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PDF-SHAPING CONTROL DESIGN

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Abstract: Stochastic control design techniques found in the chemical process control literature generally focus on the reduction of state or output variance to improve process operation. Most of these techniques address the problem of minimizing a quadratic cost function subject to linear process dynamics. In these cases, a linear feedback control law results and the disturbance, state and output PDF's are Gaussian. In this paper, the problem of control design for nonlinear stochastic processes and nonsymmetric, nonquadratic cost functions is addressed. A control design technique is presented for first-order, discrete-time processes with a single input and additive Gaussian white noise.

Keywords: Non-Gaussian processes, Nonlinear systems, Feedback control methods, Stationarity, Stochastic control, Discrete-time systems

1. MOTIVATIONS

The problem of regulatory control design for stochastic processes is typically considered from the perspectives of stability and optimality with respect to a cost function. Existing design techniques are mainly concerned with linear processes and quadratic cost functions. In this paper, a more general class of problems is reviewed and the main barrier impeding their solution is identified. As a means of both investigating and circumventing this barrier, the focus of the design problem is changed to consider the relationship between the closed-loop (CL) process dynamics and the probability density function (PDF) of the CL process. This results in a PDF-shaping control design technique that is useful both for control design and as an intermediate step in understanding the original cost function based problem.

The common approach to regulatory control design for discrete-time stochastic processes is to attempt to minimize an expectation type cost function:

$$J = E[\hat{\ell}(x_t, u_t)] \tag{1}$$

subject to the given process dynamics:

$$x_{t+1} = \hat{f}(x_t, u_t, w_t) \tag{2}$$

where $E[\cdot]$ is the expectation operator, $\hat{\ell}(\cdot,\cdot)$ is a cost functional, x_t is the process state, u_t is the manipulated variable and w_t is a random disturbance with probability density function $p_w(w_t)$. A general solution to this problem is not available; however, a number of specific solutions have been achieved. If the process (2) is linear and the PDF, $p_w(w_t)$, is Gaussian, then solutions for the set of cost functionals, $\hat{\ell}(x_t, u_t) = x_t{}^2 + \lambda u_t{}^2$, $\lambda \geq 0$, are available (Harris, 1985). In this case the control law is linear and the CL process generates a Gaussian PDF. Solutions for problems involving linear processes and with a general, nonsymmetric, nonquadratic cost functional $\ell(x_t)$ have also been developed (Harris, 1992). Again, the resulting control law is linear, so that the PDF of the CL process may be parameterized by mean and variance. A very different case is addressed in (Lee, 1990) where results for the general minimum variance cost functional $\hat{\ell}(x_t, u_t) = x_t{}^2 + \lambda u_t{}^2$ are extended to a class of nonlinear processes. In this case, the control design procedure does not cover general nonsymmetric, nonquadratic cost functionals, nor does it consider the PDF of the CL process.

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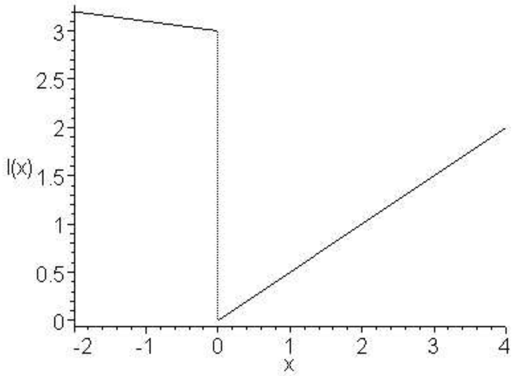

Fig. 1. An illustration of the type of nonsymmetric, nonquadratic cost functional that may arise from a quality specification.

