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FEEDBACK LINEARIZATION MPC FOR DISCRETE-TIME BILINEAR SYSTEMS

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Abstract: Unlike earlier work, this paper identifies parts of the state space where it is possible to invoke feedback linearization and uses the closed-loop paradigm [8] to accommodate input constraints while utilizing an extremely efficient online receding horizon strategy. The key to this development is a partial invariance property that is derived through the use of linear difference inclusion made possible due to the bilinear nature of the models considered here. For all other parts of the state space the proposed algorithm switches to a bilinear controller which is designed to give invariance and feasibility over low complexity polytopes. Without increasing computational complexity, the use of bilinear controllers affords extra freedom with which to maximize the region of attraction. The improvements in terms of this and closed loop output performance are shown to be very significant. *Copyright © 2002 IFAC*

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1. INTRODUCTION

I/O feedback linearization (FL) allows linear results to be grafted into the NL (NL) problem. Hence it has had some success in NMPC (e.g. Nevistic and Morari, 1995;Henson and Seborg,1997; Kravaris *et. al*, 1998) but this was limited because: (i) the penalty term in the cost is on the actual not the FL input; (ii) input constraints are not convex; (iii) unstable inverse dynamics imply instability. Limitation (i) is not significant because MPC handles constraints explicitly. Objection (ii) can be partially remedied through the use of short horizons provided that an additional stability constraint is added steering the state to a region which is invariant/feasible under a known control law. However the design of terminal control laws that result in large terminal regions is not trivial and the online computational demands are high. The difficulty with NMP (NMP) characteristics prohibits the use of FL; stable/anti-stable and inner-outer approximations (Doyle *et. al*, 1992, 1996), or equivalent synthetic outputs (e.g. Niemec and Kravaris, 2001), or interpolation (Bloemen *et. al*, 2001) offer partial remedies.

Here we show that the "closed loop paradigm" in Kouvaritakis *et. al* (2000) can be used to reduce dramatically the online computation and accommodate NMP characteristics in certain regions of the state space. The remainder of the operating region is covered by an alternative sub-optimal NL controller. The work focuses on bilinear systems, but extensions to more general classes are possible through the use (e.g. Boyd *et. al*, 1994) of Linear Difference Inclusion (LDI). For bilinear systems the calculation of invariant sets is straight-forward and defines regions, say X_{fl}, within which FL provides an implementable optimal solution. Invariance, feasibility and optimality are handled as for the linear case through the use of the "closed loop paradigm". It will be shown that X_{fl} cannot include points where there is a change in relative degree nor can it include the origin for NMP systems. These two problems are overcome through a switch to an alternative underlying control law. This law is such that feasibility/invariance computations are tractable, yet due to its NL nature it allows for the definition of much larger regions of attraction. The very significant benefits in respect of size of region of attraction as well as output performance are illustrated by means of simulation studies.

2. LINEAR SYSTEM CLOSED LOOP PARADIGM

In linear MPC, the "closed loop paradigm" allows for a significant reduction in the on-line computation at a small cost in terms of sub-optimality (Kouvaritakis *et al*, 2000; Boyd *et. al*, 1994). The key here is that the stability constraint is invoked at current time (not *N* steps ahead), and this trivializes the treatment of uncertainty. The degrees of freedom are no longer the predicted control moves, but rather perturbations on an unconstrained optimal law, and this is combined with the introduction of the vector of perturbations as states. Thus given a model (A,B,C) with input constraints $|u| \le \bar{u}$ it is possible to compute the optimal control law (w.r.t. the LQ cost), $u = -Kx$, which can be modified as $u_{k+i|k} = -Kx_{k+i|k} + c_k$ to avoid constraint violations. The prediction dynamics can be written as:

$$z_{k+1} = \Psi z_k,\ z_k^T = [x_k^T \quad f_k^T],\ f_k^T = [c_o^T \quad \dots \quad c_N^T]$$

$$\Psi = \left[\begin{array}{c|c} \Phi & B[I \quad 0 \quad \dots \quad 0] \\ 0 & T \end{array}\right],\quad \Phi = A - BK \qquad (1)$$

