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MIMO POSITIVE RESPONSE SYSTEMS UNDER OUTPUT INEQUALITY CONSTRAINTS

Guy Bornard, Costin Ene

Laboratoire d'Automatique de Grenoble, UMR CNRS 5528
ENSIEG-INPG, BP 46, 38402, St. Martin d'Hères, Cedex, France,
e-mail: Guy.Bornard@inpg.fr

Abstract: The purpose of this work is the study of the "response positiveness" properties of multi-variable non-minimum phase systems subject to output inequality constraints. In previous work the authors have introduced the notion of positive response systems and their behaviour was studied in the SISO case under output constraints. The present paper shows the MIMO case systems and discusses upon the conservation of the mentioned property linked to the number of active output constraints versus number of system input. It is shown by geometric approach that an over-controlled system has always a "positive response". *Copyright ©2002 IFAC*

Keywords: geometric approaches, predictive control, constraints, non-minimum phase systems

1. INTRODUCTION

Non-minimum phase systems raise special problems when the output is subject to constraints. In (Bornard and Ene, 2000) is was shown though that only a sub-multitude of these systems have an "inverse part" of the response that hardens the task of building a stabilizing control solution. This sub-class is called "non-positive response" or non-$\mathcal{PR}$-systems. The tough task of the control is to make the system output leave an active constraint in order to regain the admissible domain. Model Predictive Control (MPC) has emerged as a powerful and widely used control technique (Prett and Garcia, 1988), (Bitmead *et al.*, 1990). Inspired by them, some solutions were proposed in the first-mentioned reference.

The present paper treats the Multi-Input-Multi-Output (MIMO) case of time-invariant linear systems subject to output inequality type constraints. The approach is a geometrical one and great support is to be found in (Wonham, 1985) and (Commault and Dion, 1982). The principal theme of the present discussion is made upon the number of possible active output constraints versus the number of the system's input. The relationship of these numbers characterizes the system through the positive response property aspect. It is clear that one system cannot have more simultaneous active constraints than degree of freedom. The main purpose of this paper is to show than in an over-controlled case, any system inherits the "positiveness" property of its response.

The paper is organized as follows: In Section 2 a brief example introduces the problems we deal with. Model Predictive Control technique is used to build a stabilizing input solution. In Section 3 the property of "positive response" is recalled. Section 4 foreshadows the MIMO case. Possible simultaneous active constraints are observed, linked to the system's steady state space dimensions. The core of the paper is Section 5. It is shown that for over-controlled systems, the "positive response" property is generically characteristic. Conclusions and remarks end the paper.

2. RECEDING HORIZON CONSTRAINT CONTROL

The receding horizon control technique, or MPC, is exploited to find a stabilizing control solution for non-minimum systems. Special attention is paid to the cases when output constraints are active. The following example speaks for itself about the problem encountered.

Example 1. Consider the following two SISO systems with unitary steady state gain, which differ by their C matrix :

$$A = \begin{bmatrix} .8 & 0 \\ 0 & .6 \end{bmatrix}, B = \begin{bmatrix} .2 \\ .4 \end{bmatrix}, C = [1.5, -0.5] \quad (1)$$

$$A = \begin{bmatrix} .8 & 0 \\ 0 & .6 \end{bmatrix}, B = \begin{bmatrix} .2 \\ .4 \end{bmatrix}, \quad C = [3.0, -2.0] \quad (2)$$

The same protocol was applied to both systems. In the first part a free evolution of the output allows it to track a first order with delay model set-point. The second part leads to an active inequality type constraint. The third one drives the system back inside the admissible domain. One can notice that the first system output tracks well the set-point (figure 1), while the second systems needs a long time response when leaving the active constraint (figure 2). Although both systems (1) and (2) have unstable zeros, their different behaviour is linked to a "positive response" property introduced in (Bornard and Ene, 2000).

