15th Triennial World Congress, Barcelona, Spain

CIRCLE AND POPOV CRITERIA AS TOOLS FOR NONLINEAR FEEDBACK DESIGN

Murat Arcak * **Michael Larsen** ** **Petar Kokotović** ***

* *Department of Electrical, Computer and Systems Engineering, Rensselaer Polytechnic Institute, Troy, NY 12180-3590. Email: arcakm@rpi.edu*
** *Visteon Corporation, 36800 Plymouth Rd., Livonia, MI 48150. Tel: (734)-666-2714. Email: mlarsen@visteon.com*
*** *Department of Electrical and Computer Engineering, University of California, Santa Barbara, CA 93106-9560. Email: petar@ece.ucsb.edu*

Abstract: The goal of this paper is to transform classical absolute stability criteria into nonlinear design procedures which employ efficient numerical tools, such as LMI's. The paper starts with an analysis of an earlier circle criterion design, and shows that its feasibility is limited by conditions on the unstable part of the zero dynamics and on the relative degree. Then, an extended circle criterion design is developed which eliminates the relative degree obstacle. The restrictions on the zero dynamics are relaxed by using the Popov multiplier, which also reduces controller complexity. The results are illustrated on several physically motivated design examples.

Keywords: Nonlinear stabilization, absolute stability, Popov multiplier.

1. INTRODUCTION

Nonlinear control has made major advances in the last decade. As surveyed by Kokotović and Arcak [2001], there is already a significant number of constructive, that is, design-oriented results. An obstacle to wider applicability of these designs is their analytical complexity and the lack of computational tools to aid the designer.

Special system structures present opportunities to avoid controller complexity and reduce computations. This is the case with the structures consisting of a linear block in feedback with a nonlinearity, introduced in classical absolute stability studies. Absolute stability is guaranteed if the linear block has a certain input-output property, consistent with the type of feedback nonlinearity. Most important input-output properties of linear blocks, such as small-gain and passivity, can now be analyzed by efficient numerical tools for solving linear matrix inequalities (LMI's); Boyd *et al.* [1994].

Feedback passivation, that is achieving passivity by feedback, has been pursued in numerous studies as a "direct" approach, in which the passivity property is achieved with respect to the control input. However, for a wider use of absolute stability results, passivity is to be achieved with respect to the output and the input of the nonlinearity, neither of which, in general, coincides with the control input. Preliminary results on such "indirect passivation" designs have been reported by Janković *et al.* [1999], and Bernussou *et al.* [1999], who employed the *circle criterion* for feedback control. Their designs make use of the *sector property* $z\varphi(z) \geq 0$ of the feedback nonlinearity $\varphi(\cdot)$, and render the feedforward linear block *strictly*

[1] Research supported in part by the National Science Foundation under grant ECS-9812346 and the Air Force Office of Scientific Research under grant F49620-95-1-0409.

positive real (SPR), thus achieving global asymptotic stability (GAS) from the circle criterion.

In this paper we identify structural obstacles to the feasibility of the circle criterion design, and develop new design procedures which circumvent these obstacles. We first give an analytical test for the feasibility of the circle criterion design, based on the recent indirect passivation conditions derived by Arcak and Kokotović [2001]. This test reveals that feasibility is determined by the relative degree of the linear block, and the unstable part of its zero dynamics. Next, the relative degree obstacle is removed with an extended design which employs derivatives of the nonlinearity in the feedback control law. The zero dynamics restrictions are relaxed by a Popov multiplier design, which also reduces the complexity of the circle criterion design when both designs are applicable. This is illustrated on a jet engine surge subsystem example.

We review the basic circle criterion design in Section 2, and analyze its feasibility in Section 3. An extended design, presented in Section 4, removes the relative degree obstacle of the basic circle criterion design. The Popov multiplier design is presented in Section 5. The proofs are omitted due to space limitations.

