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# NONLINEAR MODEL PREDICTIVE CONTROL WITH POLYTOPIC INVARIANT SETS

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Abstract: Ellipsoidal invariant sets have been widely used as target sets in MPC. These sets can be computed by constructing appropriate Linear Difference Inclusions together with additional constraints to ensure that the ellipsoid lies within a given Inclusion Polytope. The choice of this polytope has a significant effect on the size of the computed ellipsoid, but the optimal inclusion polytope cannot in general be computed systematically. This paper shows that use of polytopic invariant sets overcomes this difficulty, resulting in larger stabilizable sets without loss of closed-loop performance. In the interests of online efficiency, consideration is focused on interpolation-based NMPC. *Copyright ©2002 IFAC*

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## 1. INTRODUCTION

The size of an invariant target set has a strong influence on the region of attraction of the associated MPC law, and the maximization of target set volume and computation of the corresponding feedback law is therefore a matter of practical importance. For linear systems this volume maximization can be cast as a convex problem which is efficiently solvable via semidefinite programming (SDP) (Boyd *et al.*, 1994). It is possible to extend the SDP approach to Nonlinear MPC (NMPC), either by using a linear dynamic approximation together with Lipschitz bounds on the error of approximation (Michalska and Mayne, 1993; Chen and Allgöwer, 1998), or by using a Linear Difference Inclusion (LDI) (Liu, 1968; Boyd *et al.*, 1994) in place of the original nonlinear system, and thus determining an invariant ellipsoid for an uncertain linear time-varying system (Kothare *et al.*, 1996; Boyd *et al.*, 1994). In both cases the invariant ellipsoid must be contained within the region of state space on which the approximation is valid. To balance the requirements for (i) non-conservative approximation of the plant dynamics, and (ii) a large domain on which the approximation is valid, it is clearly necessary to allow the region of validity of approximation to be variable, and this leads to a nonconvex problem which can be difficult to solve systematically.

It was shown in Chen *et al.* (2001) that, for some examples, invariant ellipsoids of greater volume are obtained using LDIs than is possible through the use of Lipschitz bounds. However the approach is based on LDI approximation over a polytopic domain (which for convenience will be referred to here as an Inclusion Polytope), and this causes computational problems. The choice of inclusion polytope is instrumental in determining the size of the maximum volume invariant ellipsoid, yet the the optimal choice for the inclusion polytope is by no means obvious, leaving ad hoc heuristic procedure as the only avenue for design. Furthermore the approach is overly conservative since the inclusion polytope covers a larger region than the inscribed invariant ellipsoid, and this leads to significant conservativism in the linear time-varying system used to approximate the nonlinear plant dynamics.

In order to remove conservativism, in this paper we allow the difference between the region on which the LDI representation is valid and the corresponding invariant set to be arbitrarily small. This is achieved by using low-complexity polytopic invariant sets in place of ellipsoidal invariant sets. In conjunction with low-complexity inclusion polytopes, this choice of invariant set enables the maximization of volume and computation of a corresponding linear feedback law to be performed using linear programming. Furthermore it allows the inclusion polytope to be optimized systematically through the computation of a sequence of invariant polytopes which converges to a limit that coincides with a (locally) optimal inclusion polytope.

This paper shows that: (i) low-complexity polytopic invariant sets can have significantly larger volume than invariant ellipsoidal sets; (ii) the design of maximum volume low-complexity invariant polytopes can be performed systematically through the sequential offline solution of a number of simple linear programs; and (iii) such polytopic sets can be used effectively in NMPC. In the interest of deriving efficient NMPC algorithms suitable for fast-sampling applications, we adopt the univariate interpolation framework used in Cannon and Kouvaritakis (2001). The approach is applicable to input-affine nonlinear systems. We illustrate the benefits of polytopic invariant sets as well as their effectiveness in NMPC through numerical examples.

## 2. PROBLEM STATEMENT

The algorithms developed in this paper apply to nonlinear systems with input-affine models of the form:

$$x_{k+1} = f(x_k) + G(x_k)u_k, \quad y_k = Cx_k \tag{1}$$

with $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^l$, $y_k \in \mathbb{R}^m$, and $f(0) = 0$. It is assumed that the system is subject to input constraints:

$$u_k \in \mathcal{U}, \quad \mathcal{U} = \{u : \ |u| \le \overline{u}\} \tag{2}$$

where $\overline{u}$ is the vector of control limits. The extension to state constraints and/or non-symmetric constraints is straightforward and will not be considered here.