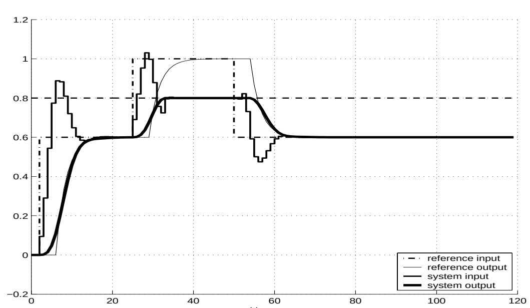

Fig. 1. Constraint on output: the standard case

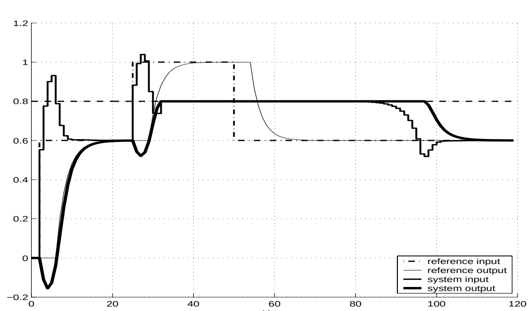

Fig. 2. Constraint on output: a difficult case

3. POSITIVE RESPONSE SYSTEMS

In this section we recall a first definition given in (Bornard and Ene, 2000):

Let us consider the linear time invariant system :

$$\begin{cases} x_{k+1} = Ax_k + Bu_k \\ y_k = Cx_k \end{cases} \quad (3)$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$ are real, time-invariant matrices. The system will be assumed throughout the paper as being controllable, observable and right invertible. Moreover, the mapping C restricted to the set of steady states is supposed injective (steady state controllability).

Definition 1. Assume that for any target steady state $\{x_{st}, u_{st}, y_{st}\}$, where $\{u_{st}\}$ and $\{y_{st}\}$ are the input and the output associated to the steady state $\{x_{st}\}$, there exists a discrete time N and a control sequence $u([0, N-1])$ such that the following conditions are fulfilled:

i) $x_{st} = x(N, u, 0)$
ii) $y_{st} = y(N, u, 0)$
iii) $y(k, u, 0) \cdot y_{st} \geq 0$ (elementwise) for $k > 0$

where $x(k, u, \xi_0)$ and $y(k, u, \xi_0) = Cx(k, u, \xi_0)$ are the solution of the system (3) at k for the input sequence u and the initialization $x_0 = \xi_0$.

Then the system (3) is said to have a positive response, or be a "positive response" system ($\mathcal{PR}$ system).

Systems not verifying this definition are said to be "non-positive response" systems (non-$\mathcal{PR}$ systems). In other words, the step response of a non-$\mathcal{PR}$ system exhibits an inverse part with respect to the input.

A more restrictive characterization of the systems can be made upon the length of the time response:

Definition 2. Let N be a positive integer. Assume that for any target steady state $\{x_{st}, u_{st}, y_{st}\}$, where $\{u_{st}\}$ and $\{y_{st}\}$ are the input and the output associated to the steady state $\{x_{st}\}$, there exists a control sequence $u([0, N-1])$ such that the following conditions are fulfilled:

i) $x_{st} = x(N, u, 0)$
ii) $y_{st} = y(N, u, 0)$
iii) $y(k, u, 0) \cdot y_{st} \geq 0$ (elementwise) for $k > 0$

where $x(k, u, \xi_0)$ and $y(k, u, \xi_0) = Cx(k, u, \xi_0)$ are the solution of the system (3) at k for the input sequence u and the initialization $x_0 = \xi_0$.

Then the system (3) is said to have a N-positive response, or be a "N-positive response" system ($N - \mathcal{PR}$ system).

This new definition prevents the numerical non-decidable cases. A very large N respecting definition 1 would not have a practical sense.

Therefore, for a practical test of the $\mathcal{PR}$-property one can chose the horizon length N close to the system time response. The test resumes to a linear programming problem that is made only once and off-line (Bornard and Ene, 2000).