While these three different control design techniques provide solutions to a number of related control problems, they do not solve the important case of design for a linear or nonlinear process with a nonsymmetric cost functional which includes the cost of control energy, $\hat{\ell}(x_t, u_t) = \ell(x_t) + \lambda u_t{}^2$. This type of problem is relevant to a number of process industries, in particular when there is a quality specification to be met. Often a product must be of at least a certain quality. If the process output is below the minimum acceptable level, it may require reprocessing, be sold for a lower profit or even discarded outright. In all of these cases there is a large cost associated with the off-spec product. Conversely, there may be significant material, energy or time costs associated with a product that consistently exceeds minimum specifications. A representation of the type of cost functional that may be associated with this type of process is given in Figure 1. Of course, for many processes there may be additional costs associated with the manipulated variable(s). In this work, these additional costs are taken to be quadratic.

In the development of the design techniques described by (Harris, 1985), (Harris, 1992) and (Lee, 1990), direct minimization of the cost function with respect to the manipulated variable, u_t, is performed. Unfortunately when working with the more general cost function:

$$J = E[\ell(x_t) + \lambda u_t{}^2] \tag{3}$$

direct minimization is not possible. Since u_t is given by a feedback control law and is thus a function of the process state, it cannot be extracted from the expectation operator. Additionally, it is unlikely that the CL process will be linear. In justifying this statement there are two cases to consider. First, if the open-loop process exhibits nonlinearities, there is no reason to expect that the optimum control law will cause the CL process to behave linearly. Second, even if the open-loop process is linear, there is no reason to expect the optimum control law (with respect to (3)) to be linear. If a CL process is nonlinear its stationary PDF will, in general, be non-Gaussian. Not only does this complicate evaluation and manipulation of the cost function (3) but it introduces an auxiliary equation to relate the dynamics of the CL process to the PDF of the CL process. For this reason, research presented in this paper focusses on developing a control design technique to shape the PDF of the CL process.

2. BACKGROUND

In this work, the type of process being considered is restricted to first-order, discrete-time nonlinear stochastic processes with additive zero mean Gaussian white noise and a single manipulated variable:

$$x_{t+1} = f(x_t, u_t) + w_t \tag{4}$$

$$p_w(w_t) = N_{0,\sigma_w}(w_t) \tag{5}$$

The notation $N_{\mu,\sigma}(\cdot)$ refers to the Gaussian PDF parameterized by a mean of μ and a variance of σ^2:

$$N_{\mu,\sigma}(\cdot) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\cdot - \mu)^2}{2\sigma^2}\right) \tag{6}$$

Additionally, in developing the control design technique, only the static feedback control law,

$$u_t = k(x_t) \tag{7}$$

is considered. Substituting this into the process (4) yields:

$$\begin{aligned} x_{t+1} &= f(x_t, k(x_t)) + w_t \\ &= \tilde{f}(x_t) + w_t \end{aligned} \tag{8}$$

Throughout this paper, the closed-loop process feedback, $\tilde{f}(x_t)$, will be referred to by the term CL feedback.

2.1 *Expectation Cost Functions*

As seen in (1) and (3) this work considers expectation type cost functions. For use of these types of cost functions to be appropriate, it is necessary that a stationary PDF exist for the CL process. It is inappropriate to discuss non-stationary processes within this framework because the expected cost of operation would be forever changing and this would necessitate an adaptive control design. Assuming the CL process is stationary, and that the feedback control strategy (7) is to be implemented, the cost function (3) may be further defined as:

$$J = \int_{-\infty}^{\infty} \left(\ell(x_t) + \lambda\,(k(x_t))^2\right) p(x_t)\, dx_t \tag{9}$$

This is a difficult cost function to work with because the stationary PDF, $p(\cdot)$, is unknown when the form of the control law, $k(\cdot)$, is unknown.

