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NEW RESULTS ON THE PARAMETRIZATION OF J-SPECTRAL FACTORS

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Abstract: This paper considers a parametrization of a class of J spectral factors of a given J spectral density. The notion of J stable functions is introduced to define a partial ordering on the set of J spectral factors. Necessary and sufficient conditions for the existence of extremal J spectral factors are given and a number of explicit factorization results is derived which serve for the algorithmic aspects of computing specific J spectral factors.

Keywords: J-spectral factors, factorization theory, linear systems

1. INTRODUCTION

The problem of finding spectral factors of a J spectral density is of paramount importance in a large variety of problems in systems and control. Indeed, J spectral factors and J spectral factorizations are at the basis of the analysis and synthesis of controller design algorithms in $\mathcal{H}_\infty$ control, they play an important role in the understanding of zero sum non-cooperative differential games in game theory, and occur in a natural way in the study of dissipative or lossless dynamical systems. Furthermore, J spectral factorizations occur in problems related to model reduction and in various problems involving questions related to representations and equivalence of dynamical systems. See, e.g. (Dym, 1994), (Ball and Helton, 1988), (M. Green and Doyle, 1990), (Green, 1992), (Meinsma, 1993), (Meinsma, 1994), (S. Weiland and de Jager, 1997), (Weiland and Gombani, 2000).

Let

$$J = J_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$$

be a signature matrix. A matrix valued rational function Φ is called a *J-spectral density* if $\Phi = \Phi^*$ and if its signature is constant and equal to J for all complex numbers on the imaginary axis. Here, as elsewhere, Φ^* is the conjugate of Φ. A matrix valued rational function W is a *J-spectral factor* of a J-spectral density Φ if

$$\Phi = WJW^*.$$

A J-spectral density Φ is often defined through the latter expression for some rational matrix valued function W and one is merely interested in J-spectral factors with specific properties. This means that W is given and we are interested in finding elements $\overline{W}$ belonging to a specific class of rational matrix valued functions such that $WJW^* = \overline{W}J\overline{W}^*$. It is the purpose of this paper to develop a number of algorithms to derive J spectral factors with specific stability properties from a given J spectral density. More specifically, we propose a partial ordering on the set of J spectral factors of a given J spectral density and establish the commutativity of a number of operators that infer J spectral factors.

For this, we first recall and introduce a number of notions that generalize the well established concepts of inner and outer functions to the context of J-inner and J outer functions. New notions of J stable and J anti-stable rational matrix valued functions are introduced and characterized in the state space context.

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The paper is organized as follows. In Section 2 the notions of J-unitary and J-inner functions are recalled and we introduce a generalization of the well known concept of an outer function which is relevant in the context of J spectral factorizations. A problem statement is given in Section 3. The main contribution in this paper consists of a series of factorization results which are collected in Section 4, and which are believed to be of independent interest. These results include a number of generalizations of the Douglas-Shapiro-Shields factorization (every rational $W \in \mathcal{H}_2$ can be factorized as $W = \overline{W}K$ with $\overline{W} \in \mathcal{H}_2^{\perp}$, and K inner with minimal degree (Fuhrmann, 1995)) and the well known outer-inner factorizations of rational matrix valued functions (every rational $W \in \mathcal{H}_\infty$ can be factorized as $W = W_{\mathrm{o}} W_{\mathrm{i}}$ with W_{o} outer, W_{i} inner (Vidyasagar, 1985)) for the context of J spectral factors. To the authors' knowledge, the theorems in Section 4 are new. Proofs are collected in section 5. Because of space limitations, we can not provide full fledged proofs in this paper. The main arguments of the proofs are merely indicated. Some conclusions are collected in Section 6.

2. DEFINITIONS

A square matrix valued function W is said to be J *unitary* if $W^*(s)JW(s) = J$ for all $s \in \mathbb{C}^0$ (the imaginary axis). It is J *inner* if it is J-unitary and, in addition, $[W(s)]^* JW(s) \leq J$ for all but finitely many points $s \in \mathbb{C}^+$ (the open right-half complex plane). Similarly, W is said to be *conjugate J-unitary* (or *conjugate J-inner*) if its conjugate, W^*, is J-unitary (J inner). In terms of indefinite inner product spaces, W is J unitary if and only if for all complex vectors v_1, v_2, and $\omega \in \mathbb{R}$,

$$\langle w_1, Jw_2 \rangle = \langle v_1, Jv_2 \rangle$$

where $w_1 = W(j\omega)v_1$ and $w_2 = W(j\omega)v_2$.