Remark 1. The above definitions still hold in the continuous case, with the corresponding transformations. In (Bornard and Ene, 2000) it was shown that in the discrete case a negative first element of the step response implies a "non-positive response" system. In the continuous case this condition is applied to the derivative at origin of the step response. Furthermore, if a system (continuous or discrete) has a "non-positive response", then it is a non-minimal phase system.

4. MIMO CONSTRAINED SYSTEMS

This section details the previous properties to MIMO systems and prepares the background for some theorems and their proofs.

Definitions 1 and 2 as well as $\mathcal{PR}$ properties from (Bornard and Ene, 2000) are considered.

Let the MIMO system (3). Let all outputs y_i, $i = 1, \dots, p$, be bordered by inequality type constraints. The steady state space $\mathcal{X}_{st} = \{x | x = x_{st}\}$ is of dimension m - the degree of freedom introduced by the m inputs u_j, $j = 1, \dots, m$, and lays inside the admissible domain. Then it comes that for any steady state correspond at most m simultaneous active output constraints, except for singularity points. If exist, constraints on other output cannot be simultaneously active, and thus may be eliminated in the considered point. In the figure 3 a representative plot is showed for a 2-input, 4-output system. Although x_{st} is the intersection point for the constraints on y_1, y_2, y_3, the constraint on y_3 is active exclusively in this point. Let $\mathcal{V}$ be a vicinity of x_{st} of any radius $\varepsilon > 0$, $\mathcal{P}$ the interior of the polyhedron formed by the constraints (i.e. the admissible domain $\mathcal{D}$). For any steady state $x'_{st} \in \{\mathcal{V} \cap \mathcal{P}\} \setminus \{x_{st}\}$ the constraint on y_3 is passive.

In the sequel, by abuse of notation and for the purpose of our study, we shall consider that the MIMO systems have at most as many outputs, which may be simultaneously constrained, as inputs: $p \leq m$.

In Section 5 it will be shown that in the case of $m > p$, a system is generically $\mathcal{PR}$. As a consequence, the $m = p$ case gives sense to the $\mathcal{PR}$ problem. This is the case of all steady states that characterize the intersection of all p active output constraints.

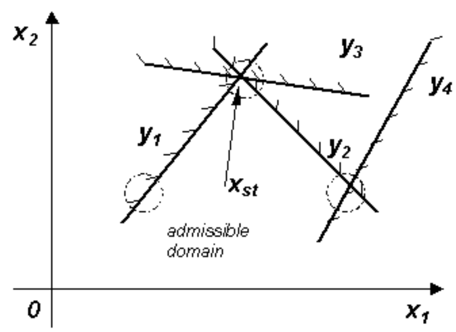

Fig. 3. Simultaneous active output constraints

5. THE OVER CONTROLLED CASE ($m > p$)

5.1 *Introduction*

Consider the MIMO system, controllable and right invertible :

$$\Sigma \begin{cases} x_{k+1} = Ax_k + Bu_k \\ y_k = Cx_k \end{cases} \tag{4}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$.

The main result of this section will be that generically, if $m > p$, then the system (4) is $\mathcal{PR}$. In order to state explicitly the theorem and to prove it, it will be useful to put the system under a so called reduced form through transformations to be defined here below.

5.2 *$\mathcal{PR}$-invariant transformations*

We start by assuming that the system (4) is under its controllability canonical form. More precisely, consider the following standard transformations :

- Change of coordinates $T : z = T^{-1}x$
- Feedback K and pre-compensation F :
  $u = Kx + Fv$, ($v \in \mathbb{R}^m$, F square, regular)

and let $\mathcal{T}^G(T, K, F)$ be the general element of the feedback group of transformations. It is obvious that :

Proposition 1. The $\mathcal{PR}$ property is invariant under the action of the feedback group of transformations $\mathcal{T}^G$ (see (Bornard and Ene, 2000) for proof). The same applies with respect to a permutation in the ordering of the output components.