2.2 *Stationary PDF's*

For the process described by (8), if the CL process is stationary, there is an associated PDF, $p(\cdot)$, determined by:

$$p(x_{t+1}) = \int_{-\infty}^{\infty} p_w\left(x_{t+1} - \tilde{f}(x_t)\right) p(x_t)dx_t \qquad (10)$$

A development of this equation is given in (Jazwinski, 1970). This is an integral equation for which there is no general analytical solution. To fully appreciate the barrier this presents to nonlinear control design, even for a relatively simple process such as (4), it is necessary to consider (5) and to expand the CL feedback:

$$\begin{aligned} &p(x_{t+1}) && (11) \\ &= \int_{-\infty}^{\infty} N_{0,\sigma_w}\left(x_{t+1} - f\left(x_t, k\left(x_t\right)\right)\right) p(x_t)dx_t \end{aligned}$$

It can be seen that control design to optimize the cost function (3) for the process (4) becomes the problem of finding a function $k(\cdot)$ to minimize (9) subject to the unknown PDF $p(\cdot)$ described by (11). In order to simplify this problem, a Gram-Charlier parameterization of $p(\cdot)$ will be introduced.

2.3 *Gram-Charlier PDF's*

For PDF's that are not too non-Gaussian, a Gram-Charlier (GC) PDF can provide a good approximation. Usually in the literature a specific GC approximation is given that works best for PDF's with zero mean and unit variance (Cramer, 1946) and (Stuart and Ord, 1994); however, when working with dynamic processes it is not always convenient to centre and normalize the variable in question. For that reason, a general GC approximation is presented that tailors the basis functions to work for PDF's with any mean and variance. The basis functions are:

$$\phi_i(x) = (-1)^i \frac{\sigma^i}{\sqrt{i!}} \frac{d^i N_{\mu,\sigma}(x)}{dx^i} \qquad (12)$$

where $N_{\mu,\sigma}(x)$ is as in (6). These GC basis functions are orthonormal with respect to the inner product:

$$\int_{-\infty}^{\infty} \frac{\phi_i(x)\,\phi_j(x)}{N_{\mu,\sigma}(x)} dx = \delta_{ij} \qquad (13)$$

Thus, when using a finite series of GC functions to approximate a PDF as:

$$\begin{aligned} p(x) &\approx c_0\phi_0(x) + c_1\phi_1(x) + \ldots + c_n\phi_n(x) \\ &\approx \mathbf{c}^T \boldsymbol{\phi}(x) \end{aligned} \qquad (14)$$

the coefficients, $\{c_i\}$, are found by:

$$c_i = \int_{-\infty}^{\infty} \frac{\phi_i(x)\,p(x)}{N_{\mu,\sigma}(x)} dx \qquad (15)$$

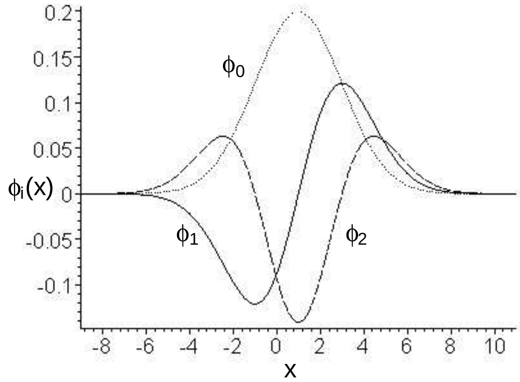

Fig. 2. The first three Gram-Charlier functions based on a mean of one and a variance of four.

It should be noted that each GC basis function is the product of a Hermite polynomial and the Gaussian PDF, i.e.,

$$\phi_i(x) = h_i(x)\,N_{\mu,\sigma}(x) \qquad (16)$$

Therefore the PDF approximation (14) may be rewritten as:

$$p(x) \approx \mathbf{c}^T \mathbf{h}(x)\,N_{\mu,\sigma}(x) \qquad (17)$$

and the coefficients found by:

$$c_i = \int_{-\infty}^{\infty} h_i(x)\,p(x)\,dx \qquad (18)$$

This corresponds to the expected value of $h_i(x)$ with respect to the PDF $p(x)$. Since $h_i(x)$ is a polynomial, the expectation looks very much like the calculation of a moment. For this reason the coefficients $\{c_i\}$ may be referred to as quasi-moments (Kuznetsov *et al.*, 1965).

