15th Triennial World Congress, Barcelona, Spain

STATE DEPENDENT DIFFERENTIAL RICCATI EQUATION FOR NONLINEAR ESTIMATION AND CONTROL

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Abstract: State-dependent Riccati equation (SDRE) methods for designing control algorithms and observers for nonlinear processes entail the use of algebraic Riccati equations. These methods have yielded a number of impressive results, however, they can be computationally quite intensive and thus far they have not yielded to those attempting to assess their stability. This paper explores an alternative, the use of state dependent *differential* Riccati equations and numerical integration to propagate their solutions forward in time. Stability is examined and examples illustrating the use of these methods are given.

Keywords: Nonlinear control, Nonlinear estimation, Riccati equation, Computational methods

1. INTRODUCTION

The design of practical observers and control algorithms in many applications entails dealing with nonlinear process dynamics. A number of methods have been developed for handling such processes. The state-dependent Riccati equation (SDRE) method, developed over the past several years, is one such method. The method is still not fully rigorous, but it has been empirically demonstrated in a number of applications. Moreover, attempts to find examples in which the method fails have generally been unsuccessful.

The method is based on "extended linearization" (Friedland, 1996; Williams et al., 1987) of the process dynamics. The dynamics are expressed by

$$\dot{x} = A(x)x + B(x)u \tag{1}$$

with observations given by

$$y = C(x)x \tag{2}$$

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The matrices $A(x), B(x),$ and $C(x)$ in (1) and (2) are not unique. Efficient selection of these matrices is known as the "parameterization problem" (Cloutier et al., 1996) and may effect the performance of the ensuing observer or control system.

The basic idea of the SDRE method is to design the observer or control law by treating the matrices as if they were constant and calculating the corresponding linear filter or quadratic control "on-line".

In the case of the controller, where all the components of state vector x are directly measured, the control law is thus:

$$u = -G(x)x \tag{3}$$

where $G(x)$ is given by

$$G = R^{-1}B'(x)M(x) \tag{4}$$

with $M(x)$ being the solution of the "state-dependent algebraic Riccati equation" (SDARE)

$$\begin{aligned} &M(x)A(x) + A'(x)M(x) \\ &-M(x)B(x)R^{-1}B'(x)M(x) + Q(x) = 0 \end{aligned} \tag{5}$$

The matrices $Q(x)$ and $R(x)$ are design matrices selected, as in case of a linear system, to give weight to the state and to the control, respectively. Because the process is nonlinear, it would not be appropriate to identify them with the integrand of a quadratic form that the control law (3) minimizes.

If the state x is not measured directly, an estimate $\hat{x}$ is used in its place in (3) through (5). The estimate is obtained by use of an "extended" observer. For a full-order observer, the estimated state $\hat{x}$ is given by

$$\dot{\hat{x}} = A(\hat{x})\hat{x} + K(\hat{x})[y - C(\hat{x})\hat{x}] \tag{6}$$

where the observer gain matrix $K(\hat{x})$ is given by the solution of the SDARE:

$$K(\hat{x}) = P(\hat{x})C'(\hat{x})W^{-1}(\hat{x}) \tag{7}$$

with

$$\begin{aligned} &P(\hat{x})A'(\hat{x}) + A(\hat{x})P(\hat{x}) \\ &-P(\hat{x})C'(\hat{x})W^{-1}(\hat{x})C(\hat{x})P(x) + V(\hat{x}) = 0 \end{aligned} \tag{8}$$

Although empirical results using the SDARE method have been quite favorable, determining the state-dependent gain matrices $G(\hat{x})$ and $K(\hat{x})$ is computationally intensive. This may not be a major problem in certain applications, especially with the speed and memory of computers continuing to increase. But many projected applications may rely on imbedded computers which may have the speed, but not the memory required to support the on-line solution of the relevant SDAREs.

Another problem is that algorithms for solution of the SDARE are inherently iterative: the number of computation steps cannot be predicted precisely and hence the operations cannot be accurately timed. There may be some reluctance on the part of system engineers to employ such an algorithm in real time when convergence within an allotted time is not guaranteed.

Furthermore, in some applications the use of the state-dependent *algebraic* Riccati equation can impose an overly restrictive requirement on the observability and controllability of the nonlinear system in question. Consider the SDARE observer. For a solution to the algebraic Riccati equation (8) to exist, the pair $[A(\hat{x}(t)), C(\hat{x}(t))]$ must be continuously observable locally for all time t, (or at least at each discrete instant t_n at which the ARE is solved), i.e., the rank of the observability test matrix

$$\mathcal{O} = [C'(\hat{x}), \ldots, (A'(\hat{x}))^{n-1}C'(\hat{x})]$$

must be n (the plant order) at every $\hat{x}$ the observer attains. This rank condition may not hold in some systems that are nevertheless observable in the more general sense.

