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GENERAL SOLUTION FOR LINEARIZED STOCHASTIC ERROR PROPAGATION IN VEHICLE ODOMETRY

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Abstract

Abstract: Although odometry is nonlinear, it yields sufficiently to linearized analysis to produce a closed-form transition matrix and a symbolic general solution for both deterministic and stochastic error propagation. The implication is that vehicle odometry can be understood at a level of theoretical rigor that parallels the well-known Schuler oscillation of inertial navigation error propagation. Response to initial conditions is shown to be expressible in closed form and is path-independent. Response to input errors can be related to path functionals. The general linearized solution for stochastic error propagation for two typical cases of odometry is derived and applied to two example trajectories. Copyright © 2002 IFAC.

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1. INTRODUCTION

This paper addresses the problem of understanding the relationship between random error present in sensor indications in odometry, and the resultant error in computed vehicle pose. The word “understanding” must be emphasized because a numerical solution to the problem of computing the resultant error is trivial. The enhanced understanding of the general case that is enabled by symbolic solutions is the motivation for the present work.

1.1. Motivation

This work is motivated by a recurrent set of questions which arise when designing and constructing position estimation systems for mobile robots for which the answer always seems to require numerical solution. How good do the sensors need to be? What kind of localization error can be expected if this particular sensor is used? Why do some errors seem to cancel out on closed paths while others reverse when you drive backwards? What is the best way to calibrate the model of this sensor?

1.2. Prior Work

Analytical analysis of error propagation in mobile robot odometry seems to have been largely ignored in the literature with a few exceptions. Early work in (Wang 1988) concentrates on improving estimates for a single iteration of the estimation algorithm by incorporating knowledge of the geometry of the path followed between odometry updates. In (Borenstein and Feng 1995) a method is presented which permits the calibration of systematic errors which are observable on rectangular closed trajectories by solving geometric relationships. In (Chong and Kleeman 1997) a solution is obtained for non systematic error on constant curvature trajectories by solving a recurrence equation. This paper presents the general solution for linearized stochastic error propagation for any trajectory and any error model. Extensions to systematic error are immediate.

1.3. Problem Description

One of the most important distinctions in position estimation is that between triangulation and dead reckoning. The essential difference from a mathematics perspective is whether the available observations project onto the states of interest, or onto their derivatives. Odometry is a form of dead reckoning. In order to take the most direct advantage of available theory, a “forced dynamics” formulation of odometry will be used.

Odometry is modelled as a nonlinear dynamical system where the measurements, normally denoted $\underline{z}(t)$, are identified with the usual control inputs $\underline{u}(t)$.

The state vector $\underline{x}(t)$ and input vector $\underline{u}(t)$ are chosen to be:

$$\underline{x}(t) = \begin{bmatrix} x(t) & y(t) & \theta(t) \end{bmatrix}^T$$
$$\underline{u}(t) = \begin{bmatrix} V(t) & \omega(t) \end{bmatrix}^T$$

The associated odometry equations are those of the "integrated heading" case:

$$\dot{\underline{x}}(t) = f(\underline{x}(t), \underline{u}(t))$$
$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \\ \theta(t) \end{bmatrix} = \begin{bmatrix} V(t)\cos\theta(t) \\ V(t)\sin\theta(t) \\ \omega(t) \end{bmatrix} \tag{1}$$

The x axis is chosen as the heading datum. This situation is illustrated below:

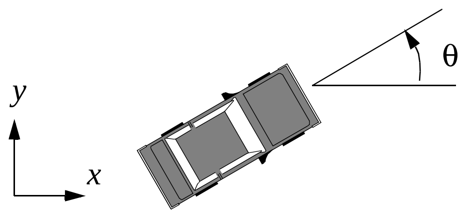

Fig 1. Coordinates for odometry.

Many alternative formulations of odometry are possible but the above formulation has two key properties. First, it is homogeneous in the inputs so the zero input response is zero. Second, it is in echelon form because any given equation depends only on the states below it in the order listed. As a result of the second property, the total solution is immediate and well-known:

$$\theta(t) = \theta(0) + \int_0^t \omega(t)dt$$
$$x(t) = x(0) + \int_0^t V(t)\cos\theta(t)dt \tag{2}$$
$$y(t) = y(0) + \int_0^t V(t)\sin\theta(t)dt$$

Since closed form solutions to integrals of general functions do not exist, the best that can be achieved is to eliminate the self reference of the state derivative to the state itself and write an explicit integral for the trajectory resulting from the input. This form above is as closed-form as a general solution can be.