Let $\Pi^0$ denote a low-complexity polytope

$$\Pi^0 = \{x : \ \|V^0x\|_\infty \le 1\}$$

for full-rank $V^0 \in \mathbb{R}^{n\times n}$. We assume that, for any inclusion polytope $\Pi^0$ within the operating region, it is possible to construct a set $\{[A_i \ B_i], \ i = 1,\ldots,p\}$ so that the model (1) satisfies the inclusion condition:

$$f(x) + G(x)u \in \mathrm{Co}\{A_ix + B_iu, \ i = 1,\ldots,p\}, \quad \forall(x,u) \in \Pi^0 \times \mathcal{U} \tag{3}$$

where Co denotes the convex hull. Under this condition, all trajectories of the system (1) corresponding to input trajectories $\{u_k\}$ for which $(x_k,u_k) \in \Pi^0 \times \mathcal{U}$ for all $k \ge 0$ are also trajectories of the LDI (Liu, 1968):

$$x_{k+1} \in \mathrm{Co}\{A_ix_k + B_iu_k, \ i = 1,\ldots,p\}. \tag{4}$$

Thus the evolution of (1) is captured by an uncertain LTV model generated by considering all possible linear combinations of a given set of linear models. Procedures for computing suitable sets of linear models exist if, for example, $f$ and $G$ are continuously differentiable (Boyd *et al.*, 1994). Consider for example the class of bilinear systems described by (1) for

$$f(x) = Ax, \quad G(x) = \begin{bmatrix} G_1x & \cdots & G_lx \end{bmatrix} \tag{5}$$

with $G_i \in \mathbb{R}^{n\times n}$. Then the LDI (4) is valid for all $x \in \Pi^0$ if $A_i$ and $B_i$ are defined as:

$$A_i = A, \quad B_i = B + G(v_i^0), \quad i = 1,\ldots,2^n \tag{6}$$

where $\{v_i^0, \ i = 1,\ldots,2^n\}$ are the vertices of $\Pi^0$.

The objective is to develop an NMPC algorithm which achieves regulation while respecting constraints, and which optimizes performance as measured by the cost

$$J_\infty = \sum_{k=1}^{\infty} \|y_k\|_\infty . \tag{7}$$

The tracking problem with or without integral action can be routinely converted into an equivalent regulation problem and will not be considered separately.

The most common paradigm for NMPC designates the control inputs over the first $N$ samples of the prediction horizon as free variables and prescribes a fixed feedback law

$$u_{k+i|k} = \kappa(x_{k+i|k}), \quad i \ge N \tag{8}$$

over the remainder of an infinite prediction horizon. The inputs $\{u_{k+i|k}, \ k = 0,\ldots,N-1\}$ are used to minimize online a predicted cost incorporating a suitable terminal penalty term. In this optimization constraints (2) are handled explicitly over the first $N$ predicted samples, and implicitly thereafter through a terminal state constraint (Chen and Allgöwer, 1998; De Nicolao *et al.*, 1998). The terminal constraint forces the predicted state $x_{k+N|k}$ to lie in a target set, which, under the model (1) and fixed control law $u = \kappa(x)$, is feasible (in the sense that $u = \kappa(x)$ satisfies constraints for all $x$ in the target set) and invariant.

A popular choice of target set is the ellipsoid:

$$\mathcal{E} = \{x : \ x^TPx \le 1\}$$

where $P$ satisfies the requirements for feasibility and invariance (Michalska and Mayne, 1993; Chen and Allgöwer, 1998). By constraining $\kappa(x)$ to be linear:

$$\kappa(x) = Kx \tag{9}$$

and approximating (1) using the LDI (4), it is possible to compute $P$ simultaneously with $\kappa(x)$ and also obtain a bound on closed-loop performance (w.r.t. a 2-norm cost) (Kothare *et al.*, 1996; Chen *et al.*, 2001). Clearly the LDI approximation is valid everywhere in $\mathcal{E}$ if

$$\mathcal{E} \subset \Pi^0 . \tag{10}$$

It follows that, under this condition invariance and feasibility of $\mathcal{E}$ for each of the linear systems comprising the LDI ensures invariance and feasibility of $\mathcal{E}$ under the nonlinear dynamics of the closed-loop model.