T is such that Tf_k has as elements $c_1, c_2, \dots, c_N, 0$. Then the minimization of the usual MPC cost is equivalent to that of norm f_k subject to input constraints which is achieved over an infinite horizon by restricting z to lie within an ellipsoid $E_z = \{z : z^T P_z z \le 1\}$, where P_z is chosen such that

$$\Psi^T P_z \Psi \le P_z,\ \tilde{K} P_z^{-1} \tilde{K}^T \le \bar{u},\ \tilde{K} = [-K \quad I \quad 0^T] \qquad (2)$$

Given (2) it can be shown (e.g. Boyd *et. al*, 1994) that E_z is in-variant and feasible. The minimization of f_k is trivial (Kouvaritakis, *et. al* 2000) and a simple scaling technique (Kouvaritakis, *et. al* 2001) can be used to effect a further reduction of the cost.

3. FL AND LINEAR DIFFERENCE INCLUSION

It is possible to extend the closed-loop paradigm to the NL case (Kouvaritakis, *et. al*, 2001), however the use of linear feedback results in small invariant feasible sets and compromises performance. Here we overcome these difficulties. For simplicity, attention is restricted to the case of SISO bilinear systems:

$$x_{k+1} = Ax_k + [B + Fx_k]u_k,\ y_k = Cx_k,\ |u| \le \bar{u} \qquad (3)$$

with $x \in R^n$. For such systems the unconstrained optimum w.r.t. to $J = \sum_{i=1,\infty} x_{k+i}^T C^T Q C x_{k+i}$ is:

$$u_k = -CAx_k / [C(B + Fx_k)] \qquad (4)$$

for all $x \notin \Pi_o = \{x : C(B + Fx) = 0\}$. The optimality of (4) follows from (3) according to which

$$x_{k+1} = \Phi(x_k)x_k,\quad y_{k+1} = 0$$
$$\Phi(x_k) = \left[A - \frac{(B + Fx_k)}{C(B + Fx_k)} CA\right] \qquad (5)$$

This may be deemed to be too aggressive and it may be preferred to force the output to decay as $y_{k+1} = py_k = pCx_k$, which can be achieved by replacing *CA* in (4) and (5) by *C*(*A*-*pI*).

The law of (10) can be implemented for all $x \notin \Pi_o$; for $x \in \Pi_o$ one could use (4) with its denominator set to a small non-zero value. However such a strategy could lead to infeasibility - (4) will not necessarily meet the input constraints - and for unstable inverse dynamics, (4) would drive the output to zero but would not stabilize the origin. To deal with feasibility we introduce the closed loop paradigm according to which (4) is perturbed to give:

$$u_k = [-CAx_k + c_k] / [C(B + Fx_k)] \qquad (6)$$

which leads to the autonomous augment state matrix

$$\Psi(x_k) = \left[\begin{array}{cc} \Phi(x_k) & B[I \quad 0 \quad \dots \quad 0] \\ 0 & T \end{array}\right] \qquad (7)$$

with $z^T \Psi^T(x) P \Psi(x) z \le z^T P z$ as the invariance condition for E_z. A convenient (albeit conservative) way to ensure the invariance condition is to invoke

$$\tilde{\Psi}^T(x) P \tilde{\Psi}(x) \le \alpha^2(x) P$$
$$\alpha(x) = CB + CFx,\quad \tilde{\Psi}(x) = \alpha(x)\Psi(x) \qquad (8)$$

For bilinear systems and $x \in E_x$, the projection of E_z onto x-space, (8) can be written as a Linear Matrix Inequality (LMI) in x and (8) can be invoked over a convex polytope Π simply by considering its vertices, $v_i, i = 1, \dots, n_v$,. This is similar to Linear Difference Inclusion (LDI) and will be referred as LDI-invariance (LDI-I) and Π will be referred to as an inclusion box. Note that $E_x = \{x : x^T P_x x \le 1\}$, $P_x = P_{xx} - P_{xc} P_{cc}^{-1} P_{cx}$, with $P_{xx}, P_{xc}, P_{cx}, P_{cc}$ being the blocks of P corresponding to the partition of z into x and f. Then the result below applies to

$$\Pi = \{x : \gamma_i^T x \le 1,\ i = 1, \dots, n_\gamma,\ n_\gamma \ge 2n\} \qquad (9)$$

