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A NONLINEAR VERSION OF BODE'S SENSITIVITY INTEGRAL: CONNECTIONS WITH ENTROPY AND NONLINEAR INNER-OUTER FACTORIZATIONS.

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Abstract: For linear systems, we show that the constraint known as Bode's sensitivity integral has an information theoretic interpretation in terms of the difference in the entropy rates between the input and output of the systems. We use this interpretation to show that, if the open loop system is globally exponentially stable, this difference is zero. For nonlinear systems that are not stable, we start to investigate the method to calculate this difference using new results on the inner/outer factorization of nonlinear systems.

Keywords: Bode, Sensitivity, Entropy, Inner, Outer

1. INTRODUCTION

In his classical monograph (Bode, 1945), Bode showed[2] that for a single-input single-output, stable, open loop system $L(z)$, the sensitivity function $S(z) = 1/(1 + L(z))$ must satisfy

$$\int_0^\infty \log |S(\mathrm{e}^{i\omega})| \, \mathrm{d}\omega = 0 \qquad (1)$$

It is well known that this integral constraint has important practical implications (Freudenberg and Looze, 1988; Doyle *et al.*, 1992). For example, consider the system depicted in Fig. 1. Here, the system L includes both the plant $P(z)$ and the controller $C(z)$. If one wishes to keep the tracking error, $|e(\mathrm{e}^{i\omega})|$, below $\epsilon < 1$ for reference signals with frequency content $\omega \in [0, \Omega]$, the sensitivity function must satisfy

$$\ln |S(\mathrm{e}^{i\omega})| < \ln \epsilon < 0, \quad \forall \omega \in [0, \Omega]$$

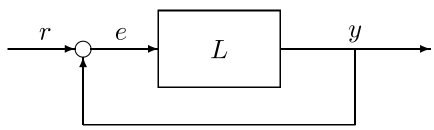

Fig. 1. Tracking problem

However, according to the constraint imposed by (1), it must follow that $\ln |S(\mathrm{e}^{i\omega})| > 0$, for some $\omega \notin (\Omega, \pi)$. In particular,

$$\begin{aligned} 0 &\leq (-\ln \epsilon)\Omega \\ &\leq (\pi - \Omega) \sup_{\omega \in (\Omega, \pi)} \ln |S(\mathrm{e}^{i\omega})| \\ &\leq (\pi - \Omega) \ln \|S\|_\infty \end{aligned}$$

Thus:

$$\|S\|_\infty \geq \exp\left(\frac{\Omega}{\pi - \Omega}(-\ln \epsilon)\right)$$

Hence, a smaller ϵ or a larger bandwidth Ω will increase the corresponding values of $|S(\mathrm{e}^{i\omega})|$ in $\omega \notin [0, \Omega]$.

Bode's result has been generalized in many ways (Freudenberg and Looze, 1988; Seron *et*

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[2] Bode's work dealt with continuous-time systems, here we present the corresponding discrete-time results.

al., 1997). For arbitrary open loop multivariable systems, Freudenberg and Looze, who showed that if $L(z)$ has unstable poles $\{p_i\}$, then the integral is now equal to:

$$\int_0^{\pi} \ln|\det S(\mathrm{e}^{i\omega})|\,\mathrm{d}\omega = \sum(-\ln|p_i|) > 0 \quad (2)$$

The appearance of the unstable poles (i.e. $|p_i| < 1$) in the right-hand side of (2) serves only to worsen the bandwidth and magnitude tradeoffs mentioned with regard to (1).

A generalization of Bode's integral relation to a class of discrete-time, nonlinear systems was considered in (Iglesias, 2001*a*) for nonlinear *minimum phase* systems. What makes possible this extension is the connection that exists between the logarithmic integral found in Bode's relationship and an information theoretic entropy. This allows for a time-domain interpretation to (1). This type of extension was first considered for linear time-varying systems in (Iglesias, 2001*b*; Iglesias, 2002). In this paper we build upon the results of (Iglesias, 2001*a*). In particular, we consider the more general case where the system is not minimum phase and present tentative result. To do this we employ some new results on nonlinear inner/outer factorizations, building upon earlier work of Ball and co-workers (Ball and Helton, 1992; Ball and van der Schaft, 1996).

We note an alternative approach to extending Bode's integral results to nonlinear systems in (Seron *et al.*, 1999).

2. PRELIMINARIES

In this section we first demonstrate how Bode's integral can be recast in information theoretic terms using the concept of entropy rates.