2. BASIC CIRCLE CRITERION DESIGN

We start with an introductory example of a single degree-of-freedom active magnetic bearing model due to Tsiotras and Velenis [2000],

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \epsilon x_3 + x_3|x_3| \\ \dot{x}_3 &= u\,, \end{aligned} \tag{1}$$

where x_1 is the rotor position, x_2 is the velocity, x_3 is the magnetic flux, and the parameter $\epsilon > 0$ represents the bias flux. Let us try to stabilize the system at $x = 0$ using a linear feedback and a copy of the nonlinearity; that is,

$$u = k_1x_1 + k_2x_2 + k_3x_3 - \beta x_3|x_3|\,. \tag{2}$$

The resulting closed-loop system in Figure 1 is the feedback interconnection of a linear block and the nonlinearity $\varphi(x_3) = x_3|x_3|$. The feasibility of the circle criterion design (2) depends on whether the parameters k_1, k_2, k_3 and β can be found to render the linear block SPR. If so, the sector property $x_3\varphi(x_3) \geq 0$ ensures GAS of the equilibrium $x = 0$ from the circle criterion. A further question is whether such a design is possible with $\beta = 0$, that is, without a nonlinear term in the control law. This would eliminate the need for knowledge about the nonlinearity, other than its sector property. As we shall see, one of our results implies that the circle criterion design for system (1) is not feasible with $\beta = 0$, which means that the nonlinear term $\beta x_3|x_3|$ is crucial.

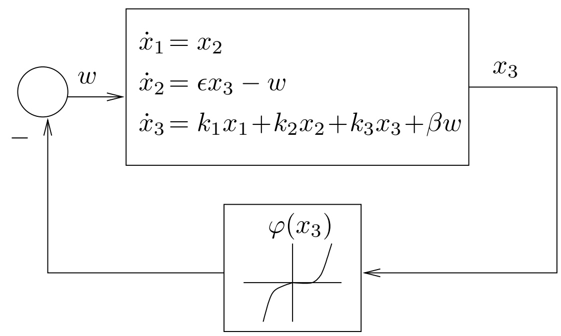

Fig. 1. The closed-loop system (1) with the circle criterion design (2).

We now formulate the circle criterion design for the system

$$\begin{aligned} \dot{x} &= Ax - G\varphi(z) + Bu \\ z &= Hx\,, \end{aligned} \tag{3}$$

where $x \in I\!R^n$, $u \in I\!R^m$, $z \in I\!R^p$, and the nonlinearity $\varphi(\cdot) : I\!R^p \to I\!R^p$ satisfies the sector property $z^T\varphi(z) \geq 0$, and is continuous, so that $\varphi(0) = 0$. We employ the control law

$$u = Kx - \beta\varphi(z) \tag{4}$$

which, with $\beta \neq 0$, requires either the knowledge of the nonlinearity, or the availability of the signal

$$w := -\varphi(z). \tag{5}$$

In view of the Positive Real Lemma, the problem of rendering the closed-loop system

$$\dot{x} = (A + BK)x + (G + B\beta)w \tag{6}$$

SPR from the input w to the output $z = Hx$ is equivalent to the existence of matrices $P = P^T > 0$ and $Q = Q^T > 0$ such that

$$(A + BK)^TP + P(A + BK) + Q \leq 0 \tag{7}$$

$$P(G + B\beta) = H^T\,. \tag{8}$$

This SPR property guarantees GAS of the equilibrium $x = 0$, because the Lyapunov function $V(x) = x^TPx$ satisfies

$$\dot{V} = -x^TQx - 2z^T\varphi(z)\,, \tag{9}$$

where the right-hand side is negative definite because of the sector property $z^T\varphi(z) \geq 0$.

Although (7)-(8) is not an LMI, multiplying (7) from both sides, and (8) from the left, by $X := P^{-1}$ results in

$$X(A+BK)^T+(A+BK)X+\tilde{Q}\leq 0 \quad (10)$$
$$(G+B\beta)=XH^T\,, \quad (11)$$

which is an LMI in $X=X^T>0$, $\tilde{Q}=\tilde{Q}^T:=X^TQX>0$, XK^T and β. This means that we can use the efficient numerical tools available for LMI's to determine whether the design is feasible and, if so, to compute K and β in the control law (4).

3. FEASIBILITY CONDITIONS

To characterize the classes of systems to which the circle criterion design is applicable, our task is to determine when there exist K, β, $X=X^T>0$ and $\tilde{Q}=\tilde{Q}^T$ satisfying (10)-(11). The case where β is constrained to be zero is of separate interest because, then, the control law (4) is linear, and the exact knowledge of the nonlinearity $\varphi(z)$ is not required for its implementation. The feasibility conditions are given for systems with a single nonlinearity and a single control input; that is, $u,z\in I\!R$. For multivariable nonlinearities, analogous results can be obtained with more cumbersome calculations.