Notwithstanding (10), the computation of the optimal $P$ and $\kappa$ with respect to the maximization of the volume of $\mathcal{E}$ and/or minimization of a worst-case bound on closed-loop performance can be cast as a convex problem which is efficiently solvable via semidefinite programming. Furthermore for fixed $V^0$, (10) can be easily incorporated into the SDP problem. However the choice of $\Pi^0$ has a significant effect on the maximum achievable volume for $\mathcal{E}$ and the optimal value for $V^0$ is by no means obvious. Allowing $V^0$ to be variable leads to a nonconvex problem, the solution of which has only been attempted on an ad hoc basis (Chen *et al.*, 2001). Moreover the constraint that $\mathcal{E}$ be invariant under the dynamics of the LDI associated with $\Pi^0$ is highly conservative. This is because the inclusion condition of (3) has to hold for all $x \in \Pi^0$, whereas for $x \in \mathcal{E}$ the model can be represented by an LDI $x_{k+1} \in \mathrm{Co}\{A_i'x_k + B_i'u_k, \ i = 1,\ldots,p'\}$ where $\mathrm{Co}\{[A_i' \ B_i'], \ i = 1,\ldots,p'\}$ is necessarily a subset of $\mathrm{Co}\{[A_i \ B_i], \ i = 1,\ldots,p\}$ since $\mathcal{E}$ is a subset of $\Pi^0$. Clearly the accuracy of the approximation of the plant dynamics on $\mathcal{E}$ given by the LDI associated with $\Pi^0$ improves as the difference $\Pi^0 - \mathcal{E}$ is reduced.

## 3. INVARIANT FEASIBLE POLYTOPES

A simple way to circumvent the difficulties caused by mismatch between the inclusion polytope and invariant set is to replace the ellipsoid $\mathcal{E}$ by a polytope $\Pi$:

$$\Pi = \{x : \|Vx\|_\infty \le \gamma\}, \quad \gamma > 0$$

with $V$ square and full-rank. Then invariance and feasibility under the closed-loop system (1,8) require

$$\|Vx_{k+1}\|_\infty - \|Vx_k\|_\infty \le 0 \quad \forall x_k \in \Pi \qquad (11)$$

$$|\kappa(x_k)| \le \overline{u} \quad \forall x_k \in \Pi. \qquad (12)$$

However the invariance condition (11) implies only Lyapunov stability, and to improve performance with respect to the cost of (7), (11) can be strengthened to

$$\|Vx_{k+1}\|_\infty - \|Vx_k\|_\infty \le -\|Cx_k\|_\infty \quad \forall x_k \in \Pi \qquad (13)$$

since we then have following obvious result.

*Lemma 1.* If (12) and (13) hold for some $\kappa$ and $V$, then the cost of (7) for the closed-loop system (1,8) has upper bound $\gamma$ for any initial condition in $\Pi$.

*Proof:* For any $x_0$ in $\Pi$, under (13) the state $x_k$ remains in $\Pi$ at all future times. The bound $\gamma$ is obtained by summing (13) over $k = 0, 1, \dots$ . □

*Remark 2.* Although (13) guarantees asymptotic convergence of the output: $\lim_{k\to\infty} y_k = 0$, it does not imply convergence of $x_k$ to 0 unless (1,8) is observable for all initial conditions in $\Pi$. To ensure convergence of the state itself it is possible to replace (13) by:

$$\|Vx_{k+1}\|_\infty - \|Vx_k\|_\infty \le -\varepsilon\|Vx_k\|_\infty \quad \forall x_k \in \Pi \qquad (14)$$

for $\varepsilon > 0$. The convergence conditions (13) and (14) imply invariance and therefore (with a slight abuse of nomenclature) define the required invariance property.

To invoke (12) and (13) or (14) to the closed-loop model (1,8), we use the LDI approximation (4). As in the ellipsoidal case, to facilitate simultaneous computation of a suitable feedback law we constrain $\kappa(x)$ to be linear (9). This choice of $\kappa(x)$ may significantly affect the volume of $\Pi$, however the approach described below can be applied to prespecified nonlinear feedback laws through the construction of suitable LDIs (see e.g. Bacic *et al.* (2001)).

The $2^n$ vertices $\{v_j\}$ of $\Pi$ are defined via

$$v_j = \gamma V^{-1} s_j, \quad j = 1, \dots, 2^n \qquad (15)$$

where $\{s_j,\ j = 1, \dots, 2^n\}$ is the set of all possible $n$-dimensional vectors whose elements are $\pm 1$. By ordering the indices so that $\{s_1, \dots, s_n\}$ are linearly independent, the the complete set of $2^n$ vertices can be parameterized in terms of the first $n$ alone. For convenience such a set of $n$ vertices will be referred to as primary vertices; all others can be expressed as linear combinations of $\{v_1, \dots, v_n\}$:

$$v_j = \begin{cases} [v_1 \ \cdots \ v_n][s_1 \ \cdots \ s_n]^{-1} s_j & j = n+1, \dots, 2^{n-1} \\ -v_{j-2^{n-1}} & j = 2^{n-1}+1, \dots, 2^n \end{cases} \qquad (16)$$

For fixed inclusion polytope $\Pi^0$, optimization of $\Pi, K$ over the primary vertices subject to invariance and feasibility can be formulated as stated below.