Consider a random variable $x \in \mathbb{R}^m$. If x can only take a finite number of values, say $x(1), \ldots, x(r)$, then the entropy of x is defined as

$$\mathcal{H}(x) := -\sum_{i=1}^{r} p_i \ln p_i$$

where p_i is the probability that $x = x(i)$. If the random variable is a continuous-type one, this can be extended to

$$\mathcal{H}(x) := -\int_{\mathbb{R}^m} f(x) \ln f(x)\,\mathrm{d}x \quad (3)$$

where $f(x)$ is the probability density function of x, and by definition we take $f(x)\ln f(x) = 0$ if $f(x) = 0$. Now consider a continuous-type random variable x_k as a function of time. The conditional entropy of order l is defined as

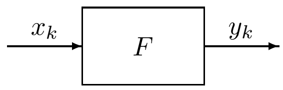

Fig. 2. General input-output system

$$\begin{aligned}\mathcal{H}(x_k|x_{k-1},\ldots,x_{k-l})\\ := -\int_{\mathbb{R}^m} f(x_k|x_{k-1},\ldots,x_{k-l})\\ \ln f(x_k|x_{k-1},\ldots,x_{k-l})\,\mathrm{d}x_k\end{aligned}$$

This is a measure for the uncertainty about its value at time k under the assumption that its l most recent values have been observed. By letting l go to infinity, the conditional entropy of x_k is defined as

$$\mathcal{H}_c(x_k) := \lim_{l\to\infty} \mathcal{H}(x_k|x_{k-1},\ldots,x_{k-l})$$

assuming the limit exists. The conditional entropy is a measure of the uncertainty about the value of x at time k under the assumption that its entire past is observed. We point out that, for a stationary signal x_k, the conditional entropy and the *entropy rate*

$$\bar{\mathcal{H}}(x) = \lim_{m\to\infty} \frac{1}{m} H(x_1,\ldots,x_m)$$

are the same.

2.1 Conditional entropy of a linear system

The following result, originally in (Kolmogorov, 1956), is well known; see (Papoulis, 1984)
Lemma 1. *Consider the system depicted by Figure 2 where $F(z)$ is the transfer function of a stable, discrete-time, linear time-invariant system. The conditional entropy of the output y_k equals:*

$$\mathcal{H}_c(y_k) = \mathcal{H}_c(x_k) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \ln|F(\mathrm{e}^{i\omega})|\,\mathrm{d}\omega$$

We also note that
Corollary 2. *If, in addition, $F(z)$ is minimum phase and $\lim_{z\to\infty} F(z) = 1$ then $\mathcal{H}_c(y_k) = \mathcal{H}_c(x_k)$.*

Suppose that $F(z) = S(z)$ where $S(z)$ is the sensitivity function, and $S(z) = (1 + L(z))^{-1}$ where $L(z)$ represents the loop transfer function. In this case, the integral of Lemma 1 is referred to as Bode's integral. We note as well that the minimum phase requirement of Corollary 2 amounts to requiring that the loop transfer function is stable, since the zeros of $S(z)$ are the poles of $L(z)$. Morever, the requirement that $\lim_{z\to\infty} S(z) = 1$ is equivalent to requiring that $L(z)$ be strictly proper.

Because of this relationship, let us denote

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln|S(\mathrm{e}^{i\omega})|\,\mathrm{d}\omega = \mathcal{H}_c(e_k) - \mathcal{H}_c(w_k) =: \mathcal{B}(S) \quad (4)$$

where $S = w \mapsto e$. The advantage of the righthand-most term in (4) is that, as it involves time-domain terms, it is well defined even for systems that do not admit a transfer function. In the next section we look at using this connection to consider nonlinear systems.

2.2 Nonlinear Systems

Suppose that we consider a general discrete-time system with input e_k and output z_k whose dynamics are governed by the following nonlinear difference equation:

$$\Sigma_L := \begin{cases} x_{k+1} &= A(x_k) + B(x_k)e_k \\ z_k &= C(x_k) \end{cases} \tag{5}$$

and suppose that u_k is obtained as unity feedback

$$e_k = w_k - z_k$$

resulting in the closed-loop system

$$\Sigma_S := \begin{cases} x_{k+1} &= A^{\times}(x_k) + B(x_k)w_k \\ e_k &= -C(x_k) + w_k \end{cases} \tag{6}$$

where $A^{\times}(x_k) = A(x_k) - B(x_k)C(x_k)$.