The same restrictions apply in the case of the SDARE controller. Here, the existence of a solution to (5) demands that the pair $[A(x(t)), B(x(t))]$ be controllable locally at every x that the plant attains, i.e.

$$rank[B(x),\ \ , \ldots, (A(x))^{n-1}B(x)] = n$$

a condition that may fail in some nonlinear systems that are nonetheless controllable.

This paper offers an alternative to the SDRE methods for control and estimation, an alternative which addresses the issues of high computational requirements and the potentially overly restrictive observability and controllability requirements. The key idea underlying these new methods is the use of *differential* rather than algebraic Riccati equations, and their real-time solution by numerical integration. Illustrative examples comparing the SDARE and new methods are given, and the stability of the new control algorithm is examined.

2. THE SDDRE METHOD

2.1 *Estimation*

The extended Kalman filter (EKF) method for state estimation has been well established in countless theoretical studies and practical applications. The discrete-time EKF method may be regarded as an equivalent to the real-time numerical integration of the linearized variance equation

$$\begin{aligned} \dot{P} &= P(\hat{x})J_x'(\hat{x}) + J_x(\hat{x})P(\hat{x}) \\ &-P(\hat{x})J_y'(\hat{x})W^{-1}J_y(\hat{x})P(\hat{x}) + V \end{aligned} \tag{9}$$

where J_x and J_y are the Jacobian matrices of the nonlinear dynamics and observation equations, respectively. Although the theoretical stability of the EKF method may not have been established rigorously (at least to the knowledge of the authors) the question of its utility is hardly questionable.

The linearized variance equation (9) is very similar but not identical to the following proposed alternative, which is heretofore referred to as the state-dependent differential Riccati equation (SDDRE):

$$\begin{aligned} \dot{P} &= P(\hat{x})A'(\hat{x}) + A(\hat{x})P(\hat{x}) \\ &-P(\hat{x})C'(\hat{x})W^{-1}C(\hat{x})P(\hat{x}) + V(\hat{x}) \end{aligned} \tag{10}$$

As with the SDARE approach, the solution of (10) is used to generate the observer gain (7). Equation (10) is very much like the equation for the "classical" extended Kalman filter (EKF). It differs from the latter in that the latter uses the differential "variance equation" and the Jacobian matrix of the dynamic nonlinear dynamics. For the EKF problem formulation

$$J_x = A(x) + \sum_i x_i \frac{\partial A(x)}{\partial x_i}$$
$$J_y = C(x) + \sum_i x_i \frac{\partial C(x)}{\partial x_i}$$

so the Jacobian matrices have terms not present in $A(x)$ and $C(x)$. The significance of differences between the EKF and the SDDRE have been studied in (Haessig, 1999).

2.2 *Control*

In the finite horizon control problem, the corresponding Riccati equation

$$\begin{aligned} -\dot{M} &= M(x)A(x) + A'(x)M(x) \\ &-M(x)B(x)R^{-1}B'(x)M(x) + Q(x) \end{aligned} \quad (11)$$

is intended for integration *backward* in time from some terminal condition. It is well-known that it is unstable when integrated forward in time, as would be required for real-time implementation.

The practical solution to this problem of instability is simply to reverse the direction of time and simply integrate (11) with the sign before $\dot{M}$ *reversed*. When this is done, the resulting Riccati differential equation:

$$\begin{aligned} \dot{M} &= M(x)A(x) + A'(x)M(x) \\ &-M(x)B(x)R^{-1}B'(x)M(x) + Q(x) \end{aligned} \quad (12)$$

is identical in form to (10). It is well known that the variance equation (10) is stable in the forward direction of time. Hence, since (12) is the same as (10) with A' in place of A and B' in place of C, any method of implementation (numerical integration) that succeeds with (10) will succeed with (12).

2.3 *Discussion*

As noted above, the SDARE controller (observer) demands the existence of local controllability (observability) at each x ($\hat{x}$). When conditions for local controllability (observability) fail, even during brief periods of incomplete controllability (observability), numerical solution of the SDARE method generally also fails to produce a usable solution. Propagation of the “Riccati” solution forward in time through use of the SDDRE (10) or (12), however, does not require local controllability or observability at each instant of time; it can simply integrate through instants or periods of local uncontrollability or unobservability.