When the linear part of (3) is controllable and observable, that is, when the triple (H,A,B) is minimal, a state and feedback transformation results in the normal form

$$\dot{\xi}_i=A_0^i\xi_i+E_0^iy_1+G_0^iw,\quad i=1,2,3, \quad (12)$$
$$\begin{aligned}\dot{y}_1&=y_2+g_1w\\ \dot{y}_2&=y_3+g_2w\\ &\vdots\\ \dot{y}_r&=u+g_rw\end{aligned} \quad (13)$$
$$z=y_1\,, \quad (14)$$

where r denotes the relative degree from the input u to the output z, and the spectra of A_0^i in the zero dynamics subsystem (12) are

$$\sigma(A_0^1)\subset\mathbb{C}^+,\ \sigma(A_0^2)\subset\mathbb{C}^0,\ \sigma(A_0^3)\subset\mathbb{C}^-. \quad (15)$$

The following lemma, proved in Arcak and Kokotović [2001], shows that the obstacles to feasibility are primarily due to the unstable part of the zero dynamics:

Lemma 1. ($\beta=0$) Consider the system (12)-(14), and the matrices $U=U^T$, $V=V^T$ defined by

$$A_0^1U+UA_0^{1\,T}=(E_0^1-G_0^1)(E_0^1-G_0^1)^T \quad (16)$$
$$A_0^1V+VA_0^{1\,T}=(E_0^1+G_0^1)(E_0^1+G_0^1)^T. \quad (17)$$

A state feedback control law $u=Kx$ that renders the closed-loop system SPR from w to $z=y_1$ exists if and only if

$$g_1>0,\quad g_2<0,\quad U-V>\frac{2}{g_1}G_0^1G_0^{1\,T}, \quad (18)$$

and, for every eigenvector p of $A_0^{2\,T}$,

$$p^*(E_0^2-G_0^2)(E_0^2-G_0^2)^Tp> \quad (19)$$
$$p^*(E_0^2+G_0^2)(E_0^2+G_0^2)^Tp\,.$$

□

We next give the feasibility conditions for the case $\beta\neq 0$:

Lemma 2. ($\beta\neq 0$) When $r=1$, a control law $u=Kx+\beta w$ that renders (12)-(14) SPR from w to $z=y_1$ exists if and only if $U>V$, and (19) holds for every eigenvector of $A_0^{2\,T}$. When $r=2$, $g_1>0$ and $U-V>\frac{2}{g_1}G_0^1G_0^{1\,T}$ are required in addition. When $r\geq 3$, all the conditions of Lemma 1 are required. □

An important implication of Lemma 2 is that, when the relative degree is $r=1$ or $r=2$, the nonlinear term $\beta\varphi(z)$ in the control law (4) renders the feasibility conditions less restrictive. However, when $r\geq 3$, the conditions of Lemma 1 and Lemma 2 are the same; that is, if the design is feasible, it is also feasible with the linear control law $u=Kx$.

Example 1. The magnetic bearing system (1) has $r=1$, and with the new variables $y_1:=x_3$, and $\xi_2:=(x_1,\,x_2)^T$, it appears in the form (12)-(14), where

$$A_0^2=\begin{bmatrix}0&1\\0&0\end{bmatrix},\ E_0^2=\begin{bmatrix}0\\ \epsilon\end{bmatrix},\ G_0^2:=\begin{bmatrix}0\\-1\end{bmatrix} \quad (20)$$

and $w:=-x_3|x_3|$. The basic circle criterion design (2) is feasible with $\beta\neq 0$ because the eigenvector of $A_0^{2^T}$ is $p=[0\ 1]^T$, which satisfies (19) for every $\epsilon>0$. A solution to the LMI (10)-(11) for $\epsilon=1$ yields the control law

$$u=-0.6777x_1-2.1724x_2-1.3706x_3-2.8833x_3|x_3|$$

which achieves GAS of the equilibrium $x=0$. However, the linear design with $\beta=0$ is not feasible because $g_1=0$. □