We note that the dynamics of Σ_L includes both the controller *and* the plant. Of course, we assume that the controller has been chosen so that the closed-loop system is stable. Precisely, we assume that x_0 is globally exponentially stable equilibrium point of $x_{k+1} = A^{\times}(x_k)$ and that $C(x_0) = 0$.

Definition 3. *The system Σ_S is* minimum phase *if for the inverse system*

$$\Sigma_{S^{-1}} := \begin{cases} x_{k+1} = A(x_k) + B(x_k)e_k \\ w_k \quad = C(x_k) + e_k \end{cases} \tag{7}$$

x_0 is also globally exponentially stable equilibrium point of $x_{k+1} = A(x_k)$, $B(x)$ is bounded, $C(x)$ is Lipschitz.

The requirement on $B(x)$ and $C(x)$ is to ensure the l^∞ input-output stability of both Σ_S and $\Sigma_{S^{-1}}$ (Lemma 5.5 in (Khalil, 1996)).

In order to deal with infinite dimensional input-output map, the concepts of *fading memory* and *nonlinear moving average theorem* are needed to truncate the map into finite dimension.

Definition 4. *(Boyd and Chua, 1985) A time invariant operator $N : l^\infty \to l^\infty$ is said to have fading memory on a subset K of l^∞ if there is a decreasing sequence $w : \mathbb{Z}_+ \to (0,1]$, $\lim_{k\to\infty} w(k) = 0$, such that for each $u \in K$ and $\epsilon > 0$ there is a $\delta > 0$ such that for all $v \in K$*

$$\sup_{k\leq 0} |u(k)-v(k)|w(-k) < \delta \to |Nu(0)-Nv(0)| < \epsilon$$

Theorem 5. *(Boyd and Chua, 1985) (NLMA Approximation Theorem): Let $\epsilon > 0$, K be any ball in l^∞, and suppose N is any time invariant operator: $l^\infty \to l^\infty$ with fading memory on K.*

Then there is a polynomial $p : \mathbb{R}^M \to \mathbb{R}$ such that for all $u \in K$

$$\left\| Nu - \hat{N}u \right\| \leq \epsilon$$

where $\hat{N}$ is the $NLMA$ operator given by

$$\hat{N}u_k := p(u_k, \cdots, u_{k-M+1})$$

2.3 Entropy Formula

Lemma 6. *(Papoulis, 1984) If*

$$\mathbf{y}_i = g_i(\mathbf{x}_1, \cdots, \mathbf{x}_n) \qquad i = 1, \cdots, n$$

are n functions of the random variables $\mathbf{x}_i$, then

$$\begin{aligned} H(\mathbf{y}_1, \cdots, \mathbf{y}_n) \leq & H(\mathbf{x}_1, \cdots, \mathbf{x}_n) \\ & + E\{\log |J(\mathbf{x}_1, \cdots, \mathbf{x}_n)|\} \end{aligned}$$

where $J(\mathbf{x}_1, \cdots, \mathbf{x}_n)$ is the Jacobian of the above transformation. Equality holds iff the transformation has a unique inverse.

Lemma 7. *(Papoulis, 1984) For random variables $\mathbf{x}_1, \cdots, \mathbf{x}_n$, we have*

$$H(\mathbf{x}_1, \cdots, \mathbf{x}_n) \leq H(\mathbf{x}_1) + \cdots + H(\mathbf{x}_n)$$

with equality, if the $\mathbf{x}_i$'s are independent.

3. MAIN RESULTS: MINIMUM PHASE CASE

We now study the case that the system is minimum phase.

Theorem 8. *Suppose that the system Σ_S given by (6) is minimum phase, and that both Σ_S and $\Sigma_{S^{-1}}$ have fading memory. Then $\mathcal{B}(S) = 0$.*

Proof. The system we are discussing has fading memory. By Theorem 5, such a system has a polynomial approximation with arbitrarily small error, and this approximation is given by

$$\hat{S}\mathbf{e}_k = p(\mathbf{e}_k, \cdots, \mathbf{e}_{k-N}) \tag{8}$$

where p is a polynomial $\mathbb{R}^{N+1} \to \mathbb{R}$.