The first three GC functions are:

$$\begin{aligned} \phi_0(x) &= N_{\mu,\sigma}(x) \\ \phi_1(x) &= \frac{x-\mu}{\sigma} N_{\mu,\sigma}(x) \\ \phi_2(x) &= \frac{x^2 - 2\mu x + \mu^2 - \sigma^2}{\sqrt{2}\sigma^2} N_{\mu,\sigma}(x) \end{aligned} \qquad (19)$$

These are plotted in Figure 2. Notice that for a distribution $p(x)$ with mean μ and variance σ:

$$\begin{aligned} c_1 &= \int_{-\infty}^{\infty} h_1(x)\,p(x)\,dx \\ &= \int_{-\infty}^{\infty} \frac{x}{\sigma} p(x)\,dx - \int_{-\infty}^{\infty} \frac{\mu}{\sigma} p(x)\,dx \\ &= \frac{\mu}{\sigma} - \frac{\mu}{\sigma} = 0 \end{aligned} \qquad (20)$$

c_2 also turns out to be zero, but not the higher quasi-moments. This is because the zeroth-order approximation for any distribution is just $\phi_0(x) = N_{\mu,\sigma}(x)$ which takes into account the mean and variance of $p(x)$. Thus, the quasi-mean, c_1, and quasi-variance, c_2, are both zero. For non-Gaussian PDF's, the higher quasi-moments will in general be non-zero.

Most often the fourth-order GC approximation is used:

$$p(x) = \left(1 + c_3 h_3(x) + c_4 h_4(x)\right) N_{\mu,\sigma}(x) \tag{21}$$

c_3 and c_4 can be shown to be directly related, respectively, to the skewness and the kurtosis of the distribution. This particular approximation is thus accurate to the fourth moment. (21) also illustrates that GC approximations of all orders are a product of a polynomial, $g(x)$, and a Gaussian PDF. This means any PDF that is a product of a polynomial and a Gaussian PDF may be exactly parameterized by a GC PDF.

The GC PDF's are used for approximation of unimodal PDF's, although it is possible to get multimodal behaviour. Additionally, there are limits on the values of the quasi-moments that may be chosen. For a GC approximation to be a true PDF, it must be non-negative everywhere. Therefore the quasi-moments must be such that the polynomial part of the approximation remains non-negative. Numerical techniques to ensure that this constraint is satisfied are given by (Jondeau and Rockinger, 1999).

3. PDF-SHAPING FEEDBACK CONTROL DESIGN

As stated above, this research is focussed on design of controllers to shape the PDF of the CL process. This is one step towards solution of the broader control design problem discussed in §1. PDF-shaping feedback control design refers to the solution of the integral equation (10). In this case, an approximate solution is made by parameterizing both the PDF and the CL feedback.

For design of PDF-shaping control laws, first the desired stationary PDF for the process must be selected. A good choice of PDF will be based on engineering concerns or process economics. In this work it is assumed that a GC PDF (17), is chosen. Control design then proceeds by substitution of the PDF into the stationary process equation (10):

$$\begin{aligned} &\mathbf{c}^T \mathbf{h}(x_{t+1}) N_{\mu,\sigma}(x_{t+1}) \\ &= \int_{-\infty}^{\infty} N_{0,\sigma_w}(x_{t+1} - \tilde{f}(x_t)) \mathbf{c}^T \mathbf{h}(x_t) N_{\mu,\sigma}(x_t) dx_t \end{aligned} \tag{22}$$