As a consequence, it can be assumed without loss of generality that the pair (A, B) of the system (4) is already under the Brunovsky standard form. This means that A and B are of the form :

$$A = \begin{bmatrix} A_1 & \cdots & 0 \\ \vdots & \ddots & \\ 0 & & A_m \end{bmatrix} \quad A_i = \begin{bmatrix} 0 & \cdots & 0 \\ 1 & \ddots & \vdots \\ \vdots & \ddots & 0 \\ 0 & \cdots & 1 \end{bmatrix} \tag{5}$$

$$B = \begin{bmatrix} B_1 & \cdots & 0 \\ \vdots & \ddots & \\ 0 & & B_m \end{bmatrix} \quad B_i = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and the C matrix has no special form.

Moreover, for reasons that will be made clear further, the size of the blocks can be made equal by extending the non maximal blocks by extra states, the A_i and B_i blocks keeping the same structure. Each block has now a size $\nu = \max(\nu_i)$, and the system (A, B, C) is controllable but generally not observable. Remark that in the sequel, the output can be reset to the original order.

Let us define now the transformations $\mathcal{S}^{Yi}$ (advance on the output i) and $\mathcal{S}^{Uj}$ (delay on the input j). The are defined by their action on the matrix C. Let $C_i^j = \left[C_{i1}^j, \ldots, C_{i\nu}^j\right]$, the jth block of the ith line C_i of C.

Suppose that the

Assumption 1. $(\mathcal{HY}_i) : C_{i1}^j = 0$ for $j = 1, \ldots, m$,

is satisfied, then the action of $\mathcal{S}^{Yi}$is defined by :

$$\begin{aligned}
\mathcal{S}^{Yi}\left(C_i^j\right) &= \left[C_{i2}^j \cdots C_{i\nu}^j, 0\right], j = 1, \ldots, m \\
\mathcal{S}^{Yi}(C_i) &= \left[\mathcal{S}^{Yi}(C_i^1), \ldots, \mathcal{S}^{Yi}(C_i^m)\right] \\
\mathcal{S}^{Yi}(C) &= \begin{bmatrix} C_1 \\ \vdots \\ C_{i-1} \\ \mathcal{S}^{Yi}(C_i) \\ C_{i+1} \\ \vdots \\ C_p \end{bmatrix}, \\
\mathcal{S}^{Yi}(\Sigma) &= (A, B, \mathcal{S}^{Yi}(C)) \\
\mathcal{S}^{Yi}(y) &= \begin{bmatrix} y_1 \\ \vdots \\ y_{i-1} \\ z\, y_i \\ y_{i+1} \\ \vdots \\ y_p \end{bmatrix},
\end{aligned} \tag{6}$$

Remark that the hypothesis $\mathcal{HY}_i$ means that the relative degree r_i of Σ restricted to the output i is greater than one.

Similarly, if the

Assumption 2. $(\mathcal{HU}_j) : C_{i\nu}^j = 0$ for $i = 1, \ldots, p$,

is satisfied, then the action of $\mathcal{S}^{Uj}$ is defined by :

$$\begin{aligned}
&\mathcal{S}^{Uj}\left(C_i^j\right) = \left[0, C_{i1}^j \cdots C_{i\nu-1}^j\right], i = 1, \ldots, p \\
&\mathcal{S}^{Uj}(C^j) = \begin{bmatrix} \mathcal{S}^{Uj}(C_1^j) \\ \vdots \\ \mathcal{S}^{Uj}(C_p^j) \end{bmatrix}, \\
&\mathcal{S}^{Uj}(C) = \left[C^1, \ldots, C^{j-1}, \mathcal{S}^{Uj}(C^j), C^{j+1}, \ldots, C^m\right] \\
&\mathcal{S}^{Uj}(\Sigma) = (A, B, \mathcal{S}^{Uj}(C)) \\
&\mathcal{S}^{Uj}(u) = \left[u_1, \ldots, u_{j-1}, z^{-1}u_j, u_{j+1}, \ldots, u_m\right]
\end{aligned} \tag{7}$$

Remark that the hypothesis $\mathcal{HU}_j$ means that delaying the input j can be achieved without increasing the state dimension.