Theorem 3.1 *E_z is LDI-I if $\alpha(v_i)$ are of the same sign for all $i = 1, \dots, n_v$, $j = 1, \dots, n_\gamma$ and*

$$\tilde{\Psi}^T(v_i) P \tilde{\Psi}(v_i) \le \alpha^2(v_i) P,\quad \gamma_j^T P_x^{-1} \gamma_j \le 1 \qquad (10)$$

Proof: For $S = P^{-1}$ condition (22a) is equivalent to

$$\left[\begin{array}{cc} \pm \alpha(x) S & S\tilde{\Psi}^T(x) \\ \tilde{\Psi}(x) S & \pm \alpha(x) S \end{array}\right] \ge 0 \qquad (11)$$

where the + sign is used only for $\alpha(x) \ge 0$ .Under the assumption that $\alpha(v_i)$ are sign definite, (10) will be equivalent to (11) which is affine in x, thereby ensuring (by superposition), that the LDI-I condition will hold everywhere in E_z provided that $E_x \in \Pi$, a condition which is guaranteed by (10b). ■

Using the techniques of Boyd *et. al* (1994), we can ensure feasibility of (6) for all $z \in E_z$ by requiring:

$$(\tilde{K}z)^2 = \left|\tilde{K}P^{-1/2}P^{1/2}z\right|^2 \le \tilde{K}P^{-1}\tilde{K}^T \le \alpha^2(v_i)\bar{u}^2 \quad (12)$$

which like (10) can be re-written as an LMI in S.

Remark 3.1 *Given the LMI nature of (10) and (12), the maximization of the volume of an E_z which is invariant and feasible under (6) reduces to a convex optimization problem (e.g.* Boyd, *et. al*, 1994*) that can be solved efficiently using SDP.*

However if $\Pi_o \cap \Pi \neq \{0\}$ then $\alpha(v_i)$ cannot all have the same sign and hence LDI-I (and feasibility) will not hold. Also, if (3) is NMP , then (8) cannot hold true at the origin, since under (4) the origin is unstable. Then FL cannot be used over all of E_z. But it is possible to define a smaller inclusion box, $\hat{\Pi}$, with vertices $\hat{v}_i$ for which (10a) gives a Partial LDI-invariance (PLDI-I). Rather than insist that $z_{k+1} \in E_z$ for all $z_k \in E_z$, we ask that $z_{k+1} \in E_z$ only for $z_k \in E_z$ associated with $x_k \in \mathrm{X}_{fl} = \{E_x \cap \hat{\Pi}\}$.

Corollary 3.1 $\mathrm{X}_{fl} = \{E_x \cap \hat{\Pi}\}$ *is PLDI-invariant if (10a) holds true at the vertices of* $\hat{\Pi}$ *rather than* Π.

Proof: As for Theorem 3.1 except that (10b) is no longer needed because by definition $\mathrm{X}_{fl} \subseteq \hat{\Pi}$. ■

Earlier approaches were either restricted to the case of minimum phase plant of definite relative degree or otherwise avoided FL, and instead proposed approximations (e.g. Doyle *et. al*, 1992; Doyle *et. al*, 1996) or the use of "statically equivalent synthetic outputs" (Niemiec and Kravaris, 2001), or used interpolation (Bloemen, *et. al*) between FL and a "stabilizing" state feedback control, $u = K_{st}(x) = -Kx$. The strategy here is to use FL whenever possible, and this is achieved through PLDI-I which enables the use of FL for all $x \in \mathrm{X}_{fl}$. The very significant benefits of this will be demonstrated in Section 6.

Two problems are outstanding: (i) X_{fl} may be small; (ii) even if the initial condition lies in X_{fl}, there is no guarantee that it will remain there under (6). A convenient solution that overcomes these is to switch to $K_{st}(x)$ for all $x \notin \mathrm{X}_{fl}$, however that would compromise performance and/or would result in small regions of attraction. In the section below we investigate the design of a NL controller that will be shown to lead to very much larger regions of attraction (as illustrated in Section 6). First we consider briefly the computation of $\hat{\Pi}$.