Example 2. The surge subsystem of an axial compressor model has been used to illustrate several nonlinear designs, including a modified ("lean") version of backstepping; Krstić *et al.* [1995]. Here we present a basic circle criterion design for the same surge model:

$$\dot{\phi}=-\psi-\frac{3}{2}\phi^2-\frac{1}{2}\phi^3 \quad (21)$$
$$\dot{\psi}=u\,, \quad (22)$$

where ϕ and ψ are the deviations of the mass flow and the pressure rise from their set points, and the control input u is the flow through the throttle with a preliminary linear feedback. Because the quadratic term in the nonlinearity $\frac{3}{2}\phi^2 + \frac{1}{2}\phi^3$ violates the sector property, we add and subtract the linear term $\frac{9}{8}\phi$ in (21), and obtain

$$\dot{\phi} = -\psi + \frac{9}{8}\phi - \varphi(\phi) \tag{23}$$

$$\dot{\psi} = u\,, \tag{24}$$

where

$$\varphi(\phi) := \frac{9}{8}\phi + \frac{3}{2}\phi^2 + \frac{1}{2}\phi^3 \tag{25}$$

satisfies the sector condition $\phi\varphi(\phi) \geq 0$.

This model has relative degree $r = 2$, and with $y_1 := \phi$ and $y_2 := -\psi + \frac{9}{8}\phi$, its normal form is (12)-(14) with no zero dynamics. The basic circle criterion design

$$u = k_1\phi + k_2\psi - \beta\varphi(\phi) \tag{26}$$

is feasible with $\beta \neq 0$ because $g_1 > 0$. However, the linear control law with $\beta = 0$ is not feasible because $g_2 > 0$. To gain further insight into the feasibility conditions, we represent (23),(24),(26) as the feedback interconnection of the transfer function

$$G(s) = \frac{s - k_2 - \beta}{s^2 - (k_2 + \frac{9}{8})s + (k_1 + \frac{9}{8}k_2)} \tag{27}$$

and the sector nonlinearity $\varphi(\cdot)$. A second order transfer function is SPR if and only if its poles are stable, and its zero is located in the interval $(\sigma, 0)$, where σ represents the sum of the poles. For the transfer function (27), we have $\sigma = k_2 + \frac{9}{8}$ and, hence, its zero $k_2 + \beta$ must satisfy

$$k_2 + \frac{9}{8} < k_2 + \beta < 0\,. \tag{28}$$

This means that the circle criterion design is not feasible with $\beta = 0$. However, with $\beta > \frac{9}{8}$, any choice of k_1, k_2 satisfying $k_2 + \beta < 0$ and $k_1 + \frac{9}{8}k_2 > 0$, renders $G(s)$ SPR and, thus, ensures GAS of the equilibrium $(\phi, \psi) = 0$. □

4. EXTENDED CIRCLE CRITERION DESIGN

In this section we extend the applicability of the basic circle criterion design to systems which violate the conditions $g_1 > 0$ and $g_2 < 0$ of Lemma 2. We first consider the relative degree two case, that is, the system

$$\begin{aligned} \dot{\xi} &= A_0\xi + E_0 y_1 - G_0\varphi(y_1) \\ \dot{y}_1 &= y_2 - g_1\varphi(y_1) \\ \dot{y}_2 &= u - g_2\varphi(y_1)\,. \end{aligned} \tag{29}$$

Because the basic circle criterion design of Section 2 is not feasible when $g_1 < 0$, we let $\tilde{g}_1 > 0$, and define

$$\tilde{y}_2 := y_2 - (g_1 - \tilde{g}_1)\varphi(y_1)\,, \tag{30}$$

which results in the new equations

$$\begin{aligned} \dot{\xi} &= A_0\xi + E_0 y_1 - G_0\varphi(y_1) \\ \dot{y}_1 &= \tilde{y}_2 - \tilde{g}_1\varphi(y_1) \\ \dot{\tilde{y}}_2 &= u - g_2\varphi(y_1) - (g_1 - \tilde{g}_1)\dot{\varphi}\,, \end{aligned} \tag{31}$$

where $\dot{\varphi}$ is available as a function of y_1 and y_2:

$$\dot{\varphi} = \frac{\partial\varphi}{\partial y_1}(y_2 - g_1\varphi(y_1)). \tag{32}$$