From the definitions, one has the equalities :

$$\begin{aligned}
y_{\mathcal{S}^{Yi}(\Sigma)}(u, 0) &= \mathcal{S}^{Yi}(y_\Sigma(u, 0)) \\
y_{\mathcal{S}^{Uj}(\Sigma)}(u, 0) &= y_\Sigma(\mathcal{S}^{Uj}(u), 0)
\end{aligned} \tag{8}$$

where $y_\Sigma(u, x_0)$ means the sequence solution of the system Σ to an input sequence u when initialized with x_0.

The following proposition can then be stated.

Proposition 2. Invariance of $\mathcal{PR}$ property under $\mathcal{S}^{Yi}$ and $\mathcal{S}^{Uj}$:
If $C_{i1}^j = 0$ for $j = 1, \ldots, m$
then $\mathcal{S}^{Yi}(\Sigma)\mathcal{PR} \Longrightarrow \Sigma\mathcal{PR}$
If $C_{i\nu}^j = 0$ for $i = 1, \ldots, p$
then $\mathcal{S}^{Uj}(\Sigma)\mathcal{PR} \Longrightarrow \Sigma\mathcal{PR}$

Proof : Assume that $\mathcal{S}^{Yi}(\Sigma)\mathcal{PR}$. There exists a $\mathcal{PR}$-admissible trajectory $(u, w = y_{\mathcal{S}^{Yi}(\Sigma)}(u, 0))$ of the system $\mathcal{S}^{Yi}(\Sigma)$. From equation (8), the corresponding trajectory for Σ is given by $(u, y = \mathcal{S}^{Yi^{-1}}(w))$ where $\mathcal{S}^{Yi^{-1}}$ is a causal operator. Since w is $\mathcal{PR}$-admissible, the same applies for y.
A similar argument stands for $\mathcal{S}^{Uj}$: There exists a $\mathcal{PR}$-admissible trajectory $(v, y_{\mathcal{S}_j^U(\Sigma)}(v, 0))$ of the system $\mathcal{S}^{Uj}(\Sigma)$, and from equation (8), one has $y_{\mathcal{S}^{Uj}(\Sigma)}(v, 0) = y_\Sigma(u, 0)$ for $u = \mathcal{S}^{Uj}(v)$, where $\mathcal{S}^{Uj}$ is a causal operator. □

The propositions (1) and (2) mean that a sequence of transformations can be applied to the system before looking for $\mathcal{PR}$-admissible trajectories.

5.3 *Reduction algorithm*

Consider again the system (4) with (A, B) under its Brunovky form, and with subsystems of equal dimension ν.

Initialization stage :
• For $i = 1, \ldots, p$ apply recursively $r_i - 1$ times the transformation $\mathcal{S}^{Yi}$, i.e. while the condition $\mathcal{HY}_i$ matches. The system obtained is given by :

$$
\begin{aligned}
\tilde{\Sigma}_0 &= (A, B, \tilde{C}_0) \\
&= (\mathcal{S}^{Y1})^{r_1} \circ \cdots \circ (\mathcal{S}^{Yp})^{r_p}(\Sigma) \\
&= \mathcal{S}_0^Y(\Sigma)
\end{aligned} \tag{9}
$$

• Let $\tilde{D}_0 = \tilde{C}_0 B$ and find a full rank square matrix F_0 such that :

$$
D_0 = \tilde{D}_0 F_0 = \left[\bar{D}_0, 0\right] \tag{10}
$$

where the submatrix $\bar{D}_0$ has s_0 columns and has a full rank (s_0).