Corollary 3.1 *The two hyperplanes defined by $C(I \pm A)^{-1}Fx = C(I \pm A)^{-1}B$ divide the x-space into four sectors of which two exclude Π_o. $\hat{\Pi}$ must lie in one of those two sectors.*

Proof: The two hyperplanes of the corollary define all points for which $\Phi(x)$ has at least one eingen-value equal to 1 or –1; monodromy arguments then show that everywhere in two of the four sectors defined by the hyperplanes, at least one eigenvalue of $\Phi(x)$ will be more than one in modulus. These two sectors include all x that satisfy $CB + CFx = 0$ except the point $x = -F^{-1}B$. $\hat{\Pi}$ cannot lie in those two sectors because, given the triangular nature of $\Psi(x)$, it is clear that invariance requires the eigenvalues of $\Phi(x)$ to lie inside the unit circle. ■

Corollary 3.1 defines two linear constraints on the vertices of $\hat{\Pi}$. To complete the computation of $\hat{\Pi}$, it is possible to invoke procedures (Bloemen *et. al*, 2001) to maximize of E_x subject to invariance and feasibility. Such procedures were based $K_{st}(x) = -Kx$ rather than FL, and did not suffer from the difficulties of loss of relative degree which now must be accommodated by generating the LMIs conditions (11). On account of the sign change of $\alpha(x)$, the resulting inclusion box Π will not have the LDI-I property but will contain $\hat{\Pi}$ with its associated PLDI-I property. Corollary 3.1 in conjunction with Π can be used to complete the construction of $\hat{\Pi}$. The computation of Π (though offline) can be intensive; one could obtain suitable choices of $\hat{\Pi}$ through the use of polytopic invariance (see Section 4) for which it is possible to deploy the sequential LP procedures developed in Cannon *et. al*(2001).

4. BILINEAR CONTROLLER

The use of LDI requires that $E_x \in \Pi$, which is ensured by (10b). This restriction raises questions as to the wisdom of the use of ellipsoidal sets, and it has been shown that for bilinear systems (Cannon *et. al*, 2001), LDI can be deployed to define invariant polytopes of much larger volume. The definition of both invariant sets (polytopic/ellipsoidal) were based on a state feedback $K_{st}(x) = -Kx$. It was seen however in Section 3 that for bilinear systems, LDI can be applied even when the controller itself is NL , e.g. (6), provided the NL ity enters as a denominator that is linear in x. Although (6) itself cannot be used outside X_{fl}, one could still use the same form of controller:

$$u = -Kx/[1 + \mu^T x] \quad (13)$$

and still be able to use LDI (in a straightforward manner). By analogy to the "bilinear transform" this form of controller will be referred as "bilinear". For simplicity of presentation in this section we shall not resort to the use of perturbations c.

The class of (13) contains $K_{st}(x)$ and the "synthetic output FL controller" (Niemiec and Kravaris, 2001) as special cases and should result in larger regions of attraction. As explained above the associated invariant set will be taken to be polytopic, say $\Pi_X = \{x : \|Wx\|_\infty \le 1\}$, $W \in R^{n\times n}$, $\det(W) \neq 0$, which are invariant iff:

$$\|W\Phi(x)\|_\infty \le \|Wx\|_\infty, \quad \Phi(x) = A - \frac{(B+Fx)K}{1+\mu^T x} \qquad (14)$$

To reduce online computation, it is not intended that (13) should be tuned online. Thus and to allow a handle on performance (14) can be strengthened to:

$$\|W\Phi(x)\|_\infty - \|Wx\|_\infty \le -\varepsilon|Cx| \qquad (15)$$

Invoking this at the different prediction instants k and summing over k establishes that the output converges to zero, the upper bound on the rate of convergence being controlled by the size of ε. As with the ellipsoidal sets of Section 3, LDI can be invoked to get a convenient test for the invariance and feasibility of Π_x under the control law of (13).

