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MULTIVARIABLE FINITE SETTLING TIME STABILISATION: PARAMETRISATION AND PROPERTIES

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Abstract: The problem of *Finite Settling Time Stabilisation (FSTS)* for multivariable, discrete-time systems is discussed in this paper. The approach is algebraic and the problem reduces to the solution of a polynomial Diophantine equation; many of the solvability conditions are expressed as standard linear algebra tests. A Kučera-Youla-Bonjiorno type parametrisation of the family of the FSTS controllers is obtained and necessary and sufficient conditions for strong FSTS are derived. Finally, solvability conditions for FST tracking and disturbance rejection are given and l_1 and l_∞ FSTS controllers are obtained using linear programming optimisation. *Copyright © 2002 IFAC*

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1. INTRODUCTION

The *Finite Settling Time Problem (FSTP)* and more specifically the deadbeat regulation is unique in discrete-time systems (Bergen and Ragazzini, 1954; Kalman, 1960). Most of the state or output deadbeat regulators are of constant state feedback type. The main aim is to shift all the poles (or almost all in the case of output feedback) of the closed loop system to the origin. As the solution of the pole placement problem in the multivariable case is not uniquely determined a variety of deadbeat controllers has been proposed based on techniques varying from procedures on selecting independent vectors from a certain vector space to the solution of discrete Riccati equations (Kučera, 1971; Lewis, 1982; O'Reilly, 1981). Kučera has pioneered the use of polynomial algebra methods for the study of time-optimal control in discrete-time systems (Kučera, 1973, 1980a, Kučera and Šebek, 1984). Many other researchers followed this approach (Eichstaedt, 1982; Wolovich, 1983) with Zhao and Kimura (1988, 1989) looking at the problem of robustness for multivariable deadbeat tracking.

The present work does not directly deal with deadbeat control but with the rather more general problem of Finite Settling Time Stabilisation. That is, all internal and external variables (signals) of the closed loop system are required to settle to a new steady state after finite time from the application of a step change to any of each inputs and for every initial condition. The state/output deadbeat regulation and the problem of deadbeat tracking are then special cases of the FSTSP.

The standard unity feedback configuration is used for the design of FSTS controllers and the powerful polynomial approach is used for the solution of FSTSP. The whole problem is reduced to the solution of a polynomial Diophantine matrix equation that guarantees not only internal stability but also internal (state) FST. The family of all FSTS controllers is parametrically expressed and testable necessary and sufficient conditions for strong stabilisation, FST tracking and disturbance rejection for a class of signals are derived. Finally, l_1 and l_∞ compensators (i.e. FSTS controllers which minimise the l_1 or l_∞ norm of the error signal) are obtained using linear programming.

Throughout the paper, $d = z^{-1}$ is the delay operator, $\mathcal{R}(\mathcal{C})$ is the field of real (complex) numbers, $\mathcal{R}[d]$ is the ring of polynomials and $\mathcal{R}(d)$ the field of rational functions in *d*. $\mathcal{D}$ denotes the closed unit disc, $\mathbf{M}[d]$, $\mathbf{M}(d)$ are the sets of polynomial and rational matrices respectively and $\mathbf{U}[d]$ is the set of unimodular polynomial matrices, all of appropriate dimensions.

2. DEFINITION – PARAMETRISATION OF FSTS CONTROLLERS

Consider the unity feedback configuration of Fig. 1 where $\mathbf{P}$ and $\mathbf{C}$ are the pulse transfer function matrices of the discrete-time plant and controller respectively, $\mathbf{u}_1$, $\mathbf{u}_2$ are the externally applied inputs and $\mathbf{y}_1$, $\mathbf{y}_2$ are the outputs of the system. It is assumed that the plant has *l* inputs and *m* outputs, $\mathbf{P}$, $\mathbf{C} \in \mathbf{M}(d)$, they are causal and $\mathcal{S}_\mathcal{P}$, $\mathcal{S}_\mathcal{C}$ are the state space descriptions of the plant and controller respectively.