Thus the Jacobian of the approximated transformation is lower triangular with coefficients

$$\begin{aligned} &J_{i,j}(\mathbf{e}_k, \cdots, \mathbf{e}_{k-N}) = \frac{\partial p(\mathbf{e}_k, \cdots, \mathbf{e}_{k-N})}{\partial \mathbf{e}_{k-j+i}} \quad i > j \\ &J_{i,j}(\mathbf{e}_k, \cdots, \mathbf{e}_{k-N}) = 0 \qquad i < j \\ &J_{i,i} = 1 \end{aligned}$$

We apply Lemma 6 to the following transformation

$$\mathbf{w}_{n-k} = p(\mathbf{e}_{n-k}, \cdots, \mathbf{e}_{n-k-N}), \quad 0 \le k \le m - N$$
$$\mathbf{y}_{n-k} = p(\mathbf{e}_{n-k}, \cdots, \mathbf{e}_{n-m}, 0, \cdots), \quad m - N \le k \le m$$

we have that

$$\begin{aligned} & H(\mathbf{w}_n, \cdots, \mathbf{w}_{n-m+N}, \mathbf{y}_{n-m+N-1}, \cdots, \mathbf{y}_{n-m}) \\ =& H(\mathbf{e}_n, \cdots, \mathbf{e}_{n-m}) + E\{\log |J(\mathbf{e}_n, \cdots, \mathbf{e}_{n-m})|\} \\ =& H(\mathbf{e}_n, \cdots, \mathbf{e}_{n-m}) \end{aligned}$$

By Lemma 7

$$\begin{aligned} & H(\mathbf{w}_n, \cdots, \mathbf{w}_{n-m+N}, \mathbf{y}_{n-m+N-1}, \cdots, \mathbf{y}_{n-m}) \\ \le & H(\mathbf{w}_n, \cdots, \mathbf{w}_{n-m+N}) + \sum_{k=1}^{N} H(\mathbf{y}_{n-m+N-k}) \end{aligned}$$

Now, divide both sides by $m+1$ and take the limit as m goes to infinity, then

$$\begin{aligned} & \lim_{m\to\infty} \frac{1}{m+1} \cdot \\ & \quad H(\mathbf{w}_n, \cdots, \mathbf{w}_{n-m+N}, \mathbf{y}_{n-m+N-1}, \cdots, \mathbf{y}_{n-m}) \\ \le & \lim_{m\to\infty} \frac{m-N+1}{m+1} \frac{H(\mathbf{w}_n, \cdots, \mathbf{w}_{n-m+N})}{m-N+1} \\ & + \lim_{m\to\infty} \frac{1}{m+1} \sum_{k=1}^{N} H(\mathbf{y}_{n-m+N-k}) \\ = & \bar{H}(\mathbf{w}) \end{aligned}$$

We have

$$\lim_{m\to\infty} \frac{1}{m+1} \sum_{k=1}^{N} H(\mathbf{y}_{n-m+N-k}) = 0$$

because all the items in the finite sum are finite numbers.

Thus,

$$\bar{H}(\mathbf{w}) \ge \bar{H}(\mathbf{e}) \tag{9}$$

Since the system is minimum phase, its inverse system is also causal, stable and has fading memory, with $\mathbf{w}_n$ as input and $\mathbf{e}_n$ as output. A similar derivation as (9) gives

$$\bar{H}(\mathbf{e}) \ge \bar{H}(\mathbf{w}) \tag{10}$$

By (9) and (10) we have $\bar{H}(\mathbf{w}) = \bar{H}(\mathbf{e})$ □

The fading memory requirement was used so as to limit the past input contributions on the output. It is natural to ask what constraints on $A(x)$, $B(x)$, $C(x)$ and $\mathbf{w}$ would satisfy this fading memory requirement. This will be the task of future research.

4. NONLINEAR INNER-OUTER FACTORIZATION

Let $G(z)$ be a transfer matrix of linear discrete time system and let the realization $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be minimal. It is well known (Khargonekar and Sontag, 1982) that a stabilizing state feedback gain F can yield a right coprime factorization (rcf) $G = NM^{-1}$ over RH^{∞}, where

$$\begin{bmatrix} M \\ N \end{bmatrix} = \left[\begin{array}{c|c} A+BF & BW \\ \hline F & W \\ C+DF & DW \end{array}\right] \tag{11}$$

and W is any nonsigular matrix. If G is stable, then M above is an outer hence the factorization $G = N(M^{-1})$ is an inner-outer factorization.