This requirement for time-continuous controllability (observability) has proven to be an issue that can preclude application of the SDARE method in problems to which the SDDRE method can be successfully applied. In (Haessig, 1999) both methods are applied to the problem of simultaneous state and parameters estimation in nonlinear systems. There the SDARE method was shown to work well in problems involving a few unknown parameters (i.e. less than 3); however, the requirement for continuous observability became an issue when estimating more than a few. Each constant parameter to be estimated adds a state equation of the form $\dot{\theta}_i = 0$ to the system dynamics. When there are two or more parameters, because the parameter dynamic equations are identical, the system is locally unobservable for all time (Haessig, 1999). This problem can be alleviated by replacing $\dot{\theta}_i = 0$ with $\dot{\theta}_i = -\lambda_i \theta_i$ where each λ_i is a small positive number different for each i. However, as the number of parameters grows, the observability test matrix tends to become ill-conditioned, causing the SDARE method to break down. This was studied by Haessig (1999) in an example given originally in (Bodson, 1993), involving a permanent magnet stepper motor governed by a fourth-order model with 5 unknown parameters. The author was unable to successfully apply the SDARE method due to the problem described above. On the other hand, the SDDRE filtering method was demonstrated through simulation to accurately estimate all 5 parameters as well as the process state.

3. STABILITY

Consider the nonlinear system (1) controlled by the SDDRE control law as given by (3), (4), and (12). In assessing the stability of the approach, one must initially consider the stability of the proposed Riccati equation (12), an equation identical in form to the standard filter Riccati equation (10). By well-known Kalman filter theory for linear time-varying systems, the convergence of the filter Riccati equation is guaranteed if the pair $[A(t), C(t)]$ is observable and the pair $[A(t), V^{1/2}]$ is controllable for all t. Thus, if conditions of local controllability and observability hold globally, i.e. if the pair $[A'(x), B'(x)]$ is observable and the pair $[A'(x), Q^{1/2}]$ is controllable for all x, then the convergence of (10) is assured.

To investigate the stability of the controlled system, consider a candidate Lyapunov function involving the inverse of the ”Riccati” solution to (12):

$$V(x(t)) = x'(t)M^{-1}x(t) \quad (13)$$

having the time derivative,

$$\begin{aligned} \dot{V}(x) &= \dot{x}'M^{-1}x + x'M^{-1}\dot{x} \\ &-x'M^{-1}\dot{M}M^{-1}x \end{aligned} \quad (14)$$

Into this is substituted the controlled system dynamics

$$\dot{x} = (A(x) - B(x)G)x = A_c(x)x \quad (15)$$

and the matrix differential equation (12) rewritten here in a more compact form

$$\dot{M} = A_c'M + MA_c + G'RG + Q \quad (16)$$

This leads to the Lyapunov time derivative function:

$$\dot{V}(x) = x'M^{-1}[MA_c' + A_cM$$
$$-MA_c - A_c'M - (G'RG + Q)]M^{-1}x \quad (17)$$

In this equation, the first two and next two terms to the right of the opening bracket both individually sum to form symmetric matrices, which are identified as follows:

$$MA_c' + A_cM = S \quad (18)$$

$$MA_c + A_c'M = T \quad (19)$$

With these substitutions,

$$\dot{V}(x) = -x'M^{-1}[T - S]$$
$$+(G'RG + Q)]M^{-1}x \quad (20)$$

Stability in this time varying system can be proved if one can show there exists fixed (time-invariant) positive definite quadratic functions bounding $V(x)$ from above and below, and one other bounding $\dot{V}(x)$ from above. The following conceptual arguments suggest their existence.

Although the system matrices given in (1) and (2) are functions of the state x, because the state is a known function of time t, it is possible to express (1) and (2) as using time dependent system matrices, i.e. $A(x(t)), B(x(t)), C(x(t))$. Consequently, it seems reasonable to expect that control concepts and properties applicable to linear time-varying systems should also be applicable in this nonlinear context. In particular, one needs to use the properties of Riccati equation solutions that depend on the observability and controllability of the system.