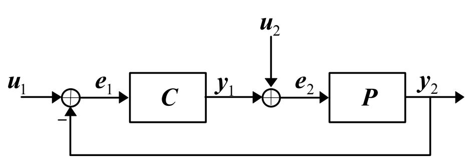

Fig. 1. The unity feedback configuration.

The finite settling time response of the closed loop system is defined as follows.

Definition 1. The discrete-time feedback system of Fig. 1 is said to exhibit:

i) An *External — Finite Settling Time (E-FST)* response, or to be Externally-FST Stable (E-FSTS) if for any step change in its inputs $\mathbf{u}_1$, $\mathbf{u}_2$ all signals $\mathbf{y}_1$, $\mathbf{y}_2$ settle to a new steady state value in finite number of steps.
ii) An *Internal — Finite Settling Time (I-FST)* response, or to be Internally-FST Stable (I-FSTS) if for every initial state vector and any step input all states settle to a new steady state in finite time. □

Note that in the above definition the values of the finite settling time and of the steady state are left free. The deadbeat response corresponds to the case of perfect tracking of step inputs in minimum number of steps and thus, it is a special case of the FST response.

Let $\mathbf{H}(\mathbf{P},\mathbf{C})$ denote the transfer function matrix of the closed loop feedback system from the input $\mathbf{u} = [\mathbf{u}_1', \mathbf{u}_2']'$ to the error vector $\mathbf{e} = ['\mathbf{e}_1', \mathbf{e}_2']'$. If the feedback system is well formed it can be shown that

$$\mathbf{H}(\mathbf{P},\mathbf{C}) = \begin{bmatrix} (\mathbf{I}+\mathbf{PC})^{-1} & -\mathbf{P}(\mathbf{I}+\mathbf{CP})^{-1} \\ \mathbf{C}(\mathbf{I}+\mathbf{PC})^{-1} & (\mathbf{I}+\mathbf{CP})^{-1} \end{bmatrix} \quad (1)$$

The following statements give the conditions for and some properties of the FST response (Karcanias and Milonidis, 1992).

Proposition 1. The feedback system of Fig. 1 exhibits an external FST response *iff* $\mathbf{H}(\mathbf{P},\mathbf{C}) \in \mathbf{M}[d]$, i.e. the closed loop system is a FIR (Finite Impulse Response) system. □

Remark 1. If $\mathcal{S}_\mathcal{P}$, $\mathcal{S}_\mathcal{C}$ are stabilisable and detectable, the fact that $\mathbf{H}(\mathbf{P},\mathbf{C})$ is polynomial implies that the feedback system is internally stable as all the controllable and observable eigenvalues are shifted to the origin of the *z*-plane (Kučera, 1979; Vidyasagar, 1985). □

Proposition 2. If $\mathcal{S}_\mathcal{P}$, $\mathcal{S}_\mathcal{C}$ are both controllable and observable, then the closed loop system exhibits a total (external as well as internal) FST response, *iff* $\mathbf{H}(\mathbf{P},\mathbf{C})$ is a polynomial matrix in *d.* □

The following theorem can be derived form Propositions 1 and 2 using the standard results for the analysis of the feedback configuration (Kučera, 1979; Vidyasagar, 1985; Milonidis 1994).

Theorem 1. Let $\mathbf{P} = \mathbf{N}\mathbf{D}^{-1} = \tilde{\mathbf{D}}^{-1}\tilde{\mathbf{N}}$, $\mathbf{C} = \mathbf{N}_\mathrm{C}\mathbf{D}_\mathrm{C}^{-1} = \tilde{\mathbf{D}}_\mathrm{C}^{-1}\tilde{\mathbf{N}}_\mathrm{C}$ be $\mathcal{R}[d]$– coprime MFDs of the plant and controller pulse transfer functions. The solution to the FST problem exists *iff*

$$\tilde{\mathbf{D}}\mathbf{D}_\mathrm{C} + \tilde{\mathbf{N}}\mathbf{N}_\mathrm{C} \in \mathbf{U}[d] \text{ or } \tilde{\mathbf{D}}_\mathrm{C}\mathbf{D} + \tilde{\mathbf{N}}_\mathrm{C}\mathbf{N} \in \mathbf{U}[d] \quad (2)$$