Theorem 4.1 Let w_l^T *denote the rows of* W *and* v_i *denote the vertices of* Π_x. *Then* Π_x *is invariant and feasible under the dynamics of (3) and the control law of (13) if for all* $i, j = 1,\dots,v$, $l = 1,\dots,n$

$$\left|w_l^T \tilde{\Phi}(v_i)v_j\right| + \varepsilon(1+\mu^T v_i)\left|Cv_j\right| \le 1 \qquad (16)$$

$$\tilde{\Phi}(x) = (1+\mu^T x)A - (B+Fx)K$$

$$\left|Kv_i\right| \le (1+\mu^T v_i)\bar{u},\ 1+\mu^T v_i > 0 \qquad (17\text{-}18)$$

Proof: For $x \in \Pi_X$ (18) implies that $1+\mu^T x \ge 0$ and (15) is equivalent to $\left|w_l^T \Phi(x)x\right| + \varepsilon|Cx| \le 1$ or $|\, w_l^T \tilde{\Phi}(x)x \,| + \varepsilon(1+\mu^T x)\,|\, Cx \,| \le 1+\mu^T x$ (upon pre-multiplication by $1+\mu^T x$) which through LDI can be made to hold everywhere in Π_x by invoking (16). Similarly, pre-multiplying (2) written for the law of (13) by $1+\mu^T x$ we get $|Kx| \le (1+\mu^T x)\bar{u}$ which is affine in x and will hold everywhere in Π_x if and only if condition (17) of the theorem is satisfied. ■

Remark 4.1 *The test of the theorem is convenient because for fixed* Π_x, *conditions (15-18) are linear or affine in* K, μ. *Also unlike ellipsoids* E_x *which are constrained to lie within an inclusion polytope,* Π, *for polytopic sets the inclusion polytope* Π *can be coincidental with the invariant polytope* Π_x. *To maximize the volume of* Π_x *or increase the value of* ε, *one has to resort to NL constrained optimisation (which of course could be performed offline). Alternatively fixing* μ *one can deploy the sequential LP approach to the problem proposed in* (Cannon *et. al*, 2001). *It is noted that* **t***he direction of the inequality in (18) is stated without loss of generality, because a reversal in this direction could be absorbed in the sign of* K. *Finally it is also noted that Theorem 4.1 can be used to generate suitable choices for* Π *and* $\hat{\Pi}$ *of Section 3: simply substitute* K, μ^T *by* $CA/(CB), CF/(CB)$.

5. MPC switching strategy

Earlier work (Bloemen *et. al*, 2001) showed that it is possible to combine the aggressive unconstrained optimal law of (4) with a cautious law, $K_{st}(x) = -Kx$, which stabilizes the origin and leads to invariance/feasibility in an ellipsoidal neighbourhood of the origin. This neigh-bourhood however can be small and can be enlarged through the use of polytopic sets (Cannon *et. al*, 2001). It will be seen below that such polytopes can be made considerably bigger through the use of the bilinear controller of (13) in place of $K_{st}(x)$. An attractive feature of the earlier work(Bloemen *et. al* 2001) is that it involves a single degree of freedom and thus trivializes online computation. The use of a single variable however can result in sub-optimality and the proposal here is to overcome this by switching between the perturbed unconstrained optimal law of (6) and the bilinear law of (13). The idea is that through (6) one can introduce ν extra degrees of freedom to get a far better handle on performance. Of course (6) can only be used for $x \in \mathrm{X}_{fl}$, but by switching to (13) it is possible to cater for all $x \in \{\Pi_x - \mathrm{X}_{fl}\}$.

Algorithm 5.1

Step 0 (offline) Design K, μ so that Π_x is invariant /feasible under (13) for a given ε. Compute P such that X_{fl} is PLDI-I and scale P so that $E_x \subseteq \Pi_x$.

Step 1 (feedback linearization) Use the procedure of Kouvaritakis, *et. al* (2000) to minimize $\|f_k\|$ subject to $z_k \in E_z$. If the solution satisfies the convergence constraint (15) and if either $x_{k+1|k} \in \mathrm{X}_{fl}$ or $x_k \in \mathrm{X}_{fl}$

implement the first element of f_k as per (6), increment k and at the next time instant repeat Step 1; otherwise goto step 2.