In an attempt to simplify this expression the orthogonality condition (13) is invoked:

$$\begin{aligned} c_0 &= 1 \\ 0 = c_1 &= \int_{-\infty}^{\infty} \frac{\tilde{f}(x_t) - \mu}{\sigma} \mathbf{c}^T \mathbf{h}(x_t) N_{\mu,\sigma}(x_t) dx_t \\ &= E\left[\frac{\tilde{f}(x_t) - \mu}{\sigma}\right] \\ 0 = c_2 &= E\left[\frac{\tilde{f}^2(x_t) - 2\mu \tilde{f}(x_t) + \sigma_w^2 - \sigma^2 + \mu^2}{\sqrt{2}\sigma^2}\right] \\ c_3 &= E\left[\frac{\tilde{f}^3(x_t) - 3\mu \tilde{f}^2(x_t) + 3\mu\left(\sigma^2 - \sigma_w{}^2\right)}{\sqrt{6}\sigma^3}\right. \\ &\qquad \left. \frac{+3\left(\sigma_w{}^2 - \sigma^2 + \mu^2\right) \tilde{f}(x_t) - \mu^3}{\sqrt{6}\sigma^3}\right] \\ &\vdots \end{aligned} \tag{23}$$

Even with the parameterization of the PDF and the benefit of the GC orthogonality condition, the equations (23) cannot be solved for $\tilde{f}(\cdot)$. Therefore, the CL feedback is parameterized with a series of basis functions:

$$\tilde{f}(x_t) = a_0 \theta_0(x_t) + a_1 \theta_1(x_t) + \ldots \tag{24}$$

Substituting this parameterization into equation (23) and performing the indicated integration yields:

$$\begin{aligned} c_0 &= 1 \\ 0 &= q_1(\mathbf{a}, \mathbf{c}, \sigma, \mu) \\ 0 &= q_2(\mathbf{a}, \mathbf{c}, \sigma, \mu) \\ c_3 &= q_3(\mathbf{a}, \mathbf{c}, \sigma, \mu) \\ c_4 &= q_4(\mathbf{a}, \mathbf{c}, \sigma, \mu) \\ &\vdots \end{aligned} \tag{25}$$

where each $q_i(\mathbf{a}, \mathbf{c}, \sigma, \mu)$ is a different function of the parameters. These equations may then be solved for the CL feedback parameters, **a**.

If this method is to successfully produce the parameters, there are a few key requirements. First, the basis functions for the CL feedback parameterization must be such that analytical solutions exist for the integrals in (23). For this reason, a series of polynomials may be a good choice as polynomials are integrable against the Gaussian PDF. Second, once the integration has been performed, it must be possible to find real-valued solutions to the equations (25). This implies that there must be at least as many parameters in the CL feedback parameterization as in the PDF parameterization. Further, the possibility of multiple solutions to equation (25) exists and so some auxiliary conditions are required to render a unique solution. Since this approximation is for a simple first-order process, which of the solutions is correct may be ascertained by inspection.

Once the CL feedback has been approximated, the feedback control law must be found by the back-substitution:

$$f(x_t, u_t) = \mathbf{a}^T \boldsymbol{\theta}(x) \tag{26}$$

3.1 *Accuracy of the Approximation*

This control design technique is approximate for the following reason. When a GC PDF of n^{th}-order is formed, the mean, variance and the first $n-2$ higher quasi-moments are selected; however, this implies that all remaining quasi-moments are zero. When the set of equations (25) is developed to design the CL feedback, only the selected GC parameters are considered. Therefore, while the design may satisfy the mean, variance and the chosen higher quasi-moments, there is no guarantee that the resultant PDF will be exactly as desired. Fortunately, the quasi-moments are somewhat like coefficients in a Taylor series; as they get higher in order, they must be very large to have a large impact on the PDF. For this reason, some post-design analysis to check the higher quasi-moments as well as simulation studies is recommended.

Alternatively there are two ways of reducing the likelihood of the higher quasi-moments causing difficulties. One way is to make a careful choice of the CL feedback parameterization. Ideally, a parameterization would be selected to create a one-to-one correspondence between CL feedback parameters and the quasi-moments. However, no such set of basis functions is currently known. A simpler way of eliminating the higher quasi-moments is to include more terms in the CL feedback parameterization. This allows for (25) to be augmented with additional equations to force some higher quasi-moments to zero.