It is well known that every matrix valued proper (i.e., analytic at infinity) rational function W can be represented as $W(s) = C(Is - A)^{-1}B + D$ for suitable (real) matrices (A, B, C, D). Any such quadruple is said to be a *state space representation* of W and we write

$$W = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right). \tag{1}$$

Here, (A, B, C, D) is not necessarily minimal in that the McMillan degree $\delta(W)$ of W is not necessarily equal to the dimension of A. W is said to be *row J stable* if there exists a positive definite solution $P = P^*$ of the equation

$$AP + PA^* + BJB^* = 0. \tag{2}$$

It is called *column J stable* if there exists a positive definite solution $P = P^*$ of

$$A^*P + PA + C^*JC = 0. \tag{3}$$

Similarly, W is called *row J anti-stable* (resp. *column J anti-stable*) if its conjugate, W^*, is column (resp. row) J stable. For short, W is called J stable (J anti-stable) if it is row J stable (column J anti-stable). It is easily seen that the notion of J stability does not depend on the specific state space representation of W. That is, row and column J stability are properties of W, not of a specific representation (A, B, C, D) of W as defined by (1).

Finally, a square rational matrix W is said to be J *outer* if it is row J stable, invertible in $\mathcal{L}_\infty$ and if its inverse W^{-1} is column J-stable. It is said to be *conjugate J outer* if its conjugate, W^*, is J outer.

Remark 2.1. The notions of J stable, J-inner and J outer functions generalize the usual notions of stable, inner and outer functions. Indeed, W is row I stable if and only if all eigenvalues of A are in $\mathbb{C}^-$ for some (and hence for all) minimal representations of W. W belongs to $\mathcal{H}_\infty$ in that case. Likewise, W is I inner if and only if W is inner in the usual sense. Furthermore, W is I outer if and only if it is a unitary element in $\mathcal{H}_\infty$, in which case it is usually called a *minimum phase function*.

Remark 2.2. Many of the concepts introduced here naturally generalize to non-square rational functions. We focus here on square rational functions mainly for reasons of exposition.

3. PROBLEM FORMULATION

Suppose that W is a given matrix valued rational function and consider the J spectral density $\Phi = WJW^*$. We consider the following diagram.

$$\begin{tikzcd}
\underline{W}_- \arrow[r, "\underline{Q}'"] & \underline{W} \arrow[r, "\underline{Q}''"] & \underline{W}_+ \\
W_- \arrow[u, "\underline{K}_-"] \arrow[r, "Q'"] & W \arrow[u, "\underline{K}"] \arrow[r, "Q''"] & W_+ \arrow[u, "\underline{K}_+"'] \\
\overline{W}_- \arrow[u, "\overline{K}_-"] \arrow[r, "\overline{Q}'"] & \overline{W} \arrow[u, "\overline{K}"] \arrow[r, "\overline{Q}''"] & \overline{W}_+ \arrow[u, "\overline{K}_+"']
\end{tikzcd}$$

The interpretation of symbols in this diagram is as follows. The left side of the diagram (W's with subscripts $-$) represent rational and invertible matrix valued functions whose inverses are J stable. Functions on the right side (W's with subscripts $+$) are characterized by the property that their inverses are J anti-stable. The functions on the top and bottom line of the diagram represent, respectively, J stable and J anti-stable rational matrix valued functions. All arrows indicate a post-multiplication with the labeled object (i.e., $\underline{W} = W\underline{K}$, etc.). The main reason for investigating this structure in the context of spectral factorizations, lies in the fact that all W's in the diagram define J spectral factors of the J spectral density $\Phi = WJW^*$ provided all arrows indicate multiplications with J-unitary rational functions. Indeed, if $\underline{W} = W\underline{K}$ with $\underline{K}$ a J-unitary rational function, then $\underline{K}^*$ is J-unitary and

$$\underline{W} J \underline{W}^* = W \underline{K} J \underline{K}^* W^* = W J W^* = \Phi$$

which shows that $\underline{W}$ is a J spectral factor of Φ.