Consider first (13) involving M^{-1}, the inverse of the Riccati solution. For stability the nonlinear system must be both controllable through the input matrix $B(x)$ and observable though the state weighting matrix $Q^{1/2}$. The effect of system controllability is to decrease or deflate M. This creates an upper bound on M resulting in an lower bound on M^{-1}, as required for Lyapunov stability. If the system is observable then the state weighting matrix essentially has the effect of inflating the M matrix, placing a lower bound on M and an upper bound on M^{-1}, also required for Lyapunov stability. Thus one can argue that the candidate Lyapunov function (13) is bounded above and below if the system is both observable and controllable.

Consider now the candidate Lyapunov function time derivative (20). Standard linear quadratic control theory indicates that the term $G'RG$ is positive definite in time-varying linear systems that are both observable and controllable. Thus the term $(G'RG + Q)$ in (20) will be positive definite.

The matrices S and T may or may not be positive definite. Whether their difference $T - S$ enhances or detracts from the positive definiteness of $T - S + (G'RG + Q)$ remains as a topic for future research.

Assessment of stability of the SDDRE observer is complicated by the presence of both x and $\hat{x}$, and left also for future research.

4. EXAMPLES

4.1 *Example 1 – Inverted Pendulum*

To illustrate the techniques discussed above a very simple nonlinear control problem is considered, namely the stabilization of an inverted pendulum, described by

$$\ddot{\theta} - \sin\theta = u \quad (21)$$

In state-space form, with $x_1 = \theta,\ x_2 = \dot{\theta}$, the matrices describing the dynamics are

$$A(x) = \begin{bmatrix} 0 & 1 \\ a(x) & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (22)$$

where

$$a(x) = \sin x_1 / x_1 \quad (23)$$

The performance matrices for this problem are taken as

$$Q = \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix}, \quad R = 1 \quad (24)$$

with $q = 0.01$.

The solution to the SDARE in this example is easily calculated analytically:

$$M = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} \quad (25)$$

with

$$m_2 = a + \sqrt{a^2 + q}$$
$$m_3 = \sqrt{2m_2}, \quad m_1 = m_3\sqrt{m_2 - a}$$

The gain matrix is

$$G(x) = \begin{bmatrix} m_2(x) & m_3(x) \end{bmatrix} \quad (26)$$

Since $\sin x_1/x_1 \rightarrow 1$ as $x_1 \rightarrow 0$, the reference condition used in generation of the fixed gain linear control about $x_1 = 0$ is $a = 1$.

In Figure 1 the SDARE method is used to compute the control gain in the upper plot, whereas linear fixed gain control is used in the lower. Clearly system performance with linear control is unacceptable as it is virtually unstable, whereas that achieved with SDARE control is much better.

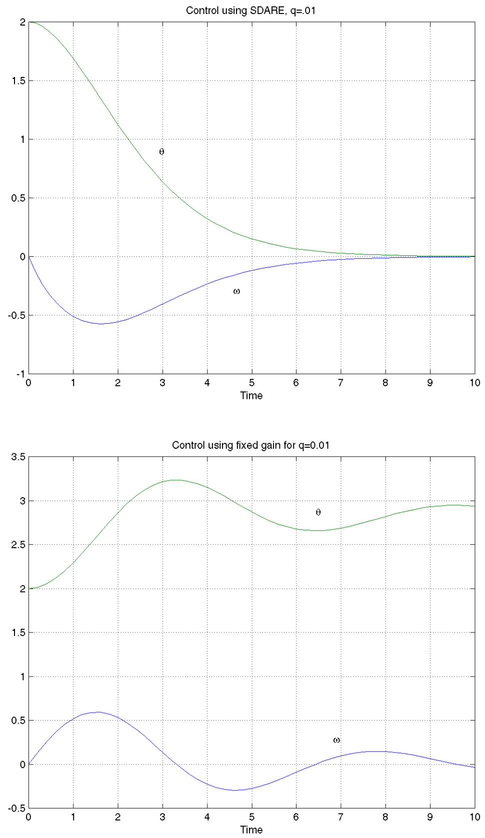

Fig. 1. *Simulated performance with linear and SDARE control. With linear control, performance verges on instability.*

The result of numerical integration of the SDDRE in this same example is shown in Figure 2. It is observed that the transient response is quite similar to the SDARE result of Figure 1 (if anything, slightly better).