• Define $\mathcal{T}$ by :

$$
\mathcal{T}(F) = \begin{bmatrix} F_{11} I_\nu & \cdots & F_{1m} I_\nu \\ \vdots & & \vdots \\ F_{m1} I_\nu & \cdots & F_{mm} I_\nu \end{bmatrix} \tag{11}
$$

and apply the transformation $S_0^T = (T_0, 0, F_0)$ to the system $\tilde{\Sigma}_0$, with $T_0 = \mathcal{T}(F_0)$. It is easy to verify that, by construction :

$$
\begin{aligned}
T_0^{-1} A T_0 &= A \\
T_0^{-1} B F_0 &= B
\end{aligned} \tag{12}
$$

Then the initialization stage results in :

$$
\begin{aligned}
C_0 &= \tilde{C}_0 T_0 \\
\Sigma_0 &= (A, B, C_0) \\
D_0 &= C_0 B = \tilde{C}_0 F_0 B
\end{aligned} \tag{13}
$$

For any trajectory ($y^0 = y_{\Sigma_0}(u^0, 0)$) of Σ_0, the corresponding trajectory ($y = y_\Sigma(u, 0)$) of Σ is given by the causal relations :

$$
\begin{aligned}
y &= (\mathcal{S}^{Y1})^{-r_1} \circ \cdots \circ (\mathcal{S}^{Yp})^{-r_p}(y^0) \\
&= (\mathcal{S}_0^Y)^{-1}(y^0) \\
u &= F_0 u^0 \\
&= \mathcal{S}_0^U(F_0 u^0)
\end{aligned} \tag{14}
$$

where $\mathcal{S}_0^U = \mathcal{I}m$

• If $s_0 = p$, then set $\kappa = 0$. The algorithm is completed.

stage l :

Assume that the $l-1$ first stages have provided the following entities :

$$
\begin{aligned}
\Sigma_{l-1} &= (A, B, C_{l-1}) \\
D_{l-1} &= C_{l-1} B \\
&= \left[\bar{D}_{l-1}, 0\right] \\
y &= (\mathcal{S}_{l-1}^Y)^{-1}(y^{l-1}) \\
u &= \mathcal{S}_{l-1}^U(F_{l-1} u^{l-1})
\end{aligned} \tag{15}
$$

where $\bar{D}_{l-1}$ has a full rank s_{l-1}. Remark that this is the case for $l = 1$.

• For $j = 1, \ldots, s_{l-1}$ apply the transformation $\mathcal{S}^{Uj}$ to Σ.

• For $i = 1, \ldots, p$ apply the transformation $\mathcal{S}^{Yi}$ to Σ.

By construction, the hypothesis $\mathcal{HU}_j$ and $\mathcal{HY}_i$ are satisfied when these transformations are applied. Remark that one has :

$$
\begin{aligned}
\tilde{C}_l &= \mathcal{S}^{Y1} \circ \cdots \circ \mathcal{S}^{Yp} \circ \mathcal{S}^{U1} \circ \cdots \circ \mathcal{S}^{Us_l}(C_{l-1}) \\
\tilde{C}_l B &= D_l = \left[\bar{D}_{l-1}, \breve{D}_l\right]
\end{aligned} \tag{16}
$$

• Find a full rank square matrix $\tilde{F}_l$ such that :

$$
D_l = \tilde{D}_l \tilde{F}_l = \left[\bar{D}_l, 0\right] \tag{17}
$$

where $\bar{D}_l$ has a full rank s_l. Clearly $s_l \geq s_{l-1}$.

Without loss of generality, the matrix $\tilde{F}_l$ can be taken of the form :

$$
\tilde{F}_l = \begin{bmatrix} \mathcal{I}_{s_{l-1}} & \tilde{F}_l \\ 0 & \bar{F}_l \end{bmatrix}
$$

providing that $D_l = \left[\bar{D}, \bar{\bar{D}}, 0\right]$, i.e. that the s_{l-1} first columns of D_{l-1} remain unchanged in D_l. As a consequence, $\tilde{F}_l$ and $\mathcal{S}_{l-1}^U$ commute.