Moreover, the family of all FSTS controllers is given by

$$\begin{aligned}\mathcal{W}(\mathbf{P}) = \{\mathbf{N}_\mathrm{C}\mathbf{D}_\mathrm{C}^{-1}\!: \mathbf{N}_\mathrm{C} = \mathbf{X}+\mathbf{DR},\ \mathbf{D}_\mathrm{C} = \mathbf{Y}-\mathbf{NR}, \\ \mathbf{R} \in \mathbf{M}[d]\ |\mathbf{Y}-\mathbf{NR}| \neq \mathbf{0}\}\end{aligned} \quad (3)$$

or

$$\mathcal{W}(\mathbf{P}) = \{\tilde{\mathbf{D}}_C^{-1}\tilde{\mathbf{N}}_C\colon\ \tilde{\mathbf{N}}_C = \tilde{\mathbf{X}} + \tilde{\mathbf{R}}\tilde{\mathbf{D}},\ \tilde{\mathbf{D}}_C = \tilde{\mathbf{Y}} - \tilde{\mathbf{R}}\tilde{\mathbf{N}},\ \tilde{\mathbf{R}} \in \mathbf{M}[d]\ \left|\tilde{\mathbf{Y}} - \tilde{\mathbf{R}}\tilde{\mathbf{N}}\right| \neq \mathbf{0}\} \tag{4}$$

where $\mathbf{R}$, $\tilde{\mathbf{R}}$ are arbitrary and $\mathbf{X}$, $\mathbf{Y}$, $\tilde{\mathbf{X}}$, $\tilde{\mathbf{Y}}$ are appropriate $\mathcal{R}[d]$ matrices satisfying the following Bezout identity

$$\begin{bmatrix} \tilde{\mathbf{Y}} & \tilde{\mathbf{X}} \\ -\tilde{\mathbf{N}} & \tilde{\mathbf{D}} \end{bmatrix}\begin{bmatrix} \mathbf{D} & -\mathbf{X} \\ \mathbf{N} & \mathbf{Y} \end{bmatrix} = \mathbf{I} \tag{5}$$

□

The pair $(\mathbf{P},\mathbf{C})$ such that $\mathbf{C} \in \mathcal{W}(\mathbf{P})$ is called FST-stable. However, not all $\mathbf{C} \in \mathcal{W}(\mathbf{P})$ are physically realisable, i.e. causal. The conditions for causality are given by Corollary 1. Before Corollary 1 is stated the following two lemmas are given necessary for its proof.

Lemma 1 (Milonidis, 1994). Let $(\mathbf{P},\mathbf{C})$ be FST-stable. Then $(\mathbf{X},\mathbf{D})$, $(\mathbf{Y},\mathbf{N})$, $(\tilde{\mathbf{X}},\tilde{\mathbf{D}})$, $(\tilde{\mathbf{Y}},\tilde{\mathbf{N}})$ are pairs of $\mathcal{R}[d]-$coprime matrices. □

Lemma 2 (Kučera, 1980b). Let $\mathbf{P}(d) = \mathbf{N}(d)\mathbf{D}^{-1}(d) = \tilde{\mathbf{D}}^{-1}(d)\tilde{\mathbf{N}}(d)$ be $\mathcal{R}[d]-$coprime MFDs of any discrete-time system. Then $\mathbf{P}(d)$ corresponds to a causal system *iff* $\det(\mathbf{D}(0)) \neq 0$ or $\det(\tilde{\mathbf{D}}(0)) \neq 0$. □

The parametrisation of the subfamily of causal FSTS controllers is given next.