3.2 *Stability of the Closed-Loop Process*

No control design technique may be considered complete without some comments about the stability or ergodicity of the resulting CL process. Following a theorem in (Tong, 1990), if the deterministic part of the CL process (8) can be shown to be globally stable then, with some additional technical conditions, the process can be shown to be ergodic. At this time it cannot be shown that this technique produces CL processes meeting the requirements of the theorem; however, there are reasons to believe that this technique ensures stable processes, at least locally.

Since the design goal is a stationary PDF, an indirect goal of creating a stable process is built into the technique. A process cannot be stationary if it is not stable. Similarly, as part of the PDF specification, the variance that the CL process is to display is a specified finite value. Again, a process cannot both be unstable and display a finite variance. Therefore to achieve a finite variance, stability must also be achieved.

As a last resort, the stability of the deterministic portion of the process may be evaluated with a Lyapunov analysis. Heuristically, if a large region of stability (large relative to the variances of the process and noise PDF's) exists, the process should display a local type of ergodicity.

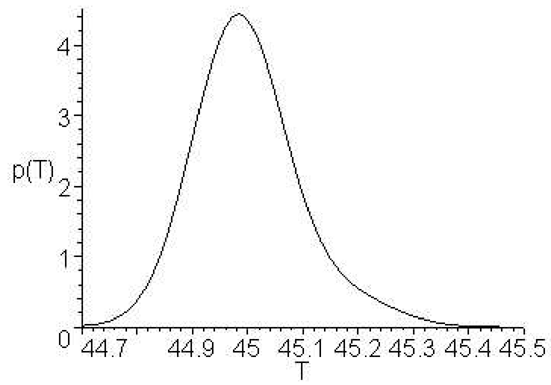

Fig. 3. The desired GC PDF with $\mu = 45$, $\sigma = 0.1$, $c_3 = 0.20$ and $c_4 = 0.15$.

4. ILLUSTRATIVE EXAMPLE

Consider liquid flowing through a well-mixed, heated tank. The heater is designed to supply heat at a constant rate of $Q = 120.0\,kW$ but due to a number of problems the actual rate of heat delivery is scattered about the set rate with a Gaussian PDF (standard deviation of $9.4\,kW$). For reasons relating to downstream operations, the temperature of the liquid at the tank outlet, T_t, is required to have the stationary PDF:

$$p(T_t) = \big(1 + 0.20 h_3(T_t) + 0.15 h_4(T_t)\big) N_{45,0.1}(T_t) \tag{27}$$

A plot of this PDF is given in Figure 3. To achieve this goal, the flow rate at the outlet, F_t, is to be manipulated in response to outlet temperature measurements. A well-tuned feedforward control strategy keeps inlet flow identical to outlet flow so that the volume of fluid in the tank remains constant at $V = 0.5\,m^3$. The inlet temperature of the fluid is constant at $T_{in} = 15\,^\circ C$ and the heat capacity and density of the fluid remain constant at $c_p = 1.5\,kJ/kg^\circ C$ and $\rho = 1250\,kg/m^3$ respectively. A discrete-time dynamic energy balance, for a time step of $\Delta t = 5\,s$, may be developed as follows. The heat in the tank at the next time step, H_{t+1}, will be equal to the heat currently in the tank, H_t, plus the sum of any additional heat additions and losses. As is standard:

$$H_t = V \rho c_p (T_t - T^*) \tag{28}$$

where T^* is a reference temperature. Heat is introduced by the heater ($H_Q = Q_t \Delta t$), and the inlet and outlet streams (H_{in} and H_{out}) account for the addition and loss of heat, respectively. Thus an overall energy balance is written:

$$H_{t+1} = H_t + H_Q + H_{in} - H_{out} \tag{29}$$

For the purposes of this example model, the heat transfer for each stream during the $5s$ control interval is taken to be:

$$H_{stream} = F_t \rho c_p (T_{stream} - T^*) \Delta t \tag{30}$$

By applying the given assumptions and rearranging the energy balance, the following first-order process for outlet stream temperature is derived:

$$T_{t+1} = T_t + \frac{F_t}{V}(T_{in} - T_t)\Delta t + \frac{Q_t \Delta t}{V \rho c_p} \tag{31}$$