Apply the transformation $(\mathcal{T}(\tilde{F}_l), 0, \tilde{F}_l)$ in order to obtain the system $\Sigma_l = (A, B, C_l = \tilde{C}_l \mathcal{T}(\tilde{F}_l))$, with the following properties :

$$
\begin{aligned}
C_l B &= D_l = \left[\bar{D}_l, 0\right], \\
&\quad (D_l = \left[\bar{D}, \bar{\bar{D}}, 0\right] \text{ full rank } s_l) \\
u_{l-1} &= \mathcal{S}^{U1} \circ \cdots \circ \mathcal{S}^{Us_l}(\tilde{F}_l\, u^l) \\
u &= \mathcal{S}_{l-1}^U \circ \mathcal{S}^{U1} \circ \cdots \circ \mathcal{S}^{Us_l}(F_{l-1}\, \tilde{F}_l\, u^l) \\
&= \mathcal{S}_l^U(F_l\, u^0) \\
y &= (\mathcal{S}^{Y1})^{-1} \circ \cdots \circ (\mathcal{S}^{Yp})^{-1}(y^{l-1}) \\
&= (\mathcal{S}^{Y1})^{-1} \circ \cdots \circ (\mathcal{S}^{Yp})^{-1} \circ (\mathcal{S}_{l-1}^Y)^{-1}(y^0) \\
&= (\mathcal{S}_l^Y)^{-1}(y^0)
\end{aligned} \tag{18}
$$

where :

$$
\begin{aligned}
\mathcal{S}_l^U &= \mathcal{S}_{l-1}^U \circ \mathcal{S}^{U1} \circ \cdots \circ \mathcal{S}^{Us_l} \\
\mathcal{S}_l^Y &= \mathcal{S}_{l-1}^Y \circ \mathcal{S}^{Y1} \circ \cdots \circ \mathcal{S}^{Yp}
\end{aligned} \tag{19}
$$

Remark that the transformations S^{Yi} commute between themselves. The same applies for the transformations S^{Uj}.

• If $s_l = p$, then set $\kappa = l$. The algorithm is completed.

5.4 *Properties of the reduced system*

By construction, the reduced system Σ_κ has the following properties :

- Σ_κ is $\mathcal{PR}$ if and only if Σ is $\mathcal{PR}$
- Σ_κ has a trivial structure at infinity (CB full rank). As a consequence, $V^* = \ker(C)$

The control making $\ker(C)$ invariant is given by $u = \begin{bmatrix} \bar{u} \\ \tilde{u} \end{bmatrix}$ where $\tilde{u}$ is free and $\bar{u}$ is given by :

$$
\begin{aligned}
\bar{u} &= -\bar{D}_\kappa^{-1} C_\kappa x \\
D_\kappa &= C_\kappa \bar{B} = \left[\bar{D}_\kappa, O\right] \\
B &= \begin{bmatrix} \bar{B} \\ \tilde{B} \end{bmatrix}
\end{aligned} \tag{20}
$$

The dynamics in the V^* subspace is governed by the system :

$$x_{k+1} = (A - \bar{B}D_\kappa^{-1}C_\kappa)x + \tilde{B}\tilde{u} \qquad (21)$$

The following hypothesis will be made :

Assumption 3. ($\mathcal{HC}$) : The system (Σ_κ) in (21) is controllable on $\ker(C_\kappa)$ with $\tilde{u}$ as input.

Remark that this condition is generic (it is equivalent to the controllability of the V^* subspace of the original system Σ, see (Wonham, 1985)).

5.5 $\mathcal{PR}$ *property for* $m > p$

Theorem 1. Assume that the system Σ of equation (4) with $m > p$ is controllable and right invertible, and that its reduced form Σ_κ matches the assumption $\mathcal{HC}$. Then Σ is $\mathcal{PR}$.