In particular, we will be interested in the J spectral factors defined by the corner points of the above diagram. These are the 'extremal' rational matrix valued functions

(1) $\underline{W}_-$: J stable with J stable inverse.
(2) $\underline{W}_+$: J stable with J anti-stable inverse.
(3) $\overline{W}_-$: J anti-stable with J stable inverse.
(4) $\overline{W}_+$: J anti-stable with J anti-stable inverse.

Note that $\underline{W}_-$ is J outer and $\overline{W}_+$ is conjugate J outer. We develop algorithms for the calculation of the extremal rational functions from a given J-spectral density $\Phi = WJW^*$, i.e., we construct the mapping

$$W \longrightarrow (\underline{W}_-, \underline{W}_+, \overline{W}_-, \overline{W}_+) \tag{4}$$

where W is represented by (1) and show under which conditions the diagram above is commutative.

4. FACTORIZATIONS

In preparing the commutativity of the diagram above, we present a number of characterizations of left and right J-unitary divisors and factorizations of matrix valued rational functions. We start with a state space characterization of J-unitary and J-inner functions.

Proposition 4.1. Let K be a square rational matrix valued function. Then the following are equivalent.

(1) K is J-unitary (J-inner).
(2) K has a representation

$$K = \left(\begin{array}{c|c} A & -P^{-1}C^*JD \\ \hline C & D \end{array}\right) \tag{5}$$

where $D^*JD = J$ and P is a non-singular (positive definite) solution of (3).

(3) K has a representation

$$K = \left(\begin{array}{c|c} A & B \\ \hline -DJB^*P^{-1} & D \end{array}\right) \tag{6}$$

where $DJD^* = J$ and P is a non-singular (positive definite) solution of (2).

(See Section 5 for proofs). In addition, one easily derives that for a J-unitary (or J-inner) function K, the non-singular (or positive definite) solutions of the Lyapunov equations (3) and (2) are each others' inverses.

The following theorem extends a result of (Fuhrmann, 1995) and of (Dym, 1994; Dym, 2001) and characterizes all J-unitary left and right divisors of a J-unitary function, both in algebraic terms as well as in a representation independent context of invariant subspaces. To formulate the result, recall that a rational function K_0 is a *left divisor of* K if there exists K_1 such that $K = K_0K_1$ and the McMillan degrees $\delta(K_0)$ and $\delta(K)$ of K_0 and K satisfy $\delta(K_0^*K) = \delta(K) - \delta(K_0)$. The notion of a *right divisor* is similarly defined. As for notation, if $\mathcal{V}$ is a subspace of a linear vector space $\mathcal{X}$, then we denote by $\Pi_{\mathcal{V}}$ the canonical insertion map that maps $\mathcal{V}$ into $\mathcal{X}$. (Equivalently, $\Pi_{\mathcal{V}}^*$ is the canonical projection that maps $\mathcal{X}$ onto $\mathcal{V}$).

Theorem 4.2. Let K be a J-unitary function of McMillan degree $\delta(K) = n$. Then

(1) K allows a representation (5), *where P is the non-singular solution of* (3), *and the following are equivalent*
   (a) There exists a J-unitary left divisor K_0 of K of McMillan degree $\delta(K_0) = k$.
   (b) There exists an A-invariant subspace $\mathcal{V}$ of dimension k such that $P_{\mathcal{V}} := \Pi_{\mathcal{V}}^* P \Pi_{\mathcal{V}}$ is non-singular.
   (c) There exists a hermitian matrix X of rank $X = k$ *such that*

$$\begin{cases} AX + XA^* + XC^*JCX = 0 \\ XPX = X \end{cases} \tag{7}$$

If either one of these equivalent conditions hold, then one such K_0 is given by

$$K_0 = \left(\begin{array}{c|c} A & -XC^*JD \\ \hline C & D \end{array}\right).$$