Corollary 1. Suppose $(\mathbf{P},\mathbf{C})$ is FST-stable and $n_p \in \mathcal{R}[d]$ is the least invariant factor of either $\mathbf{N}$ or $\tilde{\mathbf{N}}$. Then $\mathbf{C}$ is causal for

1. <u>every</u> $\mathbf{R}$, $\tilde{\mathbf{R}} \in \mathbf{M}[d]$ if $n_p(0) = 0$
2. <u>almost every</u> $\mathbf{R}$, $\tilde{\mathbf{R}} \in \mathbf{M}[d]$ if $n_p(0) \neq 0$

If $\mathcal{W}_c(\mathbf{P})$ denotes the family of causal FSTS controllers then

1. $\mathcal{W}_c(\mathbf{P}) = \mathcal{W}(\mathbf{P})$ if $n_p(0) = 0$
2. $\mathcal{W}_c(\mathbf{P}) = \{\mathbf{C}\colon\ \mathbf{C} \in \mathcal{W}(\mathbf{P}) \text{ and } \left|\mathbf{Y}(0) - \mathbf{N}(0)\mathbf{R}(0)\right| \neq 0\}$

or

$\mathcal{W}_c(\mathbf{P}) = \{\mathbf{C}\colon\ \mathbf{C} \in \mathcal{W}(\mathbf{P}) \text{ and } \left|\tilde{\mathbf{Y}}(0) - \tilde{\mathbf{R}}(0)\tilde{\mathbf{N}}(0)\right| \neq 0\}$

if $n_p(0) \neq 0$

<u>*Proof:*</u> The case of the right-coprime factorisation of $\mathbf{C}$ is considered in the proof. The proof for the left-coprime factorisation case is similar to the right one. According to Lemma 2, $\mathbf{C}$ is causal *iff*

$$\left|\mathbf{D}_C(0)\right| = \left|\mathbf{Y}(0) - \mathbf{N}(0)\mathbf{R}(0)\right| \neq 0$$

Following Lemma 1 $\mathbf{Y}$, $\mathbf{N}$ are left-coprime and so

$$\left|\mathbf{Y} - \mathbf{N}\mathbf{R}\right| = y - n_p r$$

where $y = \left|\mathbf{Y}\right|$, n_p is the least invariant factor of $\mathbf{N}$ and $r \in \mathcal{R}[d]$ (Vidyasagar, 1985). Therefore,

$$\left|\mathbf{D}_C(0)\right| = \left|\mathbf{Y}(0) - \mathbf{N}(0)\mathbf{R}(0)\right| = y(0) - n_p(0)r(0)$$

Hence

1. if $n_p(0) = 0$, then $y(0) \neq 0$ (Lemma 1). So
$$\left|\mathbf{D}_C(0)\right| = y(0)\ \forall\ \mathbf{R} \in \mathbf{M}[d]$$
2. if $n_p(0) \neq 0$, then $\left|\mathbf{D}_C(0)\right| \neq 0$ implies that $y(0) \neq n_p(0)r(0)$. So, for every $\mathbf{R} \in \mathbf{M}[d]$ such that the corresponding $r(0)$ is not equal to $y(0)/n_p(0)$, the FSTS controller $\mathbf{C}$ is causal. □

It is clear from the proof of Corollary 1 that causality of the FSTS controllers is a generic property and if the plant possesses a delay, in which case $n_p(0) = 0$, all the $\mathcal{W}(\mathbf{P})$ family is causal. Given that causality implies well posedness of the feedback configuration, then well posedness is also generic.

3. STRONG FSTS

The problem of *strong FSTS* is defined as the stabilisation of plant $\mathbf{P}$ in FST sense by a stable compensator. Testable necessary and sufficient conditions for strong FSTS are derived in this section; it turns out that the plant must have the same *parity interlacing property* (Vidyasagar, 1985) as in the case of strong stabilisation where the domains of stability of the closed loop system and the controller coincide. To prove that we need the notions of the Banach algebra $\mathcal{A}_s$ and the blocking zeros of a pulse transfer function.

Definition 2 (Vidyasagar, 1985; Simmons, 1966). $\mathcal{A}_s$ is the set of complex functions $f(d)$ over the real field which are continuous in the closed unit disc $\mathcal{D}$ and analytic in the interior of $\mathcal{D}$. □

If addition and multiplication between any two elements of $\mathcal{A}_s$ is defined pointwise and

$$\|f\| = \sup_{d \in \mathcal{D}} |f(d)|$$

then, $\mathcal{A}_s$ is a commutative Banach algebra with identity over the real field. Also, note that every polynomial with real coefficients is in $\mathcal{A}_s$ and $f \in \mathcal{A}_s$ is a unit of $\mathcal{A}_s$, *iff* $f(d) \neq 0\ \forall\ d \in \mathcal{D}$.