This paper addresses the following problem. Let the inputs to the system be corrupted by additive errors as follows:

$$\omega'(t) = \omega(t) + \delta\omega(t)$$
$$V'(t) = V(t) + \delta V(t)$$

Using these input errors and the system dynamics, determine the behavior of the associated errors in the computed vehicle pose:

$$x'(t) = x(t) + \delta x(t)$$
$$y'(t) = y(t) + \delta y(t)$$
$$\theta'(t) = \theta(t) + \delta\theta(t)$$

2. LINEARIZED ERROR DYNAMICS

Perturbative techniques linearize nonlinear dynamical systems in order to study their first order behavior. As long as errors are small, the perturbative dynamics are a good approximation to the exact behavior, and for the present purposes, will be far more illuminating. Equation (1) is linearized as follows:

$$\delta\dot{\underline{x}}(t) = F\{\underline{x}(t), \underline{u}(t)\}\delta\underline{x}(t) \tag{3}$$
$$+ G\{\underline{x}(t), \underline{u}(t)\}\delta\underline{u}(t) + L\{\underline{x}(t), \underline{u}(t)\}\delta\underline{w}(t)$$

A second input vector $\underline{w}(t)$ has been introduced to differentiate systematic from random error sources. $\underline{w}(t)$ is simply the component of $\underline{u}(t)$ which is random. By superposition, systematic and random error sources can be treated independently.

The Jacobians may depend on the state and the input, and are evaluated on some reference trajectory:

$$F(t) = \left.\frac{\partial}{\partial \underline{x}}f\right|_{\underline{x}} \qquad G(t) = L(t) = \left.\frac{\partial}{\partial \underline{u}}f\right|_{\underline{x}}$$

Although equation (3) may still be nonlinear in the state and the input, it is linear in the perturbations. If the errors are random in nature, the state covariance and input spectral density matrices can be defined:

$$P = Exp(\delta\underline{x}(t)\delta\underline{x}(t)^T)$$
$$Q = Exp(\delta\underline{w}(t)\delta\underline{w}(\tau)^T)\delta(t-\tau)$$

To express random error propagation, the second moment or "covariance" of the error is considered. The linearized propagation of covariance is derived in several texts including (Gelb 1974) and the result is known as the linear variance equation:

$$\dot{P}(t) = F(t)P(t) + P(t)F(t)^T$$
$$+ L(t)Q(t)L(t)^T$$

2.1. Transition Matrix

The linearized differential equation (3) is of the form of a time varying linear system:

$$\dot{\underline{x}}(t) = F(t)\underline{x}(t) + G(t)\underline{u}(t) + L(t)\underline{w}(t) \tag{5}$$

While the transition matrix $\Phi(t, \tau)$ (which is tantamount to a solution) is known to exist for such systems, though it may not be easy to find. However, consider a matrix exponential of the following integral of the system Jacobian:

$$\Psi(t, \tau) = \exp\left(\int_\tau^t F(\zeta)d\zeta\right) \tag{6}$$

where the matrix exponential is defined as usual by the infinite matrix power series:

$$\exp(A) = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$$

When this exponential commutes (Brogan 1974) with the system dynamics matrix:

$$\Psi(t,\tau)F(t) = F(t)\Psi(t,\tau) \tag{7}$$

it is the transition matrix which solves the associated time-varying linear system.

This is so because the transition matrix, by (one) definition, satisfies the homogeneous system differential equation:

$$\frac{d}{dt}\Psi(t,\tau) = F(t)\Psi(t,\tau) \tag{8}$$

On substituting equation (6) into (8), the commutativity property becomes necessary to establish the equality. This property of "commutative dynamics" is the key to generating a total solution to the error propagation equations of odometry. Once a closed-form expression for the transition matrix is available, everything else follows from classical theory.

2.2. Matrix Exponential

It was noted earlier that the odometry system is in echelon form. Essentially, this means that the system Jacobian $F(t)$ is strictly upper triangular:

$$F = \{f_{ij} | f_{ij} = 0 \text{ when } (j \le i)\}$$

and since $\Psi(t,\tau)$ is composed entirely of definite integrals of $F(t)$, it is also strictly upper triangular. It can be shown that the nth power (and hence all subsequent powers) of an $n \times n$ strictly upper triangular matrix vanishes. This means that the matrix exponential can be easily written by summing the first few nonzero terms. Accordingly, closed-form expressions for the transition matrix become available if it also satisfies equation (7).