Substituting in the parameter values and splitting Q into a constant and a zero mean Gaussian random variable gives:

$$T_{t+1} = T_t + F_t(150 - 10T_t) + 0.64 + w_t \tag{32}$$

where $p_w(w_t) = N_{0,0.05}(w_t)$.

As discussed in §3, it is not possible to find an exact control law to achieve the desired PDF (27). Instead, the CL feedback parameterization $\tilde{f}(T_t) = a_0 + a_1 T_t + a_2 T_t^2 + a_3 T_t^3$ is used. Taking advantage of equation (23) yields $\{a_0 = 9.181 \times 10^4,\ a_1 = -6.100 \times 10^3,\ a_2 = 1.351 \times 10^2,\ a_3 = -9.979 \times 10^{-1}\}$. Making use of equation (26) gives the feedback control law:

$$F_t = \frac{(a_0 - 0.64) + (a_1 - 1)T_t + a_2 T_t^2 + a_3 T_t^3}{150 - 10T_t} \tag{33}$$

An elementary Lyapunov-type stability analysis, based on the Lyapunov function $V = (T_t - 45)^2$, reveals that the deterministic portion of the CL process is locally asymptotically stable around $T_t = 45\,°C$. The region of stability is at least $[43.9\,°C, 46.6\,°C]$. This region of stability should be compared to the disturbances that may be encountered during operation. The controller is stable within ± 10 standard deviations of the mean temperature, equivalent to ± 20 times the standard deviation of the process noise.

Dynamic simulations of the process (32) under this control strategy confirm that the realized PDF matches the design goal. A histogram based on 10000 time steps of simulation data is given in Figure 4. The histogram takes a shape similar to that of Figure 3. In particular, the tail behaviour of simulation histogram shows the desired positive skewness.

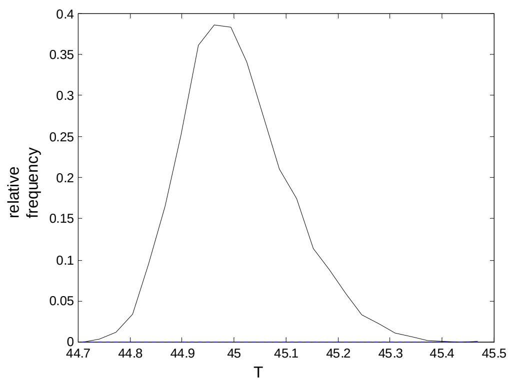

Fig. 4. A histogram based on simulation of the example process over 10000 time steps.

5. CONCLUSIONS

One problem of optimal control design for nonlinear stochastic processes has been carefully reviewed and the main barrier towards its solution identified. An approach to PDF-shaping feedback control design has been introduced as one step towards solving the more general regulatory control problem. The Gram-Charlier PDF parameterization has been proposed as a useful tool for control engineering. The GC parameterization is used to develop a PDF-shaping control design technique. The technique approximately solves the integral equation describing the stationary PDF of a closed-loop discrete-time dynamic process. The proposed technique allows for control design to cause the stationary PDF of the closed-loop process to be approximately a designer-selected GC PDF. The main idea of the technique is to develop the relationship between the control law parameters and the quasi-moments of the PDF. A short example demonstrates the mechanics of the technique. Additionally, analysis from the example suggests controlled processes designed with this technique have a good domain of local stability.

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