the choice of loop transformation is crucial for obtaining good results with the Popov-criterion. In the examples, we always use the loop transformation that yields the best estimate of the domain of attraction. Finally, note that all examples focus on planar systems for illustration purposes only: the approaches apply directly to systems of arbitrary order.

Example 1. (Pittet *et al.*, 1997). Consider (1) with

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad B_p = \begin{bmatrix} 0 \\ -5 \end{bmatrix} \quad C_q = \begin{bmatrix} 2 \\ 1 \end{bmatrix}^T$$

The domains of attraction estimated by the various methods are shown in Figure 1. Note that the piecewise quadratic Lyapunov function matches the actual region of attraction very closely, and that the dynamics in the saturated regions have unstable equilibrium points in $\pm(\,-5 \;\; 0\,)$. In fact, the level surfaces of the computed Lyapunov function are parabolic (the matrices P_1 and P_{-1} both have one negative eigenvalue) and not ellipsoidal in the saturated regions.

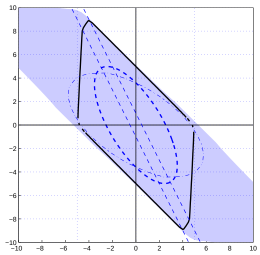

Fig. 1. Domain of attraction estimated by piecewise quadratic, Popov and circle analysis. The shaded region is the true region of attraction (obtained via simulation).

Example 2. (Romachuk, 1996). Consider (1) with

$$A = \begin{bmatrix} -0.333 & -0.86 \\ 1.42 & 0.53 \end{bmatrix} B_p = \begin{bmatrix} 0.41 \\ -2.27 \end{bmatrix} C_q = \begin{bmatrix} 0.3564 \\ 0.284 \end{bmatrix}^T$$

The domains of attractions computed using the different methods are shown in Figure 2. Note that the approach suggested in this paper produces a significantly larger domain of attraction than the method developed in (Romachuk, 1996) (not shown).

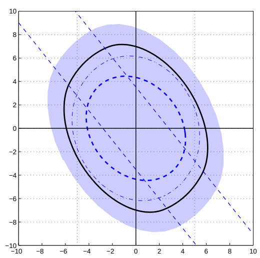

Fig. 2. Domain of attraction estimated by piecewise quadratic, Popov and circle analysis. The shaded region is the true region of attraction.

Example 3. (Johansson, 1999). Consider (1) with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0.1 \end{bmatrix} \quad B_p = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad C_q = \begin{bmatrix} 2 \\ 3 \end{bmatrix}^T$$

In this example, we apply the iterative procedure suggested in Remark 2, and find the piecewise quadratic estimate shown in Figure 3 (outermost level set). Again, the result is significantly larger than the alternatives produced by the circle and Popov criterion.

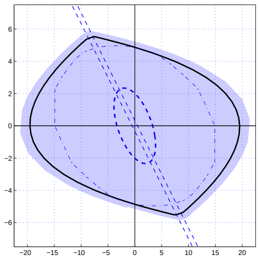

Fig. 3. The domain of attraction can be improved substantially using the iterative procedure suggested in Remark 2 (outermost level set).

6. CONCLUSIONS AND EXTENSIONS

This paper has detailed a novel procedure for estimating the domain of attraction for linear systems with saturation. The approach uses a piecewise quadratic Lyapunov function and a piecewise linear description of the closed-loop system. The central issues of how to restrict the analysis region, and how to optimize the "size" of the domain of attraction have been addressed, and the approach has been compared with recent approaches based on the circle and Popov criteria. Using several examples from the literature, we have demonstrated superior performance of the piecewise quadratic approach.

The approach presented in this paper can be extended in many directions. Firstly, the method apply directly to systems to multiple saturated feedback loops (several saturating actuators). In this case, a system with m saturation loops results in a piecewise linear model with 3^m regions. Secondly, although the application focus has been linear systems with saturation, the general approach applies to arbitrary nonlinear systems, provided that the nonlinear dynamics can be approximated by polytopic or piecewise linear models. Methods for approximating nonlinear systems using piecewise linear systems have been described in, for example, (Ohta *et al.*, 1993; Johansson, 1999). Finally, we mention that the approach can also be extended to local performance analysis. Techniques for estimating reachable sets for disturbances with bounded L_2 norm, as well as local L_2-gain estimates can be obtained by combining the techniques in (Hindi and Boyd, 1998) with those presented in (Rantzer and Johansson, 2000).

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