2.3. Solution for Commutative Dynamics

If $\Phi(t, t_0)$ is the transition matrix for the original deterministic system dynamics, then the well-known solution (Stengel 1994) is the matrix convolution integral:

$$P(t) = \Phi(t,t_0)P(t_0)\Phi^T(t,t_0) + \int_{t_0}^{t}\Phi(t,\tau)L(\tau)Q(\tau)L^T(\tau)\Phi^T(t,\tau)d\tau \tag{9}$$

The only unknown in this equation is the transition matrix. The (potentially nonsquare) input transition matrix can be defined as:

$$\tilde{\Phi}(t,\tau) = \Phi(t,\tau)L(\tau) = \Phi(t,\tau)G(\tau)$$

This matrix maps a given systematic or random error at time τ onto its net effect on the state error occurring later at time t. In effect, linearization for the purposes of studying error propagation amounts to treating errors occurring at different times independently of each other.

2.4. Error Propagation Behavior

The solution consists of a state (initial conditions) response and an input response. The state response is always path independent and hence it vanishes on any closed trajectory. The input response is the integral of a positive semidefinite matrix and hence all of its eigenvalues are nondecreasing.

3. APPLICATION TO ODOMETRY

This section will derive the error propagation equations for a few common forms of odometry.

3.1. Direct Heading Odometry

The term direct heading odometry will be used to refer to the case where a direct measurement of heading is available rather than its derivative. For example, a compass could be used to measure heading directly and a transmission encoder could be used to measure the linear velocity of the center of an axle of the vehicle.

The heading and error in heading are respectively equal at all times to the heading measurement and its error. Considering the heading to be an input, the state equations are:

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} V(t)\cos\theta(t) \\ V(t)\sin\theta(t) \end{bmatrix}$$

This system is clearly memoryless since the states do not appear on the right hand side. Perturbing it gives:

$$\frac{d}{dt}\begin{bmatrix} \delta x(t) \\ \delta y(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \delta x(t) \\ \delta y(t) \end{bmatrix} + \begin{bmatrix} \cos\theta(t) & -V(t)\sin\theta(t) \\ \sin\theta(t) & V(t)\cos\theta(t) \end{bmatrix}\begin{bmatrix} \delta V(t) \\ \delta\theta(t) \end{bmatrix}$$

The exponential of the integrated Jacobian is:

$$\Psi(t,\tau) = \exp\left(\int_\tau^t F(\zeta)d\zeta\right) = \exp[0] = I$$

This clearly satisfies equation (7), so it is the transition matrix. Substituting into the general solution in equation (9) gives:

$$P(t) = P(0) + \int_{t_0}^{t}\begin{bmatrix} c\theta & -Vs\theta \\ s\theta & Vc\theta \end{bmatrix}\begin{bmatrix} \sigma_{vv} & \sigma_{v\theta} \\ \sigma_{v\theta} & \sigma_{\theta\theta} \end{bmatrix}\begin{bmatrix} c\theta & -Vs\theta \\ s\theta & Vc\theta \end{bmatrix}^T d\tau \tag{10}$$

Equation (10) is the general linearized solution for the propagation of random error in 2D direct heading odometry for any trajectory and any error model.

3.2. Integrated Heading Odometry

In integrated heading odometry, an angular velocity indication is available which is integrated to get the heading. For example, a gyro could be used to measure heading rate and a transmission encoder, groundspeed radar, or fifth wheel encoder could be used to measure the linear velocity of the center of an axle of the vehicle. This is the case given in equation (1) repeated here for reference:

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \\ \theta(t) \end{bmatrix} = \begin{bmatrix} V(t)\cos\theta(t) \\ V(t)\sin\theta(t) \\ \omega(t) \end{bmatrix}$$

Also, define the notation for curvature with $\omega(t) = \kappa(t)V(t)$. Perturbing this, gives:

$$\frac{d}{dt}\begin{bmatrix} \delta x(t) \\ \delta y(t) \\ \delta\theta(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -V(t)\sin\theta(t) \\ 0 & 0 & V(t)\cos\theta(t) \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \delta x(t) \\ \delta y(t) \\ \delta\theta(t) \end{bmatrix} + \begin{bmatrix} \cos\theta(t) & 0 \\ \sin\theta(t) & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \delta V(t) \\ \delta\omega(t) \end{bmatrix}$$