For a nonliner system Σ_G which is affine in its input

$$\Sigma_G = \begin{cases} x_{k+1} &= A(x_k) + B(x_k)u_k \\ y_k &= C(x_k) + D(x_k)u_k \end{cases} \tag{12}$$

where A, B, C and D are smooth mappings with appropriate dimensions. we have the similar factorization $G = N \circ M^{-1}$.

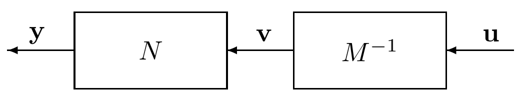

Fig. 3. Nonlinear Factorization

With the input output signals illustrated in the above figure, we can write their state space representation.

$$\Sigma_M = \begin{cases} p_{k+1} = & [A(p_k) + B(p_k)F(p_k)] \\ & +B(p_k)W(p_k)v_k \\ u_k = & F(p_k) + W(p_k)v_k \end{cases} \tag{13}$$

$$\Sigma_N = \begin{cases} q_{k+1} = & [A(q_k) + B(q_k)F(q_k)] \\ & +B(q_k)W(q_k)v_k \\ y_k = & [C(q_k) + D(q_k)F(q_k)] \\ & +D(q_k)W(q_k)v_k \end{cases} \tag{14}$$

Definition 9. *A nonlinear system* Σ *is* lossless *with respect to the supply rate* $\frac{1}{2}u_k^T u_k - \frac{1}{2}y_k^T y_k$ *if there exists a function* $V(x_k) \ge 0$ *(the* storage function*) such that*

$$V(x_{k+1}) - V(x_k) = \tfrac{1}{2}u_k^T u_k - \tfrac{1}{2}y_k^T y_k$$

and $V(0) = 0$.

This definition serves as a discrete time analogue of losslessness for continuous system in (Ball and van der Schaft, 1996).

The choice of this storage function leads a lossless global exponentially stable system to an *input-output conservative* system, thus an inner. Suppose sequence $\{y_1, y_2, \cdots\} \in L_2^m$ is the output responding to an input sequence $\{u_1, u_2, \cdots\} \in L_2^p$, then

$$\sum_{k=0}^{n} y_k^T y_k = \sum_{k=0}^{n} [u_k^T u_k + 2V(x_{k+1}) - 2V(x_k)]$$
$$= \sum_{k=0}^{n} u_k^T u_k + 2V(x_{n+1})$$

Global exponentially stability implies that

$$\lim_{n\to\infty} x_{n+1} = 0$$

hence

$$\lim_{n\to\infty} V(x_{n+1}) = 0$$

It follows that

$$\sum_{k=0}^{\infty} y_k^T y_k = \sum_{k=0}^{\infty} u_k^T u_k$$

and, therefore

$$\langle y, y\rangle_{L_2^p} = \langle u, u\rangle_{L_2^m}$$

which defines input-output conservativeness of the system.

Based on this definition, we have the following result which mirrors (Byrnes and Lin, 1994, Theorem 2.6),

Theorem 10. *A system G of the form (12) is lossless if and only if there exist a C^2 storage function $V \geq 0$, $V(0) = 0$ such that,*

$$V(A(x_k)) = V(x_k) - \tfrac{1}{2}C^T(x_k)C(x_k) \qquad (15)$$

$$\frac{\partial V(\alpha)}{\partial \alpha}\Big|_{\alpha=A(x_k)} B(x_k) = -C^T(x_k)D(x_k) \qquad (16)$$

$$B^T(x_k)\frac{\partial^2 V(\alpha)}{\partial \alpha^2}\Big|_{\alpha=A(x_k)} B(x_k) = I - D^T(x_k)D(x_k) \qquad (17)$$

and $V(A(x_k) + B(x_k)u_k)$ is quadratic in u_k.

By setting

$$\begin{aligned} V(x_k) &= \tfrac{1}{2}x_k^T X x_k \\ A(x_k) &= Ax_k, \qquad B(x_k) = B \\ C(x_k) &= Cx_k, \qquad D(x_k) = D \end{aligned}$$

we can recover the losslessness conditions (Zhou *et al.*, 1996, Corollary 21.19) of the linear case. In particular, note that

$$\begin{aligned} (15) \Rightarrow & \tfrac{1}{2}(Ax_k)^T X (Ax_k) \\ &= \tfrac{1}{2}x_k^T X x_k - \tfrac{1}{2}(Cx_k)^T(Cx_k) \\ \Rightarrow & A^T X A + C^T C - X = 0 \\ (16) \Rightarrow & \alpha^T X\Big|_{\alpha=Ax_k} B = -(Cx_k)^T D \\ \Rightarrow & A^T X B + C^T D = 0 \end{aligned}$$

and

$$(17) \Rightarrow B^T X B = I - D^T D$$

The main result of this section now follows.