(2) K allows a representation (6), *where P is the non-singular solution of* (2), *and the following are equivalent*
   (a) There exists a J-unitary right divisor K_0 of K of McMillan degree $\delta(K_0) = k$.
   (b) There exists an A^*-invariant subspace $\mathcal{V}$ of dimension k such that $P_{\mathcal{V}} := \Pi_{\mathcal{V}}^* P \Pi_{\mathcal{V}}$ is non-singular.
   (c) There exists a hermitian matrix X of rank $X = k$ *such that*

$$\begin{cases} A^*X + XA + XBJB^*X = 0 \\ XPX = X \end{cases} \tag{8}$$

If either one of these equivalent conditions hold, then one such K_0 is given by

$$K_0 = \left(\begin{array}{c|c} A & B \\ \hline -DJB^*X & D \end{array}\right).$$

Not all J-unitary functions admit J-unitary factors of lower McMillan degree. Indeed, consider

$$K = \left(\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1/2 & -1 & 1 & 0 \\ \hline -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right).$$

Then K is J-unitary and (3) holds with the non-singular matrix

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

However, $P_{\mathcal{V}} = 0$ for all non-trivial A-invariant subspaces. Hence, K has no (non-trivial) J-unitary left divisors. In a similar way one also shows that K has no non-trivial J-unitary right divisors.

The next theorem provides necessary and sufficient conditions for a J-unitary function to be factorizable in a J-inner and a conjugate J-inner factor.

Theorem 4.3. Let K be J-unitary and represented by (5) *where P is a non-singular solution of* (3)*. Let n_+ and n_- be the number of positive and negative eigenvalues of P, respectively. Then*

(1) There exists J-inner functions $\overline{K}$ and $\underline{K}$ such that

$$K = \overline{K}\,\underline{K}^*$$

if and only if there exists an A-invariant subspace $\mathcal{V}$ of dimension n_+ such that $P_{\mathcal{V}} := \Pi_{\mathcal{V}}^* P \Pi_{\mathcal{V}}$ is positive definite.

(2) There exist J-inner functions $\overline{K}$ and $\underline{K}$ such that

$$K = \underline{K}^*\overline{K}$$

if and only if there exists an A-invariant subspace $\mathcal{V}$ of dimension n_- such that $P_{\mathcal{V}} := \Pi_{\mathcal{V}}^* P \Pi_{\mathcal{V}}$ is negative definite.

The following result is an extension of the Douglas-Shapiro-Shields factorizations to the case of J-unitary rational functions. See also (Fuhrmann, 1995).

Theorem 4.4. Let W be represented by (1) *and let P be a non-singular solution of* (2)*. Define the J-unitary function*

$$K = \left(\begin{array}{c|c} A & B \\ \hline -JB^*P^{-1} & I \end{array}\right)$$

and suppose that there exists a factorization $K = \underline{K}^*\overline{K}$ with the properties of statement 2 of Theorem 4.3. Then

(1) $\underline{K}$ is the minimal degree J-inner function such that

$$\underline{W} := W\underline{K}$$

is J stable. Moreover,

$$\underline{W} = \left(\begin{array}{c|c} A + BJB^*X & B \\ \hline C + DJB^*X & D \end{array}\right)$$

is a representation of $\underline{W}$ where $X \leq 0$ is the minimum non-positive definite solution of $A^*X + XA + XBJB^*X = 0$.

(2) $\overline{K}$ is the minimal degree J-inner function such that

$$\overline{W} := W\overline{K}^*$$

is J anti-stable. Moreover,

$$\overline{W} = \left(\begin{array}{c|c} A + BJB^*X & B \\ \hline C + DJB^*X & D \end{array}\right)$$

is a representation of $\overline{W}$ where $X \geq 0$ is the maximum non-negative definite solution of $A^*X + XA + XBJB^*X = 0$.

Theorem 4.4 therefore provides a tool for the calculation of J stable and J anti-stable spectral factorizations from an arbitrary rational matrix valued function. In particular, this result can be applied for the construction of all *vertical* mappings (labeled K) in the diagram of Section 3. Note that necessary and sufficient conditions for the existence of each of these mappings are stated in Theorem 4.3.

A similar result applies for the *horizontal* mappings (labeled Q) in the diagram of Section 3. The statements are as follows.