*Theorem 3.* With the constant $\alpha_\gamma$ set to zero, the following nonlinear program defines the maximum volume polytope $\Pi \subseteq \Pi^0$ which is invariant and feasible for (4) under linear feedback $u = Kx$.

$$\min_{\substack{\gamma \\ v_j, w_j,\ j=1,\dots,n}} \ -\log\det\big([v_1 \ \cdots \ v_n]\big) + \alpha_\gamma \gamma \qquad (17)$$

subject to the following inequality constraints invoked for $i = 1, \dots, p,\ j = 1, \dots, 2^{n-1}$:

$$\|V(A_i v_j + B_i w_j)\|_\infty \le \gamma - \|Cv_j\|_\infty \qquad (18)$$

$$|w_j| \le \overline{u} \qquad (19)$$

$$\|V^0 v_j\|_\infty \le 1 \qquad (20)$$

where, for $j = n+1, \dots, 2^{n-1}$, $v_j$ is defined by (16) and $w_j = [w_1 \ \cdots \ w_n][s_1 \ \cdots \ s_n]^{-1} s_j$. The optimal linear feedback gain can be recovered the from the solution for $\{v_j, w_j,\ j = 1, \dots, n\}$ via

$$K = [w_1 \ \cdots \ w_n][v_1 \ \cdots \ v_n]^{-1}. \qquad (21)$$

*Proof:* For the LDI (4), condition (13) requires that $\|V(A_i x + B_i u)\|_\infty - \|Vx\|_\infty \le -\|Cx\|_\infty$, $i = 1, \dots, p$, $\forall x \in \Pi$. These constraints can be expressed as linear inequalities in $x, u$ by introducing slack variables $\mu, \eta$: $|V(A_i x + B_i u)| \le (\eta - \mu)\mathbf{1}$, $|Vx| \le \eta\mathbf{1}$, $|Cx| \le \mu\mathbf{1}$ (where $\mathbf{1} = [1 \ \cdots \ 1]^T$ with dimension dependent on context). To ensure that a set of symmetric linear inequalities are satisfied for all $x$ in the polytope $\Pi$, it is necessary and sufficient to invoke them at the vertices $\{v_j,\ j = 1, \dots, 2^{n-1}\}$. This is done in (18) and (19) which invoke invariance and feasibility respectively. Moreover the volume of $\Pi$ is $2^n\gamma/\det(V)$, which from (15) is proportional to the determinant of the matrix of primary vertices $[v_1 \ \cdots \ v_n]$. □

*Remark 4.* The optimization of Theorem 3 is over not only the primary vertices of $\Pi$, but also over the controller gain $K$, which is defined in terms of $\{(v_j, w_j),\ j = 1, \dots n\}$ via $Kv_j = w_j$.

*Remark 5.* Theorem 3 employs the invariance condition of (13) but can instead be asserted for (14) by setting $\gamma = 1$ and replacing the RHS of (18) by $1 - \varepsilon$.

*Remark 6.* For $\alpha_\gamma = 0$, the objective (17) may lead to a large volume polytope at the cost of a large $\gamma$ which in turn implies a poor bound on performance. In this case a compromise between performance and size of polytope can be reached by choosing suitable $\alpha_\gamma > 0$.

The invariance constraint (18) is nonconvex since it involves the product of $V$, which is a linear function of $\gamma[v_1 \ \cdots \ v_n]^{-1}$, with terms that depend linearly on the variables $\{v_j\}$ and $\{w_j\}$. This constraint can be formulated as a bilinear constraint by introducing additional variables $q$ and invoking membership of $\Pi$ via

$$x \in \Pi \quad \text{iff} \quad x = \begin{bmatrix} v_1 & \cdots & v_{2^{n-1}} \end{bmatrix} q, \quad \mathbf{1}^T |q| \leq 1 \quad (22)$$

However, due to the bilinearity of this condition, the optimization of Theorem 3 remains a nonconvex problem which can raise considerable computational demands. This difficulty can be circumvented by breaking the optimization into a sequence of simpler problems, each of which is concerned with optimizing only a single vertex $v_k$, the remainder of the primary vertices being fixed at the values computed at the previous optimization, denoted below as $\{v_j^0\}$. Thus let $c_k$ denote the vector that is orthogonal to all except the $k$th primary vertex $v_k$, then the optimization of Theorem 3 for fixed $\Pi^0$ can be performed by solving a sequence of linear programs (LPs) as indicated below.