With the feedback transformation

$$u = \tilde{u} - (\tilde{g}_1 - g_1)\dot{\varphi}(y_1, y_2)\,, \tag{33}$$

system (31) becomes

$$\begin{aligned} \dot{\xi} &= A_0\xi + E_0 y_1 - G_0\varphi(y_1) \\ \dot{y}_1 &= \tilde{y}_2 - \tilde{g}_1\varphi(y_1) \\ \dot{\tilde{y}}_2 &= \tilde{u} - g_2\varphi(y_1) \end{aligned} \tag{34}$$

where $\tilde{g}_1 > 0$. Thus, the new variable $\tilde{y}_2$ and the feedback transformation (33) eliminated the restriction $g_1 > 0$ from Lemma 2. With the design $\tilde{u} = K\tilde{x} - \beta\varphi(y_1)$, where $\tilde{x} = [\xi^T y_1\, \tilde{y}_2]^T$, and K and β are obtained from the LMI (10)-(11) for (34), the final form of our control law is

$$u = Kx - \beta\varphi(y_1) - (\tilde{g}_1 - g_1)\dot{\varphi}(y_1, y_2)\,, \tag{35}$$

where $x = [\xi^T y_1\, y_2]^T$; that is, the nonlinear term in $\tilde{y}_2$ is incorporated in $\beta\varphi(y_1)$.

When the relative degree is three or more, a repeated application of the procedure above eliminates both $g_1 > 0$ and $g_2 < 0$ from Lemma 2. Indeed, with the change of variables (30) and

$$\begin{aligned} \tilde{y}_i = y_i &- (g_1 - \tilde{g}_1)\varphi^{(i-2)}(y_1, \cdots, y_{i-1}) \\ &-(g_2 - \tilde{g}_2)\varphi^{(i-3)}(y_1, \cdots, y_{i-2})\,, \quad i = 3, \cdots, r, \end{aligned} \tag{36}$$

where $\varphi^{(k)}$ denotes the k-th time derivative of $\varphi(y_1)$, which is available as a function of $y_1, \cdots, y_{k+1}$; and, with the preliminary feedback

$$\begin{aligned} u = \tilde{u} &+ (g_1 - \tilde{g}_1)\varphi^{(r-1)}(y_1, \cdots, y_r) \\ &+(g_2 - \tilde{g}_2)\varphi^{(r-2)}(y_1, \cdots, y_{r-1})\,, \end{aligned} \tag{37}$$

system (12)-(14) becomes

$$
\begin{aligned}
\dot{\xi} &= A_0\xi + E_0 y_1 - G_0\varphi(y_1) && (38)\\
\dot{y}_1 &= \tilde{y}_2 - \tilde{g}_1\varphi(y_1) &&\\
\dot{\tilde{y}}_2 &= \tilde{y}_3 - \tilde{g}_2\varphi(y_1) &&\\
\dot{\tilde{y}}_3 &= \tilde{y}_4 - g_3\varphi(y_1) && (39)\\
&\vdots &&\\
\dot{\tilde{y}}_r &= \tilde{u} - g_r\varphi(y_1)\,. &&
\end{aligned}
$$

Choosing $\tilde{g}_1$ and $\tilde{g}_2$ to satisfy (18), and applying the state and input transformations (36)-(37), we obtain a control law of the form

$$
\begin{aligned}
u = Kx - \beta\varphi(y_1) - \beta_1\dot{\varphi}(y_1, y_2) - \cdots \quad (40)\\
-\beta_{r-1}\varphi^{(r-1)}(y_1, \cdots, y_r)\,.
\end{aligned}
$$

The following theorem summarizes our derivations:

Theorem 1. (Feasibility of the extended circle criterion design) Consider the system (3), represented as in (12)-(14), where $w = -\varphi(z)$, r is the relative degree from the control input u to the output z; and let U and V be defined by (16) and (17), respectively. A control law of the form (40) renders the closed-loop system SPR from the input w to the output z if and only if $U > V$, and (19) holds for every eigenvector of $A_0^{2^T}$. If, in addition, (18) holds, then the linear control law $u = Kx$ is also feasible. □

It is of interest to interpret the conditions in the above theorem as structural causes for feasibility or infeasibility of the extended circle criterion design. The relative degree r determines the form of the control law and the restrictions under which the linear control law is also feasible. As can be expected, for a higher relative degree, the complexity of the control law is also higher. Another important observation is that the obstacles to feasibility are primarily due to the system's unstable zeros (eigenvalues of A_0^1), because the corresponding matrices U and V defined by (16) and (17) are required to satisfy $U > V$. The zeros on the imaginary axis (eigenvalues of A_0^2) are also an obstacle due to the requirement that every eigenvector of A_0^2 must satisfy (19). When (12)-(14) is minimum phase, that is when A_0 is Hurwitz, the control law (40) is always feasible.