Theorem 11. *A stable discrete-time nonlinear system having a state space representation*

$$\Sigma_G = \begin{cases} x_{k+1} &= A(x_k) + B(x_k)u_k \\ y_k &= C(x_k) + u_k \end{cases}$$

where $u_k \in \mathbb{R}^m$ and $y_k \in \mathbb{R}^m$, A, B and C are smooth mappings with appropriate dimensions, has an inner-outer factorization where the outer factor Φ has state-space representation

$$\Sigma_\Phi : \begin{cases} x_{k+1} &= A(x_k) + B(x_k)u_k \\ y_k &= -Z^{1/2}(x_k)F(x_k) + Z^{1/2}(x_k)u_k \end{cases} \qquad (18)$$

where

$$Z(x) = I + B^T(x)\frac{\partial V^2(\alpha)}{\partial \alpha^2}\Big|_{\alpha=A(x)+B(x)F(x)} B(x)$$

$F(x)$ is a solution to

$$\frac{\partial V(\alpha)}{\partial \alpha}\Big|_{\alpha=A(x)+B(x)F(x)} = -[C(x)+F(x)]^T Z^{-1/2}(x)$$

with $A(x) + B(x)F(x)$ is stable, and $V(x) \geq 0$ satisfies

$$V(A(x) + B(x)F(x)) + \frac{1}{2}\|C(x) + F(x)\|^2 = V(x)$$
$$V(0) = 0$$

In order to employ theorem 8, the fading memory requirement is now a obstacle – we don't know whether the outer factor has fading memory given that the system itself has fading memory. we conjecture that it is true because the inner factor preserve the norm. If it is the case, we have the following results.

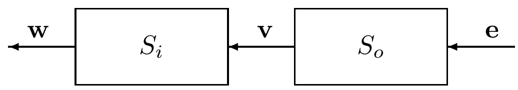

Fig. 4. Inner-Outer Factorization

Theorem 12. *Suppose that the system Σ_S given by (6) is globally exponentially stable, has fading memory, and A, B and C are smooth mappings with appropriate dimensions, then*

$$\begin{aligned} \mathcal{B}(S_o) &:= \mathcal{H}_c(v_k) - \mathcal{H}_c(e_k) \\ &= \lim_{m\to\infty} \frac{1}{2m}\mathcal{E}\Big\{\sum_{l=0}^{m} \ln|\det Z(x_{k-l})|\Big\} \end{aligned}$$

where the signals is defined as in Figure 4 and

$$Z(x) = I + B^T(x)\frac{\partial V^2(\alpha)}{\partial \alpha^2}\Big|_{\alpha=A(x)+B(x)F(x)} B(x)$$

$F(x)$ is a solution to

$$\frac{\partial V(\alpha)}{\partial \alpha}\Big|_{\alpha=A(x)+B(x)F(x)} = -F^T(x)Z^{-1/2}(x)$$

with $A(x) + B(x)F(x)$ is stable, and $V(x) \geq 0$ satisfies

$$V(A(x) + B(x)F(x)) + \frac{1}{2}\|F(x)\|^2 = V(x),$$
$$V(0) = 0$$

Proof. Only note that the Jacobian of the outer factor is just the product of $Z^{-1/2}(x_l)$ for all $l \leq k$. $\square$

Corollary 13. *If V is quadratic, then $\mathcal{B}(S_o) > 0$.*

5. DISCUSSION

The result in Theorem 8 corresponds to Bode's original result. It states that, irrespective of the choice of controller, and provided that the loop system L is stable so that S is minimum phase, *the uncertainty cannot be reduced.* In fact, one can see that Bode's original result is not a consequence of the linearity of feedback, but of causality. In the general case, where the loop gain is not stable so that we can only make S stable, we have shown how to factor the sensitivity operator S into minimum phase part and all pass part. We have shown that the entropy difference caused by the minimum phase part is nonnegative when the system satisfies fading memory requirements. Further work are needed, however. First: to characterize the difference in entropy rates between the system input and the output of the inner factor. This latter part is zero for linear systems, and we conjecture the same for nonlinear systems. Second is to investigate further the conditions for a minimum phase system to have fading memory.

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