Proof : Consider any steady state x^s such that $y_i^s > 0$ for $i = 1, \ldots, p$ ($y^s = Cx^s$). Suppose that there exists an admissible trajectory of Σ_κ from some $x^q \in \ker(C_\kappa)$ to x^s. Because of the controllability of $\ker(C_\kappa)$, there exists a trajectory of Σ_κ from 0 to x^q in $\ker(C_\kappa)$, i.e. with $y = 0$. Then the concatenation of these two pieces constitutes an admissible trajectory from 0 to x^s for Σ_κ. The corresponding trajectory of Σ is thus also admissible, that would achieve the proof.

The state $x^q \in \ker(C_\kappa)$ and the trajectory from x^q to x^s have to be constructed. Consider the backward system :

$$\Sigma^B \begin{cases} \hat{x}_{k-1} = \hat{A}x_k + \hat{B}v_k \\ \hat{y}_k = C\hat{x}_k \end{cases} \qquad (22)$$

with :

$$\hat{A} = \begin{bmatrix} \hat{A}_1 & \cdots & 0 \\ \vdots & \ddots & \\ 0 & & \hat{A}_m \end{bmatrix} \quad \hat{A}_i = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ & & \ddots & 1 \\ 0 & & \cdots & 0 \end{bmatrix}$$
$$\hat{B} = \begin{bmatrix} \hat{B}_1 & \cdots & 0 \\ \vdots & \ddots & \\ 0 & & \hat{B}_m \end{bmatrix} \quad \hat{B}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \qquad (23)$$

where the dimension ν of the blocks and the matrix C remain the same as in (4).

For any trajectory $(\{\hat{x}_k, v_{k+1}, \hat{y}_k\},\ k = N-1, \ldots, 0)$ of Σ^B, consider the corresponding trajectory $(\{x_k, u_{k-1}, y_k\}, k = 1, \ldots, N)$ of Σ obtained by setting :

$$\begin{aligned} x_0 &= \hat{x}_0 \\ u_{j_{k-1}} &= \hat{x}_{j_{k-1}}\ j = 1, \ldots, m,\ k = N, \ldots, 1 \end{aligned}$$

It is not difficult to verify that the trajectories $(\{x_k, y_k\}, k = 1, \ldots, N)$ and $(\{\hat{x}_k, \hat{y}_k\}, k = 1, \ldots, N)$ coincide.

Consider, for some given N, the output sequence $y^B = (y_{N-1} = y^s, \ldots, y_{N-\nu+1} = y^s, y_{N-\nu} = 0)$. Inspecting the form of $\hat{A}$ and $\hat{B}$ remembering the reduction algorithm shows clearly that Σ being right invertible, the same applies for Σ^B. Then there exists an input sequence $v_N, \ldots, v_{N-\nu+1}$ such that the sequence y^B is the corresponding output of the backward system Σ^B initialized at $x_N = x^s$. Since $y_{N-\nu} = 0$, then $x_{N-\nu} \in \ker(C_\kappa)$ and the backward trajectory just defined is a trajectory of Σ_κ for some input sequence. The state $x_{N-\nu}$ is the desired x_q.

Taking N large enough, this point can be accessed by Σ_κ from $x = 0$ under the constraint $y = 0$. An admissible trajectory has been built up for the reduced system Σ_κ. Σ_κ is $\mathcal{PR}$, then from the propositions (1) and (2) applied recurrently, Σ is $\mathcal{PR}$. □

6. CONCLUSIONS

This paper studies the MIMO non-minimum phase systems in the presence of output constraints. The "positive response" property previously defined is put into question for these systems. It is shown through a geometric approach that in the over-controlled case ($m > p$: more inputs than simultaneous active output constraints) one deals with $\mathcal{PR}$ properties, and that the undecidable case is for $m = p$.

From a practical point of vue, "non-positive response" systems can be encountered in thermal processes, like steam generators of power plants.

7. REFERENCES

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