Definition 3. The blocking zeros of the plant transfer function $\mathbf{P}(d)$ are all $z_i \in \mathcal{C}$ such that $\mathbf{P}(z_i) = 0$. Therefore, the blocking zeros of $\mathbf{P}(d)$ are the zeros of the least invariant factor $n_p(d)$ of either $\mathbf{N}$ or $\tilde{\mathbf{N}}$. □

The conditions for strong FSTS are given by the next theorem.

Theorem 2. The plant $\mathbf{P}$ is strongly stabilisable in FST sense *iff* $|\mathbf{D}(d)|$ has the same sign at all real blocking zeros σ_i of $\mathbf{P}$ inside the closed unit disc $\mathcal{D}$.

Proof: An outline of the proof is given here. The detailed proof can be found in Milonidis (1994) and follows similar lines with that given in Vidyasagar (1985). The blocking zeros z_i of $\mathbf{P}(d)$ are the zeros of the least invariant factor $n_p(d)$ of $\mathbf{N}(d)$. Then, for every $\mathbf{X},\mathbf{Y} \in \mathbf{M}[d]$ such that

$$\tilde{\mathbf{D}}\mathbf{Y}+\tilde{\mathbf{N}}\mathbf{X}=\mathbf{I} \Rightarrow (\tilde{\mathbf{D}}\mathbf{Y}+\tilde{\mathbf{N}}\mathbf{X})(z_i)=\mathbf{I} \text{ i.e. } \tilde{\mathbf{D}}(z_i)\mathbf{Y}(z_i)=\mathbf{I}$$

so

$$|\tilde{\mathbf{D}}(z_i)||\mathbf{Y}(z_i)|=1 \text{ or } |\tilde{\mathbf{D}}(z_i)|y(z_i)=1 \qquad (6)$$

where $y(d)=|\mathbf{Y}(d)|$. Therefore, $y(z_i)$ and $|\tilde{\mathbf{D}}(z_i)|$ have the same sign at the blocking zeros of **P**. Also for every $\mathbf{C} \in \mathcal{W}(\mathbf{P})$, $\mathbf{D_C}=\mathbf{Y}-\mathbf{NR}$. Hence

$$|\mathbf{D_C}(d)|=|\mathbf{Y}-\mathbf{NR}|=y(d)-r(d)n_p(d) \qquad (7)$$

For **C** to be stable $|\mathbf{D_C}(0)|=f(d)$ has to be a polynomial unit of $\mathcal{A}_s$. We distinguish the following two cases.

1. $n_p(d)\neq 0 \ \forall\ d \in \mathcal{D}$. Then, $n_p(d)$ is a unit of $\mathcal{A}_s$. Therefore, there is a $q(d)\in\mathcal{A}_s$ such that $y-qn_p=h$ is a polynomial unit of $\mathcal{A}_s$. Then it can be found $r(d), f(d)\in\mathcal{R}[d]$ and $f(d)\neq 0$ $\forall d\in\mathcal{D}$ such that $y(d)-r(d)n_p(d)=f(d)$ (Karcanias and Milonidis, 1992).
2. $n_p(d)$ has zeros in $\mathcal{D}$. Then, according to Eqn. 7 $f(d)$ interpolates $y(d)$ and its derivatives at the blocking zeros of **P** inside the unit disc $\mathcal{D}$. Therefore, $y(d)$ has to have the same sign at the real blocking zeros of **P** inside $\mathcal{D}$. The conclusion of the proof of Theorem 2 follows from Eqn. 6 (Karcanias and Milonidis, 1992). □

Theorem 2 can be restated as the following corollary that expresses the so-called *parity interlacing property.*

Corollary 2. There always exists a stable FSTS controller *iff* the number of poles of **P** inside any interval of successive real blocking zeros of **P** inside the unit disc $\mathcal{D}$ is even. □

4. COMPUTATION OF $\mathcal{W}(\mathbf{P})$ AS A SOLUTION TO A LINEAR ALGEBRA PROBLEM

According to Theorem 1 the computation of the family $\mathcal{W}(\mathbf{P})$ of all FSTS compensators requires only the computation of a particular solution of the Diophantine equation (2). Such a particular solution can be obtained solving a linear algebra problem over the field of real numbers as the following theorem shows.