5. THE POPOV MULTIPLIER DESIGN

The SPR requirement imposed by the circle criterion on the linear block $G(s)$ has been relaxed by various "multipliers" due to Popov [1960], Zames and Falb [1968], and other authors, who exploit additional properties of the sector nonlinearity $\varphi(\cdot)$ to establish passivity of the feedback path in Figure 2 from input u_2 to output y_2. Thus, the SPR requirement is imposed on the feedforward path $G(s)M(s)$, rather than on $G(s)$.

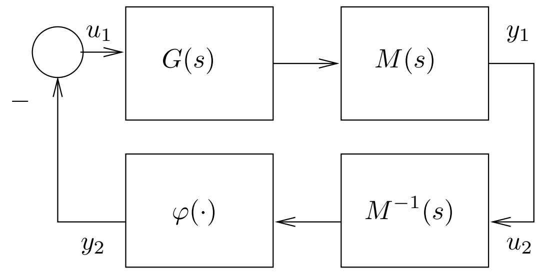

Fig. 2. Feedback interconnection of $G(s)$ and nonlinearity $\varphi(\cdot)$. The multiplier $M(s)$ is introduced to relax the SPR restriction on $G(s)$.

We now proceed with a Popov multiplier design, when $M(s) = 1 + \eta s$. In the closed-loop system (3)-(4), we denote

$$G(s) = H(sI - A - BK)^{-1}(G + B\beta)\,, \quad (41)$$

and, as a state-space realization of $(1 + \eta s)G(s)$, we use

$$
\begin{aligned}
\mathcal{A} &= A + BK & \mathcal{B} &= G + B\beta \quad (42)\\
\mathcal{C} &= H[I + \eta(A + BK)] & \mathcal{D} &= \eta H(G + B\beta).
\end{aligned}
$$

From the Positive Real Lemma the SPR property of $(1 + \eta s)G(s)$ means

$$\begin{bmatrix} \mathcal{A}^T P + P\mathcal{A} + Q & P\mathcal{B} - \mathcal{C}^T \\ \mathcal{B}^T P - \mathcal{C} & -\mathcal{D} - \mathcal{D}^T \end{bmatrix} \leq 0\,. \quad (43)$$

Suppose that K, η and β satisfy this SPR condition for some $P = P^T > 0$, $Q = Q^T > 0$, and that $\varphi(\cdot)$ is a vector nonlinearity satisfying $z^T\varphi(z) \geq 0$. Then, GAS of $x = 0$ follows from the Lyapunov function

$$V(x) = x^T P x + 2\int_0^z \varphi^T(\sigma)d\sigma \quad (44)$$

whose derivative for the closed-loop system is negative definite:

$$\dot{V} \leq -x^T Q x - \frac{2}{\eta} z^T \varphi(z)\,. \quad (45)$$

A difficulty in a design based on (43) is due to the presence of the additional parameter η. The attempt to convert (43) into an LMI using $X := P^{-1}$, $\tilde{Q} = XQX$, yields

$$\begin{bmatrix} X\mathcal{A}^T + \mathcal{A}X + \tilde{Q} & \mathcal{B} - X\mathcal{C}^T \\ \mathcal{B}^T - \mathcal{C}X & -\mathcal{D} - \mathcal{D}^T \end{bmatrix} \leq 0 \quad (46)$$

which is not an LMI jointly in K, β and η, because it is bilinear due to the products ηX and $\eta\beta$.