Where the first matrix is the system Jacobian:

$$F(t) = \begin{bmatrix} 0 & 0 & -V(t)\sin\theta(t) \\ 0 & 0 & V(t)\cos\theta(t) \\ 0 & 0 & 0 \end{bmatrix}$$

The following expressions for the coordinates of the endpoint from the perspective of the point $[x((\tau), y(\tau))]$ are defined:

$$\Delta x(t, \tau) = [x(t) - x(\tau)]$$
$$\Delta y(t, \tau) = [y(t) - y(\tau)]$$

Next, the integrated system Jacobian is:

$$R(t, \tau) = \int_\tau^t F(\zeta)d\zeta = \begin{bmatrix} 0 & 0 & -\int_\tau^t V(\xi)\sin\theta(\xi)d\xi \\ 0 & 0 & \int_\tau^t V(\xi)\cos\theta(\xi)d\xi \\ 0 & 0 & 0 \end{bmatrix}$$

Since $R^2(t, \tau) = 0$, the exponential of the integrated system Jacobian is:

$$\Psi(t, \tau) = I + R(t, \tau) = \begin{bmatrix} 1 & 0 & -\Delta y(t, \tau) \\ 0 & 1 & \Delta x(t, \tau) \\ 0 & 0 & 1 \end{bmatrix}$$

Now, this matrix satisfies equation (7) because $\Psi F = F\Psi = F$. Therefore, this matrix is the transition matrix $\Phi(t, \tau) = \Psi(t, \tau)$.

Substituting into the general solution in equation (9) gives:

$$P(t) = IC_s + \int_0^t \begin{bmatrix} c\theta & -\Delta y(t, \tau) \\ s\theta & \Delta x(t, \tau) \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \sigma_{vv} & \sigma_{v\omega} \\ \sigma_{v\omega} & \sigma_{\omega\omega} \end{bmatrix}\begin{bmatrix} c\theta & -\Delta y(t, \tau) \\ s\theta & \Delta x(t, \tau) \\ 0 & 1 \end{bmatrix}^T d\tau \qquad (11)$$

Where the initial state response IC_s is given by:

$$\begin{bmatrix} 1 & 0 & -y(t) \\ 0 & 1 & x(t) \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \sigma_{xx}(0) & \sigma_{xy}(0) & \sigma_{x\theta}(0) \\ \sigma_{xy}(0) & \sigma_{yy}(0) & \sigma_{y\theta}(0) \\ \sigma_{x\theta}(0) & \sigma_{y\theta}(0) & \sigma_{\theta\theta}(0) \end{bmatrix}\begin{bmatrix} 1 & 0 & -y(t) \\ 0 & 1 & x(t) \\ 0 & 0 & 1 \end{bmatrix}^T$$

Equation (11) is the general linearized solution for the propagation of random error in 2D integrated heading odometry for any trajectory and any error model.

3.3. Intuitive Interpretation

It is clear now that the solution could have been written by inspection. The initial conditions affect the endpoint error in a predictable manner and the remaining terms amount to an addition of the effects felt at the endpoint at time t of the errors occurring at each time τ between the start and end as illustrated in figure 2:

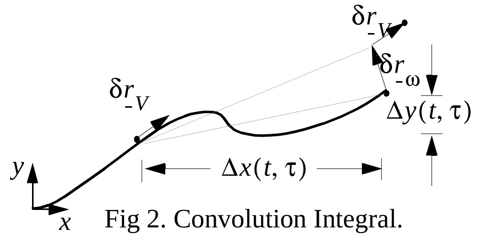

Fig 2. Convolution Integral.

The matrix relating input systematic errors occurring at time τ to their effect at time t is:

$$d\begin{bmatrix} \delta x(t) \\ \delta y(t) \\ \delta\theta(t) \end{bmatrix} = \begin{bmatrix} c\theta & -\Delta y(t, \tau) \\ s\theta & \Delta x(t, \tau) \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \delta V(\tau) \\ \delta\omega(\tau) \end{bmatrix} d\tau$$

Therefore, the covariance relationship is:

$$d\begin{bmatrix} \sigma_{xx}(t) & \sigma_{xy}(t) & \sigma_{x\theta}(t) \\ \sigma_{xy}(t) & \sigma_{yy}(t) & \sigma_{y\theta}(t) \\ \sigma_{x\theta}(t) & \sigma_{y\theta}(t) & \sigma_{\theta\theta}(t) \end{bmatrix} = \begin{bmatrix} c\theta & -\Delta y(t, \tau) \\ s\theta & \Delta x(t, \tau) \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \sigma_{vv} & \sigma_{v\omega} \\ \sigma_{v\omega} & \sigma_{\omega\omega} \end{bmatrix}\begin{bmatrix} c\theta & -\Delta y(t, \tau) \\ s\theta & \Delta x(t, \tau) \\ 0 & 1 \end{bmatrix}^T d\tau$$

These expressions are exactly what equation (11) is integrating. Linearization amounts to treating all errors as if they were independent in the sense that the endpoint is not changed to reflect the result of previous errors as the integral proceeds forward through time.