Step 2 (bilinear control) Implement (13), increment k and at the next time instant go to Step 1.

Theorem 5.1 *For any $x_0 \in \Pi_x$, under Algorithm 5.1 the state of (3) will remain bounded and within Π_x, while the output will converge to zero.*

Proof: Since X_{fl} is PLID-I we have that Step 1 will retain the next x in $E_x \subset \Pi_x$, whereas Step 2 will retain x in Π_x. Thus x will remain bounded. With respect to the output we have that if Algorithm 5.1 remains in Step 1 only, then by the "closed loop paradigm", it is known that the vector of perturbations will decrease monotonically, so that Step 1 will, at some future instant, revert to the unconstrained optimal of (4) and from then on will maintain the output at zero irrespective of whether the state converges to the origin (or any other equilibrium point) or not. On the other hand if Step 2 is entered and x remains outside X_{fl}, then by (15) we have that the output will converge to zero. Finally invoking (15) in Step 1 implies that the same conclusion could be drawn even in the case when Algorithm 5.1 switches between Steps 1 and 2. ■

Corollary 5.1 *For $I - A$ full rank and under Step 1, system (3) has one equilibrium point at the origin and up to $n-1$ equilibrium points given by*

$$x_i^* = \lambda M(s_i\gamma)_i, \quad s_i = \frac{B^T B}{B^T(I - A - \lambda_i F)M\gamma_i} \qquad (19)$$

where (λ_i, γ_i) are the $n-1$ solution pairs of $[\lambda NFM - N(I-A)M]\gamma = 0$ (a generalized eigen equation) and N, M are full rank matrix representations of the left and right annihilators of B, C.

Proof: After algebraic manipulation it can be shown that the equilibrium condition $x = \Phi(x)$ implies

$$\begin{aligned} &[I + ab^T]x = 0 \\ &a = (I - A)^{-1}(B + Fx),\ b^T = CA/[C(B + Fx)] \end{aligned} \qquad (20)$$

which implies that $b^T a = -1, \quad x = \lambda a$ for a scalar λ. This in turn (after further manipulation) implies

$$B + Fx = (I - A)M\gamma, \quad (I - A)M\gamma = B + \lambda FM\gamma \quad (21)$$

Pre-multiplication of (21a) by N and B^T gives two conditions; one being the eigen-condition of the theorem and the other $B^T[I - A - \lambda F]M\gamma = B^T B$. Thus (λ, γ) is a generalized eigen-pair and the eigenvector must be scaled by the factor s_i defined in (19b); (19a) follows from (20) and (21a). Note that $s_i = 0$ iff $(I - A - \lambda_i)M = 0$, but then (21b) will not hold and such an i will not lead to equilibrium. ■

Remark 5.1 *The corollary implies that if x remains in X_{fl}, then Algorithm 5.1 can converge to a point x_i^* which is not the origin. This will only be possible if $A - (B + Fx_i^*)CA/(CB + CFx_i^*)$ has all eigenvalues inside the unit circle. This applies to both minimum and NMP systems, but in the latter case, equilibrium at the origin will not be stable.*

Remark 5.2 *The bilinear law of (13) can be thought off as a back up for (4) in the sense that whenever (4) runs into invariance/ feasibility problems (due to NMP characteristics or loss of relative degree, or due to release from initial conditions outside X_{fl}), then Algorithm 5.1 switches over to the bilinear control law. With this in mind and in the interest of maximizing the volume of X_{fl}, it is in fact possible to define PLDI-I without condition (12); however under such circumstances it would be necessary to check the feasibility of (13) with the view to switching to Step 2 whenever (13) violates constraints.*