Rather than solving (46) using a bilinear matrix inequality, suggested by Safonov *et al.* [1994], a more direct application of the results in this paper is to treat (46) as a one-parameter family of LMI's. Starting with $\eta = 0$, which is infeasible, a one-parameter search for increasing values of η allows us to relax the zero dynamics conditions of Theorem 1:

Theorem 2. (Feasibility of the Popov multiplier design) Consider the system (3), represented as in (12)-(14), where $w = -\varphi(z)$, and the relative degree from the control input u to the output z is $r \geq 2$. Let

$$\bar{E}_0^i = (I + \eta A_0^i)^{-1} E_0^i, \quad i = 1, 2, \qquad (47)$$

and let U and V be defined by (16)-(17), with E_0^1 replaced with $\bar{E}_0^1$. A control law of the form (40) is feasible for the Popov multiplier design if and only if there exists $\eta > 0$ such that $U > V$ and (19) holds for every eigenvector of $A_0^{2^T}$, with E_0^2 replaced with $\bar{E}_0^2$. If, in addition, $g_1 > 0$, then the linear control law $u = Kx$ is also feasible. □

When $\eta = 0$, (47) implies $\bar{E}_0^i = E_0^i$; that is, we recover the zero dynamics conditions (19) and $U > V$ of Theorem 1. These conditions are relaxed in the Popov multiplier design, because they only need to hold for some $\eta \geq 0$ in (47), rather than for $\eta = 0$ as required in the extended circle criterion design. Theorem 2 restricts the relative degree by $r \geq 2$ because, if $r = 1$, then y_1 in Figure 2 contains a throughput term from u_1, and the Popov multiplier design is not applicable.

The following example shows that, even when a circle criterion design is feasible, the Popov multiplier design may lead to a simpler control law:

Example 3. Consider the axial compressor surge subsystem in Example 2, where a linear control law was not feasible for a circle criterion design. It is feasible for a Popov multiplier design because $g_1 > 0$ as in Theorem 2. Indeed, with the choice of linear feedback gains

$$k_1 = 1 + \left(\frac{9}{8} + \frac{1}{\eta}\right)^2 \quad k_2 = -\frac{2}{\eta} - \frac{9}{8}, \quad (48)$$

it is easy to verify that

$$(1 + \eta s)G(s) = \frac{(1 + \eta s)(s - k_2)}{s^2 - (k_2 + \frac{9}{8})s + (k_1 + \frac{9}{8}k_2)}$$

is SPR for all $\eta > 0$, and, hence, the linear control law

$$u = k_1\phi + k_2\psi$$

achieves GAS of (21)-(22). □

6. CONCLUSION

We have studied feasibility of the basic circle criterion design, and revealed structural obstacles arising from the relative degree and the unstable part of the zero dynamics. The relative degree obstacle has been removed with an extended scheme which employs derivatives of the nonlinearity in the feedback control law. The Popov multiplier has relaxed the conditions on the zero dynamics. To further improve the design, a promising research direction is to employ other multipliers, such as the one due to Zames and Falb [1968].

References

Arcak, M. and P.V. Kokotović (2001). Feasibility conditions for circle criterion designs. *Systems and Control Letters* **42**(5), 405–412.

Bernussou, J., J.C. Geromel and M.C. de Oliveira (1999). On strict positive real systems design: guaranteed cost and robustness issues. *Systems and Control Letters* **36**, 135–141.

Boyd, S., L. El Ghaoui, E. Feron and V. Balakrishnan (1994). *Linear Matrix Inequalities in System and Control Theory*. Vol. 15 of *SIAM Studies in Applied Mathematics*. SIAM. Philadelphia, PA.

Janković, M., M. Larsen and P.V. Kokotović (1999). Master-slave passivity design for stabilization of nonlinear systems. In: *Proceedings of the 18th American Control Conference*. San Diego, CA. pp. 769–773.

Kokotović, P.V. and M. Arcak (2001). Constructive nonlinear control: a historical perspective. *Automatica* **37**(5), 637–662.

Krstić, M., I. Kanellakopoulos and P. Kokotović (1995). *Nonlinear and Adaptive Control Design*. John Wiley & Sons, Inc.. New York.

Popov, V.M. (1960). Criterion of quality for non-linear controlled systems. In: *Preprints of the First IFAC World Congress*. Butterworths. Moscow. pp. 173–176.

Safonov, M.G., K.C. Goh and J.H. Ly (1994). Control system synthesis via bilinear matrix inequalities. In: *Proceedings of the 1994 American Control Conference*. Baltimore, MD. pp. 45–49.

Tsiotras, P. and E. Velenis (2000). Low-bias control of AMB's subject to saturation constraints. In: *Proceedings of the 2000 IEEE International Conference on Control Applications*. Anchorage, Alaska.

Zames, G. and P.L. Falb (1968). Stability conditions for systems with monotone and slope-restricted nonlinearities. *SIAM Journal of Control and Optimization* **6**, 89–108.