4. ERROR MODELS

The general results are functions of the reference trajectory and the error models. Specific trajectories and error models will be assumed here in order to get specific results.

4.1. Specific Error Models

For direct heading, a “motion dependent” random walk (where variance grows linearly with distance) will be assumed. A constant spectral probability density for the compass leads to a time dependent random walk contribution to the position coordinates. For integrated heading, a motion dependent random walk encoder variance will be assumed as well as a constant gyro bias stability. For differential heading, two potentially different motion dependent random walk variances will be assumed. These assumptions are summarized in the following table.

Table 1. Error Sources

Odometry Class	Error Sources
Direct Heading	$\sigma_{vv} = \alpha_{ss}\lvert V\rvert$
	$\sigma_{\theta\theta} = const$
Integrated Heading	$\sigma_{vv} = \alpha_{ss}\lvert V\rvert$
	$\sigma_{\omega\omega} = \sigma_{gg}$
	$\sigma_{v\theta} = 0$

The absolute value signs appear to keep variance positive regardless of the direction of motion.

5. SOLUTIONS ON PARTICULAR TRAJECTORIES

Using the above assumed errors, error propagation is completely determined by the trajectory followed. This section gives closed-form propagation equations for linear and constant curvature trajectories.

5.1. Straight Trajectory

A linear trajectory, starting at the origin, parallel to the x axis is defined by the following inputs:

$$\omega(t) = 0 \qquad V(t) = arbitrary$$

and the associated solution to equation (1):

$$x(t) = s(t) \qquad y(t) = 0 \qquad \theta(t) = 0 \qquad (12)$$

The solution for direct heading becomes:

$$P(t) = P(0) + \alpha_{ss}\begin{bmatrix} s & 0 \\ 0 & 0 \end{bmatrix} + \lvert V\rvert\sigma_{\theta\theta}\begin{bmatrix} 0 & 0 \\ 0 & s \end{bmatrix} \qquad (13)$$

Covariance remains diagonal. Alongtrack variance increases linearly with distance. Crosstrack variance also increases linearly under the assumption that velocity is constant.

The solution for integrated heading becomes:

$$P(t) = IC_s + \alpha_{ss}\begin{bmatrix} s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \sigma_{gg}\begin{bmatrix} 0 & 0 & 0 \\ 0 & s^2t/3 & st/2 \\ 0 & st/2 & t \end{bmatrix}$$

Constant velocity was assumed for some elements of the second term. Heading variance is linear in time as was intended. Heading covariance with crosstrack is linear in distance and time. Alongtrack variance is (to first order) linear in distance rather than time whereas crosstrack variance is cubic in time (or distance for constant velocity).

5.2. Constant Curvature Trajectory

A constant curvature (arc) trajectory, starting at the origin, initially parallel to the x axis is defined by the following inputs:

$$\omega(t) = \kappa(t)V(t) = V(t)/R$$

$$V(t) = arbitrary$$

and the associated solution to equation (1):

$$\theta(s) = \kappa s$$

$$x(s) = R\sin(\kappa s)$$

$$y(s) = R[1 - \cos(\kappa s)]$$

The solution for direct heading becomes:

$$P(t) = P(0) + \frac{\alpha_{ss}R}{2}\begin{bmatrix} \theta + \frac{s2\theta}{2} & s^2\theta \\ s^2\theta & \theta - \frac{s2\theta}{2} \end{bmatrix} + \lvert V\rvert\sigma_{\theta\theta}\begin{bmatrix} \theta - \frac{s2\theta}{2} & -s^2\theta \\ -s^2\theta & \theta + \frac{s2\theta}{2} \end{bmatrix} \qquad (14)$$

The overall behavior is a sum of linearly increasing, first harmonic, and second harmonic terms. A constant probability ellipse will steadily increase in size while rotating twice per orbit of the original trajectory. The integrated heading moment matrices for arc trajectories are defined in Table 2.