*Theorem 7.* With the constant $\alpha_\gamma$ set to zero, the maximum volume polytope $\Pi \subseteq \Pi^0$ which is invariant and feasible for (4) under linear feedback $u = Kx$ can be computed by solving the following LP successively for the individual vertices $v_k$, $k = 1, \ldots, n$.

$$\min_{\substack{\gamma, v_k \\ w_j,\ j=1,\ldots,n \\ Q_i, N_i,\ i=1,\ldots,p}} \quad c_k^T v_k + \alpha_\gamma \gamma \quad (23)$$

subject to the following linear constraints invoked for $i = 1, \ldots, p$, $j = 1, \ldots, 2^{n-1}$:

$$A_i[v_1 \ \cdots \ v_n] + B_i[w_1 \ \cdots \ w_n] = [v_1 \ \cdots \ v_n] N_i \quad (24)$$

$$N_i[s_1 \ \cdots \ s_n]^{-1}[s_1 \ \cdots \ s_{2^{n-1}}] = [s_1 \ \cdots \ s_n]^{-1}[s_1 \ \cdots \ s_{2^{n-1}}] Q_i \quad (25)$$

$$\mathbf{1}^T |Q_i e_j| \leq (\gamma - \|Cv_j\|_\infty)\mathbf{1}^T \quad (26)$$

$$|w_j| \leq \overline{u} \quad (27)$$

$$\|V^0 v_j\|_\infty \leq 1 \quad (28)$$

($e_j$ denotes the $j$th column of the identity matrix), where the $k$th rows of $N_i$ for $i = 1, \ldots, p$ and the primary vertices $v_j$ for $j \neq k$ are constants defined by

$$e_k^T N_i = e_k^T N_i^0 \quad (29)$$

$$v_j = v_j^0, \quad (30)$$

and where $\{v_j^0,\ j = 1, \ldots, n\}$ and $\{N_i^0,\ i = 1, \ldots, p\}$ are defined by the previously computed LP solution.

*Proof:* Given the definition of $c_k$, it is easy to show that $c_k^T v_k$ is proportional to the determinant of the matrix of primary vertices, $\det([v_1 \ \cdots \ v_n])$, and hence proportional to the volume of $\Pi$. Constraints (24–26) are equivalent to the the invariance condition (18) re-written using the membership condition of (22), while (27–28) are identical to the feasibility and inclusion constraints of (19–20). Note also that the constraints of (24–28) are linear in the optimization variables $\gamma$, $v_k$, $w_k$, and $\{Q_i, N_i,\ i = 1, \ldots, p\}$. For (25–28) this is obvious, whereas for (24) it is due to (29–30) which fix the $k$th rows of $N_i$, $i = 1, \ldots, p$ and all except the $k$th column of $[v_1 \ \cdots \ v_n]$ to constant values. The solution of the previously computed LP in the implied sequence of optimization problems is necessarily feasible for the current LP, since only constraints (29–30) are changed when a new LP is defined, and these are by definition feasible for the previously computed solution. Therefore the maximization of volume through (23) with $\alpha_\gamma = 0$ leads to a sequence of solutions corresponding to feasible invariant polytopes $\Pi$ of monotonically increasing volume. It follows that the sequence of LP solutions is guaranteed to converge to a (possibly local) solution of the nonlinear program (17–20). □

*Remark 8.* The optimal feedback gain is recoverable from the solution for $\{v_j, w_j,\ j = 1, \ldots, n\}$ via (21).

*Remark 9.* Theorem 7 is stated with reference to the invariance condition of (13) but can be asserted for that of (14) instead by setting $\gamma = 1$ and replacing the RHS of (26) by $1 - \varepsilon$.

*Remark 10.* To ensure that the relative weighting of $\gamma$ and $\det([v_1 \ \cdots \ v_n])$ is the same for different LPs in the implied sequence, $c_k$ should be defined through Laplace's expansion in terms of cofactors of $[v_1^0 \ \cdots \ v_n^0]$ whenever $\alpha_\gamma > 0$ is employed in the objective (17).