Theorem 3. Let $\mathbf{P}=\mathbf{N}\mathbf{D}^{-1}\in\mathcal{R}^{m\times\ell}(d)$ with $(\mathbf{D},\mathbf{N})$ right $\mathcal{R}[d]$-coprime and $[\mathbf{N}' \quad \mathbf{D}']'$ column reduced; ν_i $i=1,\ldots,l$ are the right minimal indices of $\mathbf{P}(d)$, $\nu=\max\{\nu_i\}$, μ_i $i=1,\ldots,m$ are the left minimal indices of $\mathbf{P}(d)$, $\mu=\max\{\mu_i\}$. Then if $\mathbf{C}=\tilde{\mathbf{D}}_\mathbf{C}^{-1}\tilde{\mathbf{N}}_\mathbf{C}$ and n is the maximum column degree of $[\tilde{\mathbf{D}}_\mathbf{C} \quad \tilde{\mathbf{N}}_\mathbf{C}]$, $\tilde{\mathbf{D}}_\mathbf{C}$, $\tilde{\mathbf{N}}_\mathbf{C}$ meet

$$\tilde{\mathbf{D}}_\mathbf{C}\mathbf{D}+\tilde{\mathbf{N}}_\mathbf{C}\mathbf{N}=\mathbf{I} \text{ for } n\geq\nu-1 \qquad (8)$$

Proof: $\mathbf{D}$, $\mathbf{N}$, $\tilde{\mathbf{D}}_\mathbf{C}$, $\tilde{\mathbf{N}}_\mathbf{C}$ can be written as

$$\mathbf{D}=\mathbf{D_0}+\mathbf{D_1}d+\cdots+\mathbf{D}_\mu d^\mu$$
$$\mathbf{N}=\mathbf{N_0}+\mathbf{N_1}d+\cdots+\mathbf{N}_\mu d^\mu$$
$$\tilde{\mathbf{D}}_\mathbf{C}=\tilde{\mathbf{D}}_\mathbf{C0}+\tilde{\mathbf{D}}_\mathbf{C1}d+\cdots+\tilde{\mathbf{D}}_\mathbf{Cn}d^n$$
$$\tilde{\mathbf{N}}_\mathbf{C}=\tilde{\mathbf{N}}_\mathbf{C0}+\tilde{\mathbf{N}}_\mathbf{C1}d+\cdots+\tilde{\mathbf{N}}_\mathbf{Cn}d^n$$

Then Eqn. 8 can be rewritten as

$$[\tilde{\mathbf{D}}_\mathbf{C0}\tilde{\mathbf{N}}_\mathbf{C0}\cdots\tilde{\mathbf{D}}_\mathbf{Cn}\tilde{\mathbf{N}}_\mathbf{Cn}]\underbrace{\begin{bmatrix} \mathbf{D_0} & \cdots & \mathbf{D}_\mu & & & & \\ \mathbf{N_0} & \cdots & \mathbf{N}_\mu & & & \mathbf{0} & \\ & \mathbf{D_0} & \cdots & \mathbf{D}_\mu & & & \\ & \mathbf{N_0} & \cdots & \mathbf{N}_\mu & & & \\ & & & \ddots & & & \\ & \mathbf{0} & & & \mathbf{D_0} & \cdots & \mathbf{D}_\mu \\ & & & & \mathbf{N_0} & \cdots & \mathbf{N}_\mu \end{bmatrix}}_{S_{n+1}}$$

$$=[\mathbf{I_l} \ \ \mathbf{0_l} \ \ \mathbf{0_l} \ \cdots \ \mathbf{0_l}] \qquad (9)$$

Using the rank properties of the toeplitz matrix S_{n+1} (Bitmead *et al.*, 1978), it can be easily proved that Eqn. 9 has always a solution if $n\geq\nu-1$ (Chen, 1984). □

5. FST TRACKING AND DISTURBANCE REJECTION

The problem of tracking is one of the important problems in control system design where the output of a system has to follow a particular set of inputs. In the case of FSTS it is required that the output $\mathbf{y_2}$ tracks the input $\mathbf{u_1}$ in finite time, i.e. the error signal $\mathbf{e_1}$ is a polynomial vector. In this case most of the controllers for deadbeat tracking can be considered as time-optimum FST tracking controllers. The conditions for FST tracking are given next.