6. Illustrative example

For the purposes of comparison we select the same example from Bloemen *et.* a(2001) for which:

$$A = \begin{bmatrix} 0.28 & -0.78 \\ -0.78 & -0.59 \end{bmatrix} \quad B = \begin{bmatrix} 0.71 \\ 1.62 \end{bmatrix} \quad F = \begin{bmatrix} 0.34 & 0.36 \\ 0.41 & -0.65 \end{bmatrix}$$

$$C = C_1 \quad \text{or} \quad C = C_2, \qquad C_1 = [-0.69 \quad 0.86], C_2 = [-.69 \quad 0.2]$$

which is open loop unstable and minimum phase for C_1 but NMP (with a zero at 2.3) for C_2; the two choices of C will be referred to as SZD/ UZD (Stable/Unstable Zero Dynamics). Throughout $\nu = 4$, $\varepsilon = 0.01$ and the choice of P was based on the box Π of [7] with $\Gamma = [\gamma_1 \ \ldots \ \gamma_n]^T = F/3$ in conjunction with Corollary 3.1. The simulation results for different initial conditions are shown in Figures 1 (UZD) and 2 (SZD). E_x as required by Algorithm 5.1 have been scaled so as to fit inside the bilinear controller inclusion boxes, Π_x. The dashed lines that are nearly aligned with the major axes of E_x indicate the kernel of C, whereas the other dashed lines give plots of Π_o. The remaining two lines on the figures correspond to the "hyperplanes" of Corollary 3.1. Of these the ones associated with the –1 eigenvalue of $\Phi(x)$ are inactive in that they do not intersect E_x whereas the lines associated with

+1 divide E_x into two parts of which the top/bottom corresponds to X_{fl} for the UZD/SZD case. The instantaneous positions of the state vectors on the various trajectories are marked with a cross or a circle depending on whether at those points Algorithm 5.1 uses Step 1 or 2, respectively. Not unreasonably, the algorithm resorted to the use of the bilinear controller for $x \notin E_x$ and indeed for almost all $x \notin \mathrm{X}_{fl}$. Conversely, for all the x that belong to X_{fl}, the algorithm has deployed FL resulting in a closed loop cost (the sum of the squares of the output), which is considerably better than of either of the two interpolation schemes (here referred to as Interpolation A and B) proposed in [7]:

Table 1 Comparison of closed loop costs

Initial condition	Int. A	Int. B	Al. 5.1
$[0.5 \quad 1.7]$ - SZD	0.4124	0.6643	0.1173
$-[0.5 \quad 0.8]$ - SZD	0.0321	0.0592	0.0022
$[0.5 \quad 1.7]$ - UZD	0.3717	0.2866	0.0051
$-[0.5 \quad 0.8]$ - UZD	0.0910	0.0981	0.0606

It is noted that as per Remark 5.1 it is possible that the algorithm drives the state to non-zero equilibrium points (while retaining the output at zero); this was the case for the initial condition $x_o = [0.8 \quad 0.2]$ for which x tends asymptotically to $[0.28 \quad 0]$.

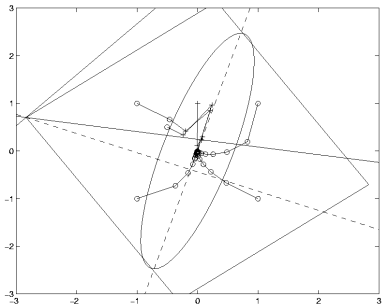

Figure 1: Closed loop trajectories of the SZD plant.

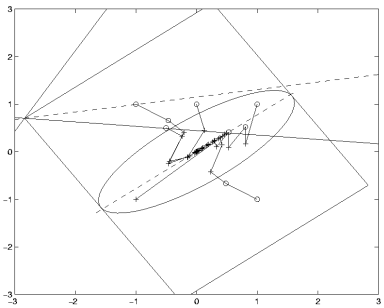

Figure 2: Closed loop trajectories of the UZD.

The algorithm provide very significant reductions in cost but it also enjoys much larger regions of attraction due to the use of: (i) the polytopic sets Π_x which result in sets of larger volume (than that possible for ellipsoids); (ii) the bilinear controller of (13). The benefits afforded by the bilinear controller are considerable: the area of largest Π_x is 18.12 (SZD) and 18.23 (UZD) whereas that of the largest possible Π_{Kx}, is 10.28 (SZD) and 11.32 (UZD).

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