Although the optimization of (17–20) and its realization as a sequence of LPs in Theorem 7 is based on an LDI representation of the plant model, it also ensures that feasibility (12) and invariance (13) are satisfied for the actual nonlinear dynamics since $\Pi \subseteq \Pi^0$ is invoked via (20). However, to optimize $\Pi$ and $K$ for the nonlinear model it is necessary to allow the inclusion polytope $\Pi^0$ to be variable. The following theorem shows that $\Pi^0$ can be optimized systematically by solving a sequence of optimizations of form (17–20) (or (23–28)) in which, at the $i$th iteration, a new inclusion polytope $\Pi_{i+1}^0$ is defined to be a scaled version of a previously computed feasible invariant polyope $\Pi_i$.

*Theorem 11.* Let $\Pi_i = \{x : \ \|V_i x\|_\infty \leq \gamma_i\} \subseteq \Pi_i^0$ be invariant and feasible under linear feedback $u = K_i x$ for the LDI associated with the inclusion polytope $\Pi_i^0$. Then there exists a scaling factor $\rho_i \in (0, 1]$ such that the optimization of Theorem 3 (or Theorem 7), with inclusion polytope $\Pi^0 = \Pi_{i+1}^0$ defined via

$$\Pi_{i+1}^0 = \{x : \ \rho_i \|V_i x\|_\infty / \gamma_i \leq 1\},$$

yields a new invariant and feasible polytope $\Pi_{i+1}$ of volume greater than or equal to $\Pi_i$.

*Proof:* Denote the LDI associated with $\Pi_i^0$ as $x_{k+1} \in \mathrm{Co}\{A_i x_k + B_i u_k,\ i = 1, \ldots, p\}$ and suppose that $\Pi_{i+1}^0$ is chosen to be equal to $\Pi_i$, i.e. $\rho_i = 1$. Then $\Pi_{i+1}^0 \subseteq \Pi_i^0$ since $\Pi_i \subseteq \Pi_i^0$, and it follows that the nonlinear model (1) is represented by an LDI $x_{k+1} \in \mathrm{Co}\{A_i' x_k + B_i' u_k,\ i = 1, \ldots, p'\}$ for all $x_k \in \Pi_{i+1}^0$, where $\mathrm{Co}\{[A_i' \ B_i'],\ i = 1, \ldots, p'\}$ is necessarily a subset of $\mathrm{Co}\{[A_i \ B_i],\ i = 1, \ldots, p\}$. Therefore $\Pi_{i+1} = \Pi_i$ will be a feasible, but possibly suboptimal, solution to the optimization of Theorem 3 (or Theorem 7) with $\Pi^0 = \Pi^0_{i+1}$. This implies that, by choosing scaling parameters $\rho_i \le 1$, solutions $\Pi_{i+1}$ will be obtained for which $\mathrm{vol}(\Pi_{i+1}) \ge \mathrm{vol}(\Pi_i)$. □

Theorem 11 provides the systematic means for optimizing (sequentially) the volume of an invariant feasible set. Key to this is the possibility of allowing the Inclusion Polytope $\Pi^0_{i+1}$ to become coincidental with the previously computed invariant/feasible polytope $\Pi_i$. Such a sequential approach clearly is not possible for the case of ellipsoidal sets which have to be contained within a given Inclusion Polytope.

## 4. NMPC ALGORITHM

Online computational demands can be a limiting factor in NMPC of systems with fast dynamics, such as electromechanical systems for which the sampling interval may need to be of the order of milliseconds. This can prohibit the use of long horizons and hence make the characterization of the degrees of freedom as future control moves computationally unviable. An effective alternative is to interpolate over a linear mix of predicted trajectories. The simplest form of interpolation involves a single interpolation variable, allowing the mix of two predicted trajectories: one that provides the guarantee of recursive feasibility (and hence stability) and another that steers the algorithm to the current optimal whenever feasible.

It is easy to show that the unconstrained optimal control law for the system (1) with respect to the cost (7) is given by

$$u_{\mathrm{opt}} = -\left[CG(x)\right]^{-1}CAx \quad \forall x \in \Omega_\delta \tag{31}$$

where $\Omega_\delta = \left\{x : \underline{\sigma}[CG(x)] \ge \delta\right\}$ with $\underline{\sigma}(\cdot)$ denoting the minimum singular value of $(\cdot)$, and $\delta$ being an arbitrarily small positive number. For such a choice of $\delta$, $\Omega_\delta$ can be avoided through the use of an arbitrarily small perturbation of the control law (31). Although optimal for (7), (31) does not ensure convergence to the origin or even boundedness of the closed-loop system state; both these problems become relevant in the case of nonminimum phase dynamics. Moreover (31) may violate system constraints.