Theorem 4. Let $(\mathbf{P},\mathbf{C})$ an FST-stable pair with $\mathbf{P}=\mathbf{N}\mathbf{D}^{-1}=\tilde{\mathbf{D}}^{-1}\tilde{\mathbf{N}}\in\mathcal{R}^{m\times\ell}(d)$ and $\mathbf{u_1}=\tilde{\mathbf{D}}_\mathbf{u}^{-1}\tilde{\mathbf{N}}_\mathbf{u}$ where $(\tilde{\mathbf{D}}_\mathbf{u},\tilde{\mathbf{N}}_\mathbf{u})$ left coprime MFDs. Then $\mathbf{y_2}$ tracks $\mathbf{u_1}$ in FST sense, *iff* either of the following two equivalent conditions are satisfied:

1. $\tilde{\mathbf{D}}_\mathbf{u}$ is a right divisor of $\mathbf{D_C}\tilde{\mathbf{D}}$
2. There exist $\mathbf{Q},\mathbf{R}\in\mathbf{M}[d]$ such that

$\mathbf{Q\widetilde{D}_u} + \mathbf{NR\widetilde{D}} = \mathbf{Y\widetilde{D}}$, where $\mathbf{Y}$ is a particular solution to $\mathbf{\widetilde{D}Y} + \mathbf{\widetilde{N}X} = \mathbf{I}$.

Proof: From Eqn. 1 $\mathbf{e_1} = (\mathbf{I} + \mathbf{PC})^{-1}\mathbf{u_1}$ and due to Theorem 1

$$\mathbf{e_1} = \mathbf{D_C \widetilde{D} \widetilde{D}_u^{-1} \widetilde{N}_u}$$

1. For FST tracking $\mathbf{e_1}$ must be polynomial; hence $\mathbf{D_C \widetilde{D}\widetilde{D}_u^{-1}\widetilde{N}_u} \in \mathcal{R}^{\mathrm{mx1}}[d]$ and because $\mathbf{\widetilde{D}_u}$, $\mathbf{\widetilde{N}_u}$ are coprime $\mathbf{D_C\widetilde{D}\widetilde{D}_u^{-1}} \in \mathbf{M}[d]$ (Vidyasagar, 1985). Therefore $\mathbf{D_C\widetilde{D}\widetilde{D}_u^{-1}} = \mathbf{Q} \in \mathbf{M}[d]$, or
$$\mathbf{D_C\widetilde{D}} = \mathbf{Q\widetilde{D}_u} \tag{10}$$
2. From Theorem 1 $\mathbf{D_C} = \mathbf{Y} - \mathbf{NR}$ where $\mathbf{Y}$ is a particular solution to $\mathbf{\widetilde{D}Y} + \mathbf{\widetilde{N}X} = \mathbf{I}$. Then Eqn. 10 becomes $\mathbf{Q\widetilde{D}_u} = (\mathbf{Y} - \mathbf{NR})\mathbf{\widetilde{D}}$, or
$$\mathbf{Q\widetilde{D}_u} + \mathbf{NR\widetilde{D}} = \mathbf{Y\widetilde{D}} \tag{11}$$

□

The solutions $\mathbf{R}$ to Eqn. 11 give the parametrisation of all compensators for tracking in FST sense. The conditions given in Theorem 4 as well as those in the following Theorem 5, are testable as they are linear equations with respect to the elements of the polynomial matrices $\mathbf{Q}$ and $\mathbf{R}$.