Theorem 4.5. Let W be represented by (1) *and suppose that W is invertible with*

$$W^{-1} = \left(\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right) = \left(\begin{array}{c|c} A - BD^{-1}C & BD^{-1} \\ \hline -D^{-1}C & D^{-1} \end{array}\right).$$

Let $\hat{P}$ be a non-singular solution of $\hat{A}\hat{P} + \hat{P}\hat{A}^* + \hat{B}J\hat{B}^* = 0$ and suppose that the J-unitary function

$$Q = \left(\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline -J\hat{B}^*\hat{P}^{-1} & I \end{array}\right)$$

admits a factorization $Q = Q'^*Q''$ with Q' and Q'' J-inner (cf. statement 2 of Theorem 4.3). Then

(1) Q' is the minimal degree J-inner function such that

$$W_- := WQ'^*$$

has a J stable inverse. In particular, W_- is J-outer whenever W is J stable.

(2) Q'' is the minimal degree J-inner function such that

$$W_+ := WQ''$$

has a J anti-stable inverse.

Basically, Theorem 4.4 and Theorem 4.5 provide an algorithm for the calculation of each of the mappings in the diagram of Section 3.

Note that Statement 1 of Theorem 4.5 implies that

$$W = W_o W_i$$

with $W_o := W_-$ a J-outer and $W_i := JQ'J$ a J-inner rational function. That is, W admits a J-outer – J-inner factorization under the conditions stated in Theorem 4.5.

We conjecture here that the entire diagram is commutative under the condition that W, when represented by (1), allows a non-singular hermitian solution P of (3) (or, equivalently, of (2)) of n_+ and n_- positive and negative eigenvalues, such that there exists A-invariant subspaces $\mathcal{V}_+$ and $\mathcal{V}_-$ of dimensions n_+ and n_-, respectively, with $P_{\mathcal{V}_+} := \Pi_{\mathcal{V}_+}^* P \Pi_{\mathcal{V}_+}$ positive definite and $P_{\mathcal{V}_-} := \Pi_{\mathcal{V}_-}^* P \Pi_{\mathcal{V}_-}$ negative definite.

5. PROOFS

In this section we briefly sketch the main ideas behind the proofs of the results.

Proof of Proposition 4.1
See, e.g. (Alpay and Rakowski, 1995) ■

Proof of Theorem 4.2
Statement 1:

($a \Rightarrow b$): Suppose $K = K_0 K_1$ with K_0 as indicated. Then $K_1 = J K_0^* J K$ is J-unitary and hence, by Proposition 4.1, K_0 and K_1 admit representations

$$K_0 = \left(\begin{array}{c|c} A_0 & -X_0 C_0^* J D_0 \\ \hline C_0 & D_0 \end{array}\right), \quad K_1 = \left(\begin{array}{c|c} A_1 & -X_1 C_1^* J D_1 \\ \hline C_1 & D_1 \end{array}\right) \tag{9}$$

where both X_0 and X_1 are non-singular. The product $K = K_0 K_1$ is then represented by

$$\begin{aligned} K &= \left(\begin{array}{cc|c} A_1 & 0 & -X_1 C_1^* J D_1 \\ -X_0 C_0^* J D_0 C_1 & A_0 & -X_0 C_0^* J D_0 D_1 \\ \hline D_0 C_1 & C_0 & D_0 D_1 \end{array}\right) \\ &= \left(\begin{array}{c|c} A & -P^{-1} C^* J D \\ \hline C & D \end{array}\right) \end{aligned}$$

so that, possibly after a suitable change of basis,

$$A = \begin{pmatrix} A_1 & 0 \\ -X_0 C_0^* J D_0 C_1 & A_0 \end{pmatrix}, \; C = \begin{pmatrix} D_0 C_1 & C_0 \end{pmatrix},$$

$D = D_0 D_1$ and P satisfies (3). Let $\mathcal{V} := \operatorname{span} \begin{pmatrix} 0 & I_k \end{pmatrix}^\top$. Then $\dim \mathcal{V} = k$, $\mathcal{V}$ is A-invariant and $\Pi_{\mathcal{V}}^* (3)\, \Pi_{\mathcal{V}}$ reads

$$A_0^* P_{\mathcal{V}} + P_{\mathcal{V}} A_0 + C_0^* J C_0 = 0 \tag{10}$$

where $P_{\mathcal{V}} := \Pi_{\mathcal{V}}^* P \Pi_{\mathcal{V}}$. Since K_0 is J-unitary, Proposition 4.1 yields that $P_{\mathcal{V}} = X_0^{-1}$, i.e., $P_{\mathcal{V}}$ is non-singular.