To retain optimality whenever possible while ensuring closed-loop stability, predicted control laws can be generated by combining (31) with a "detuned" feedback law, $u_{\mathrm{det}} = \kappa(x)$, which stabilizes (1). Suitable $\kappa(x)$ and a corresponding feasible invariant polytopic set can be designed as described in section 3, and it will therefore be assumed here that $\kappa(x) = Kx$.

The implied interpolation gives predictions:

$$\begin{aligned} u_{k|k}(\lambda_k) &= \lambda_k K x_k - (1-\lambda_k)[CG(x_k)]^{-1}Cf(x_k) \\ x_{k+1|k} &= f(x_k) + G(x_k)u_{k|k}(\lambda_k) \end{aligned} \tag{32}$$

In the interest of optimality, $\lambda_k$ should be as small as possible subject to $u_{k|k}$ satisfying system constraints and under the further constraint that a feasible control move $u_{k+1|k}$ will exist at the next sampling instant. Under these conditions it is possible to establish a recursive guarantee of feasibility and hence stability, with the additional guarantee that the algorithm will converge to the unconstrained optimal feedback law.

*Algorithm 1.* At every time instant $k$ minimize $\lambda_k$ subject to the predictions of (32) satisfying (2) and
  *a.* the inequality of (13); or
  *b.* the inequality of (14);
implement the corresponding input prediction $u_{k|k}$.

Denote the polytopes defined through Theorem 3 or 7 using invariance condition (13) or (14) as $\Pi_a$ and $\Pi_b$ respectively. Then the closed-loop stability properties of Algorithms 1a,b are as given below.

*Theorem 12.* Algorithm 1a stabilizes the origin of (1) and, for all initial conditions in $\Pi_a$, steers the output to zero while ensuring that the state remains within $\Pi_a$. Algorithm 1b asymptotically stabilizes the origin of (1) with region of attraction containing $\Pi_b$.

*Proof:* From the definitions of $\Pi_a$, $\Pi_b$ and $K$, it follows that for any initial condition in $\Pi_a$ $[\Pi_b]$ there exist values of $\lambda_k \le 1$ for which Algorithm 1a [1b] is feasible. Furthermore, feasibility is ensured at the next sampling instant through condition (13) [(14)], which ensures that remains in $\Pi_a$ $[\Pi_b]$ at all future times. The sum of inequality (13) [(14)] over all future time instants implies that the infinite sum of $\|Cx_k\|_\infty$ $[\|Vx_k\|_\infty]$ will be bounded by $\gamma$ $[1/\varepsilon]$, thereby implying that the output [state] converges to the origin. □

*Remark 13.* Invariance condition (13) is less stringent than (14) and thus results in larger regions of attraction, and/or reduced closed-loop cost. However Algorithm 1b may be preferred in the case where non-zero equilibrium states exist in the kernel of $C$.

## 5. NUMERICAL EXAMPLES

*Example 1.* This example illustrates the advantages of feasible invariant polytopic sets over ellipsoidal sets in terms of volume enlargement of the relevant stabilizable sets. To allow direct comparison with previous work on the computation of terminal regions for NMPC, the model is adapted from a continuous-time bilinear system used in Chen and Allgöwer (1998) and Chen *et al.* (2001). An approximate discrete-time realization of this model, for a sampling interval of $T$ is given by:

$$x_{k+1} = \begin{bmatrix} 1 & T \\ T & 1 \end{bmatrix} x_k + T\left\{\mu \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (1-\mu)\begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix} x_k\right\} u_k$$

where $\mu = 0.9$, and $T = 0.01$ is chosen. As in the continuous-time case, the model is unstable at $(x,u) =$