Another problem usually encountered in control system design is that of disturbance rejection. If $\mathbf{u_2}$ is the disturbance signal to be rejected in FST sense then, $\mathbf{y_2}$ has to reach a zero steady state after finite time, i.e. $\mathbf{y_2} \in \mathcal{R}^{\mathrm{mx1}}[d]$. The criteria for disturbance rejection in FST sense are given by the following theorem. The proof is similar to that of Theorem 4 and is omitted.

Theorem 5. Let $(\mathbf{P}, \mathbf{C})$ an FST-stable pair with $\mathbf{P} = \mathbf{ND}^{-1} = \mathbf{\widetilde{D}}^{-1}\mathbf{\widetilde{N}} \in \mathcal{R}^{m\times\ell}(d)$ and $\mathbf{u_2} = \mathbf{\widetilde{D}_u^{-1}\widetilde{N}_u}$ where $(\mathbf{\widetilde{D}_u}, \mathbf{\widetilde{N}_u})$ left coprime MFDs. Then $\mathbf{u_2}$ is rejected at the output $\mathbf{y_2}$ in FST sense, *iff* either of the following conditions hold true:

3. $\mathbf{\widetilde{D}_u}$ is a right divisor of $\mathbf{N\widetilde{D}_C}$
4. There exist $\mathbf{Q}, \mathbf{R} \in \mathbf{M}[d]$ such that $\mathbf{Q\widetilde{D}_u} + \mathbf{NR\widetilde{N}} = \mathbf{N\widetilde{Y}}$, where $\mathbf{\widetilde{Y}}$ is a particular solution to $\mathbf{\widetilde{Y}D} + \mathbf{\widetilde{X}N} = \mathbf{I}$.

□

5. l_1 AND l_∞ FSTS CONTROLLERS

The linear nature of the FSTSP and its reduction to the solution to a Toeplitz type set of equations enables us to impose design constraints of the form of $l_1 / l_\infty -$norm minimisation and to treat the whole problem as a standard linear programming problem. The main result for the existence of l_1 / l_∞ FSTS controllers is given by the following theorem (Milonidis and Karcanias, 1997).

Theorem 6. Let $(\mathbf{P}, \mathbf{C})$ be an FST-stable pair, n the polynomial degree of $\begin{bmatrix}\mathbf{D}'_C & \mathbf{N}'_C\end{bmatrix}'$ and μ, ν the observability, controllability indices of $\mathbf{P}$. Then, there always exists an FSTS controller which minimises the l_1 or l_∞ norm of the error signal $\mathbf{e_1}$ due to a step change to all inputs $\mathbf{u_1}$ *iff* $n \geq \nu - 1$; such an FST controller is given as the solution to a linear programming problem. □

Due to the nature of the linear programming, the l_1 / l_∞ FSTS controller will give sub-optimum solutions to the time-optimum FSTS problem and 'good' tracking properties as well. Control strategies like shaping of the error signal $\mathbf{e_1}$ or of the control signal $\mathbf{e_2}$ can also be implemented using linear programming techniques (Dahleh and Pearson, 1988; Milonidis and Karcanias, 1997).

6. CONCLUSIONS

The work presented in this paper is an extension of the single variable case treatment of the *Total Finite Settling Time Stabilisation Problem* (Karcanias and Milonidis, 1992) to the case of multivariable discrete time systems. An algebraic approach for the study of T-FSTS has been developed and it has led to a YJBK parametrisation of FSTS controllers. The subfamily of causal FSTS controllers has been parametrised in terms of a relatively simple condition and its computation is reduced to the solution of a set of Toeplitz type linear equations. The *parity interlacing property* on strong stabilisation has been shown to be also valid in the case of strong FSTS where the stability domains of controller and closed loop system are different.

The advantage of the approach is that the family of FSTS controllers may be computed without resorting to the solution of a polynomial Diophantine matrix equation. The additional conditions for FSTS tracking and FSTS disturbance rejection as well as minimisation of the $l_1 / l_\infty -$norm of the error signal, may be imposed as additional constraints on the design parameters and the whole minimisation problem can be reduced to a standard linear programming problem. The parametrisation of the multivariable FSTS controllers according to McMillan degree and the design of robust two parameter FSTS compensators are a subject under investigation at the moment.

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