($b \Rightarrow c$): Let $\mathcal{V}$ be a k-dimensional A-invariant subspace. Define $A_{\mathcal{V}} = \Pi_{\mathcal{V}}^* A \Pi_{\mathcal{V}}$, $C_{\mathcal{V}} := C \Pi_{\mathcal{V}}$ and $P_{\mathcal{V}} := \Pi_{\mathcal{V}}^* P \Pi_{\mathcal{V}}$. Since P satisfies (3) and $P_{\mathcal{V}}$ is non-singular, it follows that

$$\begin{aligned} 0 &= \Pi_{\mathcal{V}}^* [A^* P + P A + C^* J C] \Pi_{\mathcal{V}} \\ &= A_{\mathcal{V}}^* P_{\mathcal{V}} + P_{\mathcal{V}} A_{\mathcal{V}} + C_{\mathcal{V}}^* J C_{\mathcal{V}} \\ &= P_{\mathcal{V}}^{-1} A_{\mathcal{V}}^* + A_{\mathcal{V}} P_{\mathcal{V}}^{-1} + P_{\mathcal{V}}^{-1} C_{\mathcal{V}}^* J C_{\mathcal{V}} P_{\mathcal{V}}^{-1}. \end{aligned}$$

Define $X := \Pi_{\mathcal{V}} P_{\mathcal{V}}^{-1} \Pi_{\mathcal{V}}^*$. Then $\operatorname{rank} X = \dim \mathcal{V} = k$ and $XPX = \Pi_{\mathcal{V}} P_{\mathcal{V}}^{-1} P_{\mathcal{V}} P_{\mathcal{V}}^{-1} \Pi_{\mathcal{V}}^* = \Pi_{\mathcal{V}} P_{\mathcal{V}}^{-1} \Pi_{\mathcal{V}}^* = X$. Pre- and post-multiplying the last displayed equation with $\Pi_{\mathcal{V}}$ and $\Pi_{\mathcal{V}}^*$, resp., and using the identity $A \Pi_{\mathcal{V}} = \Pi_{\mathcal{V}} A_{\mathcal{V}}$ yields that $0 = XA^* + AX + XC^* JCX$. This proves (7).

($c \Rightarrow b$): Let X be a rank k hermitian solution of (7). Then $\mathcal{V} := \operatorname{im} X$ has $\dim \mathcal{V} = k$ and since $AX = -X(A^* + C^* JCX)$ it follows that $A\mathcal{V} \subseteq \mathcal{V}$, i.e., $\mathcal{V}$ is A-invariant. Moreover, $XPX = X$ implies that $\Pi_{\mathcal{V}}^* X \Pi_{\mathcal{V}} P_{\mathcal{V}} \Pi_{\mathcal{V}}^* X \Pi_{\mathcal{V}} = \Pi_{\mathcal{V}}^* X \Pi_{\mathcal{V}}$ where $P_{\mathcal{V}}$ has dimension $k \times k$. Since $\operatorname{rank} \Pi_{\mathcal{V}}^* X \Pi_{\mathcal{V}} = \operatorname{rank} X = k$, we must have that $\operatorname{rank} P_{\mathcal{V}} = k$, i.e, $P_{\mathcal{V}}$ is non-singular.