$(0,0)$ and its linearization about the origin is uncontrollable but stabilizable. The system is subject to input and state constraints:

$$|u| \le 2, \qquad |x| \le [4\ 4]^T. \tag{33}$$

A feasible invariant ellipsoid $\mathcal{E} = \{x : x^T P x \le 1\}$ can be computed by maximizing vol($\mathcal{E}$) over $P$, $K$ and $\Pi^0$ (the inclusion polytope) using heuristic search to determine $\Pi^0$ in conjunction with SDP to compute $P, K$ for given $\Pi^0$, as prescribed in Chen *et al.* (2001). This leads to the ellipsoid shown in Fig. 1 along with the associated inclusion polytope $\Pi^0$ (dashed line). The values of $V^0$, $P$ and $K$ are as follows:

$$V^0 = \begin{bmatrix} 0.4545 & 0 \\ 0 & 0.5000 \end{bmatrix}, \quad P = \begin{bmatrix} 0.4643 & 0.3805 \\ 0.3805 & 0.5618 \end{bmatrix},$$
$$K = [-1.2628\ \ -1.4109] \quad \text{vol}(\mathcal{E}) = 9.2208$$

which is comparable to the maximum volume ellipsoid and corresponding linear feedback gain obtained in Chen *et al.* (2001), for which vol($\mathcal{E}$) = 9.0835. However, the approach of Theorems 7 and 11 with $\alpha_\gamma = 0$ converges to the maximum volume feasible invariant polytope of Fig. 1 in solid line, for which:

$$V^0 = \begin{bmatrix} 0.1638 & -0.3931 \\ 0.6066 & 0.6066 \end{bmatrix}, \quad V = V^0, \quad \gamma = 1,$$
$$K = [-1.2131\ \ -1.2128] \quad \text{vol}(\Pi) = 11.8405$$

i.e. vol($\Pi$) is 28% larger than vol($\mathcal{E}$). Furthermore the computation of $\Pi$ via Theorem 11 is easier to implement than the computation of $\mathcal{E}$, for which there is no systematic means of updating $\Pi^0$.

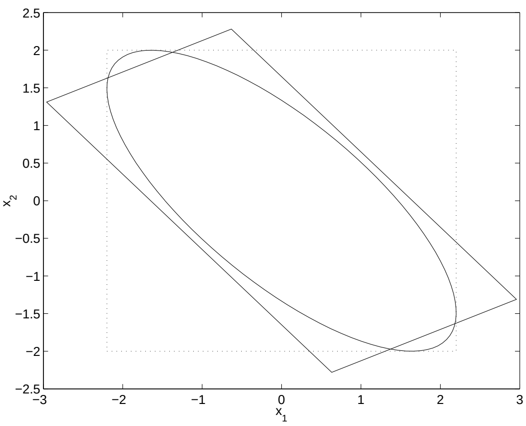


Fig. 1. Comparison of maximum area feasible invariant polytopic and ellipsoidal sets for Example 1.

*Example 2.* This example compares the closed-loop performance of interpolation NMPC based on ellipsoidal target sets and polytopic target sets. In Cannon and Kouvaritakis (2001), interpolation NMPC is applied to the bilinear plant model:

$$x_{k+1} = \begin{bmatrix} 0.28 & -0.78 \\ -0.78 & -0.59 \end{bmatrix} x_k + \left( \begin{bmatrix} 0.71 \\ 1.62 \end{bmatrix} + \begin{bmatrix} 0.34 & 0.36 \\ 0.41 & -0.65 \end{bmatrix} x_k \right) u_k$$
$$y_k = \begin{bmatrix} -0.69 & 0.20 \end{bmatrix} x_k$$

which is unstable and nonminimum phase at $(x,u) = (0,0)$, and has input constraint:

$$|u| \le 0.5.$$

Computing the maximum volume feasible polytopic set satisfying invariance condition (13) by setting $a_\gamma = 0$ results in $K = [0.1515\ 0.1514]$ and vol($\Pi$) = 11.5937. Maximizing the volume of a feasible invariant ellipsoid over $P$, $K$, and $V^0$ yields $K = [0.0291\ 0.2561]$ and vol($\mathcal{E}$) = 9.4634. However, despite $\Pi$ covering an area of state space 22% larger than $\mathcal{E}$, interpolation MPC based on the polytopic set $\Pi$ provides comparable closed-loop performance with that based on $\mathcal{E}$. Table 1 gives closed-loop costs averaged over 100 initial conditions equispaced around the boundary of $\mathcal{E}$ ($\mathcal{E}$ is contained entirely within $\Pi$). The $\ell_1$ cost is given by (7) whereas the $\ell_2$ cost refers to the index $\sum_{k=1}^{\infty} \|y_k\|_2^2$.

Table 1. Closed-loop costs of Interpolation MPC for Example 2

| | Volume | Average $\ell_1$ cost | Average $\ell_2$ cost |
|---|---|---|---|
| $\mathcal{E}$ | 9.4634 | 1.1443 | 0.4131 |
| $\Pi$ | 11.5937 | 1.1245 | 0.4160 |

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