Table 2. Integrated Heading Moment Matrices

$$M_{11} = \begin{bmatrix} \theta + \frac{s2\theta}{2} & s^2\theta & 0 \\ s^2\theta & \theta - \frac{s2\theta}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad M_{12} = \begin{bmatrix} -R\left(\theta + \frac{s2\theta}{2}\right) & \frac{1}{2}R(s\theta^2 - \{c\theta - 1\}^2) & s\theta \\ \frac{1}{2}R(s\theta^2 - \{c\theta - 1\}^2) & R\left(-\theta + 2s\theta - \frac{s2\theta}{2}\right) & (1 - c\theta) \\ s\theta & (1 - c\theta) & 0 \end{bmatrix}$$

$$M_{22} = \begin{bmatrix} R\left[\theta\left(1 + \frac{c2\theta}{2}\right) - \frac{3}{2}\left(\frac{s2\theta}{2}\right)\right] & \left[\theta\left(\frac{s2\theta}{2}\right) + \frac{3}{2}\left(\frac{c2\theta}{2}\right) - c\theta + \frac{1}{4}\right] & -[s\theta - \theta c\theta] \\ R\left[\theta\left(\frac{s2\theta}{2}\right) + \frac{3}{2}\left(\frac{c2\theta}{2}\right) - c\theta + \frac{1}{4}\right] & R\left[\theta\left(1 - \frac{c2\theta}{2}\right) + \frac{3}{2}\left(\frac{s2\theta}{2}\right) - 2s\theta\right] & [\theta s\theta + c\theta - 1] \\ -[s\theta - \theta c\theta] & [\theta s\theta + c\theta - 1] & \theta/R \end{bmatrix}$$

The solution for integrated heading then becomes:

$$P(t) = IC_s + \frac{\alpha_{ss}R}{2}M_{11}(t) + \frac{\sigma_{gg}R}{\omega}M_{22}(t) \qquad (15)$$

Constant velocity was assumed. Heading variance $\sigma_{\theta\theta}$ increases linearly with time as was intended. The covariances of translation with heading $\sigma_{y\theta}$ and $\sigma_{x\theta}$ include a pure oscillation plus another oscillation at the fundamental frequency whose amplitude increases linearly with heading, distance, or time. Translational covariance σ_{xy} includes pure oscillations at the fundamental and second harmonic frequencies. One term is a second harmonic oscillation whose amplitude grows linearly with heading, distance, or time. The translational variances σ_{yy} and σ_{xx} include terms of similar character to σ_{xy} (but there is no fundamental term) but they also include a pure linear term in distance, heading, or time which does not oscillate. Both the gyro and the encoder variances cause these linear terms in the translational variances.

5.3. Simulation

For illustration purposes, the time evolution of the elements of the state covariance in the constant curvature integrated heading case are provided in the following figure.

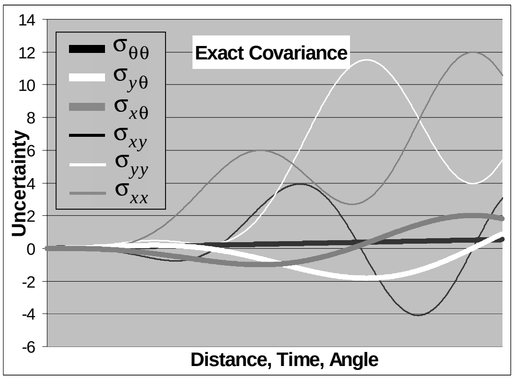

Fig 3. Integrated Heading Odometry Error on Arc Trajectory.

6. CONCLUSIONS

It has not been the intention here to demonstrate that the covariance propagation models are valid approximations of reality. The entire community invokes this assumption routinely in using optimal estimation techniques based on the Kalman filter (Smith and Cheesemen 1986). The purpose here has been to show that the equations which are commonly solved numerically can be solved in closed form.

It has been shown that closed-form solutions exist for sensor error propagation in commonly encountered forms of odometry. This property, combined with the property that the system Jacobian is upper triangular, means that a total solution to odometry error propagation exists in symbolic form. As a result, the entire theory of linear systems can be applied to odometry in symbolic form provided the error models and the trajectory are expressed symbolically.

Resultant state estimation error is always a combination of the state response and the input response. The former is always path independent and vanishes on any closed trajectory. The latter can be reduced to expressions involving path functionals.

In addition to their pedagogic value, the results of this paper can be used in design to determine acceptable levels of sensor error. They can also be used to calibrate Kalman filter sensor uncertainty models in a very principled manner.

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