($b \Rightarrow a$): In a suitable basis of the state space, K is represented by (5) with

$$A = \begin{pmatrix} A_1 & 0 \\ Z & A_0 \end{pmatrix}, \; C = \begin{pmatrix} C_1 & C_0 \end{pmatrix}$$

where A_0 has dimension $k \times k$ with $k = \dim \mathcal{V}$. The remainder of the proof is based on the (non-trivial) observation that it is not restrictive to assume that P is block diagonal. With this assumption, set $X_0 = \mathcal{P}_{\mathcal{V}}^{-1}$ and X_1 equal to the inverse of the $(1, 1)$ block of P. $P_{\mathcal{V}}$ then satisfies (10) which, by Proposition 4.1, yields that

$$K_0 = \left(\begin{array}{c|c} A_0 & -X_0 C_0^* J \\ \hline C_0 & I \end{array}\right)$$

is J-unitary. Moreover, the $(2, 1)$ block of (3) reads

$$P_{\mathcal{V}} Z + C_0^* J C_1 = 0.$$

Hence, $Z = -P_{\mathcal{V}}^{-1} C_0^* J C_1 = -X_0 C_0^* J C_1$ which implies that K is the cascade of the functions K_0 (with $D_0 = I$) and K_1 (with $D_1 = D$) as defined in (9).

Statement 2: apply Statement 1 to K^*. ■

Proof of Theorem 4.3
if: Infer from Theorem 4.2 that $K = K_0 K_1$ where K_0 and K_1 are represented by (9) with $X_0 = P_{\mathcal{V}}^{-1} > 0$ and $X_1 < 0$. Then K_0 is J-inner by Proposition 4.1, and one verifies that K_1^* is also J inner. Set $\overline{K} = K_0$ and $\underline{K} = K_1^*$.

only if: Immediate from statement 1 of Theorem 4.2. ■

Proof of Theorem 4.4
We only sketch the proof of statement 1, as the second statement is proven in a similar way. Suppose that W, P and K satisfy the hypothesis and suppose there exist a factorization $K = \underline{K}^* \overline{K}$ with both $\underline{K}$ and $\overline{K}$ J-inner. Apply Theorem 4.2 and Theorem 4.3 to infer explicit expressions for $\underline{K}$ and $\overline{K}$. In particular, infer that

$$\underline{K}^* = \left(\begin{array}{c|c} A & BJ \\ \hline -JB^* X_- & J \end{array}\right)$$

with $X_- \leq 0$ the minimum non-positive definite solution of $A^* X + XA + XBJB^* X = 0$, is a representation of a conjugate J-inner left divisor of K. Hence,

$$\begin{aligned} \underline{W} := W\underline{K} &= \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \left(\begin{array}{c|c} -A^* & X_- BJ \\ \hline JB^* & J \end{array}\right) \\ &= \left(\begin{array}{cc|c} -A^* & 0 & X_- BJ \\ BJB^* & A & BJ \\ \hline DJB^* & C & D \end{array}\right) \\ &= \ldots = \\ &= \left(\begin{array}{c|c} A + BJB^* X_- & B \\ \hline C + DJB^* X_- & D \end{array}\right) \end{aligned}$$

where we skipped a tedious but straightforward series of equivalence transformations. It is then straightforward to verify that $\underline{W}$ is J row stable with $(X_+ - X_-)^{-1}$ being a positive definite solution of the corresponding Lyapunov equation. Here, X_+ denotes the maximal non-negative definite solution of $A^* X + XA + XBJB^* X = 0$. ■

Proof of Theorem 4.5

Similar to the proof of Theorem 4.4 but for W^{-1} instead of W. Details are omitted here. ■

6. CONCLUSIONS

The conclusions of this paper can be summarized as follows.

(1) A partial ordering on the set of J spectral factors of a J spectral density has been proposed by introducing the notion of J stability for rational matrix valued functions. This leads to the investigation of the commutativity of the diagram in Section 3.

(2) The set of all left and right J-unitary divisors of a J-unitary rational matrix valued function has been characterized. The importance of this result lies in the understanding of the structure of J inner and conjugate J-inner factors of J-unitary functions.

(3) It has been shown how J stable and J anti-stable factors of a given rational function can be obtained. Similarly, factors whose inverses are J stable or J anti-stable have been characterized. Necessary and sufficient conditions for the existence of these factors have been derived.

(4) A generalization of the well known inner-outer factorization of stable rational functions to a J-inner J outer factorization can be inferred from the factorization results derived here.

(5) A conjecture has been formulated for the commutativity of the diagram of Section 3.

(6) Although we have restricted attention to square rational functions in this paper, extensions to the non-square case are possible.

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