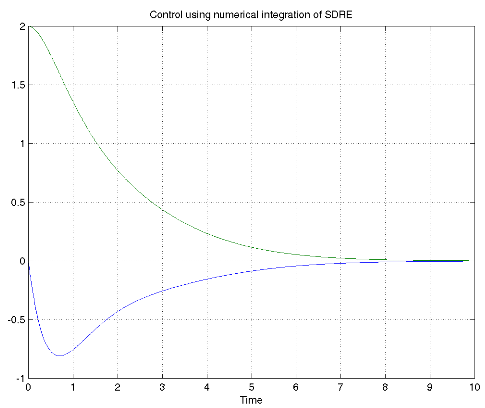

Fig. 2. *Simulated performance with SDDRE control*

4.2 *Example 2*

The computational advantage of the subject method is illustrated through its application to an example taken from (Cloutier et al., 1996), the second order multivariable system:

$$\dot{x}_1 = x_1 - x_1^3 + x_2 + u_1 \tag{27}$$

$$\dot{x}_2 = x_1 + x_1^2 x_2 - x_2 + u_2 \tag{28}$$

Both the SDARE and SDDRE methods are applied to this system and compared on the basis of the computational resources required to generate the Riccati solution and control gains, and the quality of the resulting transient response.

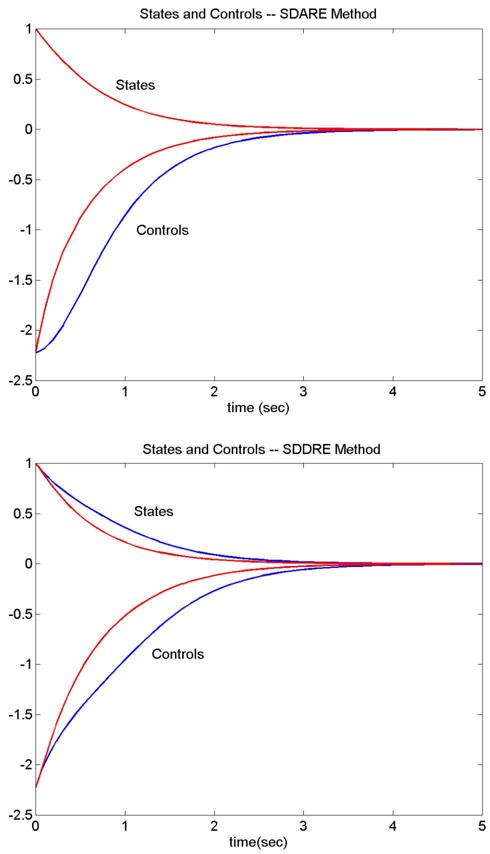

Fig. 3. *Comparison of the algebraic and differential Riccati based methods in Example 2, with SDARE control on top and SDDRE control below. With SDARE control the state variables are equal and thus appear as a single curve.*

In both cases the following A matrix parameterization is employed:

$$A = \begin{bmatrix} 1 - x_1^2 & 1 \\ 1 & x_1^2 - 1 \end{bmatrix} \tag{29}$$

and the state and control weighting matrices that appear in the performance index are set to $Q = I_2$ and $R = 2I_2$, respectively.

The results achieved using the SDARE and SDDRE methods are compared in Figure 3, with the SDARE and SDDRE results in the upper and lower plots, respectively. One notes that the transient solutions are only slightly different and qualitatively similar – the

$\dot{M}$ term present in the Riccati differential equation clearly alters the solution, but does not change the transient response in any substantive way. (The SDDRE solution requires an initial condition, which was set equal to the solution given by the SDARE at $t = 0$.)

Computational load is compared on the basis of FLOPS, the number of floating point operations required to compute a new "Riccati" solution and controller gain. This was done using the Matlab FLOP function. In the SDARE case, the number of FLOPS consumed to compute a new G were measured. This was done by querying the FLOPS function immediately before and after the Matlab lqr function, which returned G. On average, 850 FLOPS were required on each call to lqr. In the SDDRE case, the FLOP effort was measured by assuming that Euler integration would propagate the Riccati solution one step forward in time. Thus, the FLOP function was queried before and after calculation of both the $\dot{M}$ and G in (12) and (4). This was measured to be 140 FLOPS. To this another 8 FLOPS were added for the Euler propagation of the solution($M_i = M_{i-1} + \dot{M}T$), which brings the total to 148 FLOPS, or approximately a 1/6 of that consumed by the lqr function. No advantage was taken of symmetry which would further reduce the number of calculations. Clearly, even in this low order example, the potential for significant reduction in computational loading is evident.

5. CONCLUSION

A new approach for the design of controllers and observers for nonlinear systems has been introduced. These methods are similar to existing state-dependent Riccati equation methods, but differ in that *differential* Riccati equations are used to generate controller and observer gains. The stability of the resulting control algorithm is evaluated.

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