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COMPUTATION OF FREQUENCY RESPONSE GAIN OF SAMPLED-DATA SYSTEMS

Hisaya Fujioka *

* *Kyoto University, Kyoto 606-8501, Japan*

Abstract: This paper provides a condition to determine the frequency response gain of sampled-data systems is less than a given positive number for its bisection computation. In contrast to existing conditions, there is no assumption on the norm of the related state space compression operators. The derived condition also unifies the computation of the frequency response gain of sampled-data systems and the induced norm of state-space compression operators.

Keywords: sampled-data systems, frequency responses, algorithms, computer-aided control systems design, robust control

1. INTRODUCTION

Sampled-data control theory has been successfully developed in the last decade. Now we can take into account of intersample behavior of sampled-data systems in both analysis and synthesis problems. See (Chen and Francis, 1995) and references therein.

Among them, one of the most important results is the introduction of the notion of the frequency response to sampled-data systems. Two types of definitions are introduced in (Yamamoto and Khargonekar, 1996) and (Araki *et al.*, 1996), and their equivalence is studied in (Yamamoto and Araki, 1994).

In spite of their contribution to analysis and synthesis problem in sampled-data control theory, it is hard to compute the gain of the frequency response. Several upper and lower bounds, and approximations are found in (Hara *et al.*, 1995; Hagiwara *et al.*, 2001; Yamamoto *et al.*, 1999; Fujioka and Ito, 2001). For exact computation, a bisection algorithm with an assumption on the norm of the related state space compression operators is first proposed in (Hara *et al.*, 1995). Another bisection algorithm is proposed in (Ito *et al.*, 2001), where the assumption is fairly weakened but the exact value of the norm of the compression operators are required.

This paper also deals with the computation of the frequency response gain of sampled-data systems. To be more concrete, we will propose a condition to determine that the gain is less than a given positive number or not, which is immediately applicable for a bisection computation of the gain. In comparison to existing conditions for bisection computation in (Hara *et al.*, 1995; Ito *et al.*, 2001), the proposed condition does not assume any conditions on the related state space compression operators, and it does not use the value of the norm neither.

The proposed condition is closely related to results in (Dullerud, 1999), where the induced norm of a compression operator is checked. In fact, the condition in this paper unifies the computation of the induced norm of a compression operator and the frequency response gain of sampled-data systems.

2. PROBLEM FORMULATION

The problem setup in this paper is the standard one in the sampled-data control theory: Consider a sampled-data feedback control system T depicted in Fig. 1, where G_c is a continuous-time system with a state space representation:

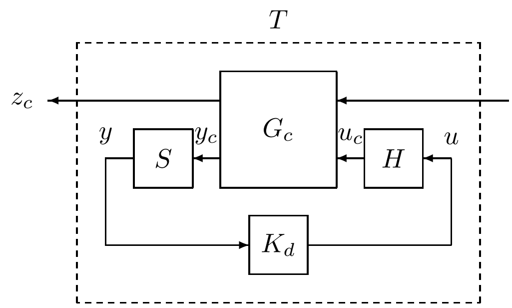

Fig. 1. Sampled-Data Feedback Systems

$$G_c : \begin{bmatrix} \dot{x}_c(t) \\ z_c(t) \\ y_c(t) \end{bmatrix} = \begin{bmatrix} A_c & B_{c1} & B_{c2} \\ C_{c1} & D_{c11} & D_{c12} \\ C_{c2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_c(t) \\ w_c(t) \\ u_c(t) \end{bmatrix}.$$

K_d is a discrete-time system with a realization:

$$K_d : \begin{bmatrix} x_K[k+1] \\ u[k] \end{bmatrix} = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \begin{bmatrix} x_K[k] \\ y[k] \end{bmatrix}.$$

Sample and hold devices are respectively denoted by S and H:

$$S : y_c \mapsto y, \quad y[k] = y(kh),$$
$$H : u \mapsto u_c, \quad u(kh+\tau) = u[k], \quad \tau \in [0,\, h),$$

where $h > 0$ is the fixed sampling period and $k = 0$, 1, 2, ...

Let $\boldsymbol{W}$ be the lifting operator (Yamamoto, 1994; Bamieh and Pearson, 1992) that maps a function f on $[0,\, \infty]$ to a function-space valued sequence $\boldsymbol{f} := \{\boldsymbol{f}[k]\}_{k=0}^{\infty}$:

$$\boldsymbol{W} : f \mapsto \boldsymbol{f} : \quad \boldsymbol{f}_k(\tau) := f(kh+\tau), \quad \tau \in [0,\, h],$$

where $\boldsymbol{f}_k := \boldsymbol{f}[k]$. Now consider a lifted system $\boldsymbol{T} := \boldsymbol{W} T \boldsymbol{W}^{-1}$ as depicted in Fig. 2. $\boldsymbol{T}$ is the feedback connection of $\boldsymbol{G}$ and K_d, where

$$\boldsymbol{G} := \begin{bmatrix} \boldsymbol{W} & 0 \\ 0 & S \end{bmatrix} G_c \begin{bmatrix} \boldsymbol{W}^{-1} & 0 \\ 0 & H \end{bmatrix}.$$

A state space representation of $\boldsymbol{G}$ is given by

$$\boldsymbol{G} : \begin{bmatrix} x[k+1] \\ \boldsymbol{z}[k] \\ y[k] \end{bmatrix} = \begin{bmatrix} A & \boldsymbol{B}_1 & B_2 \\ \boldsymbol{C}_1 & \boldsymbol{D}_{11} & \boldsymbol{D}_{12} \\ C_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x[k] \\ \boldsymbol{w}[k] \\ u[k] \end{bmatrix} \tag{1}$$

where $x[k] := x_c(kh)$, $\boldsymbol{z} := \boldsymbol{W} z_c$, and $\boldsymbol{w} := \boldsymbol{W}^{-1} w_c$. Matrices and operators in (1) are defined as follows:

$$A := \mathrm{e}^{A_c h}, \quad B_2 := \int_0^h \mathrm{e}^{A_c \xi} B_{c2}\, \mathrm{d}\xi, \quad C_2 := C_{c2},$$

$$\boldsymbol{B}_1 v = \int_0^h \mathrm{e}^{A_c(h-\xi)} B_{c1} v(\xi)\, \mathrm{d}\xi,$$

$$(\boldsymbol{D}_{11} v)(\tau) = C_{c1} \int_0^\tau \mathrm{e}^{A_c(\tau-\xi)} B_{c1} v(\xi)\, \mathrm{d}\xi + D_{c11} v(\tau),$$

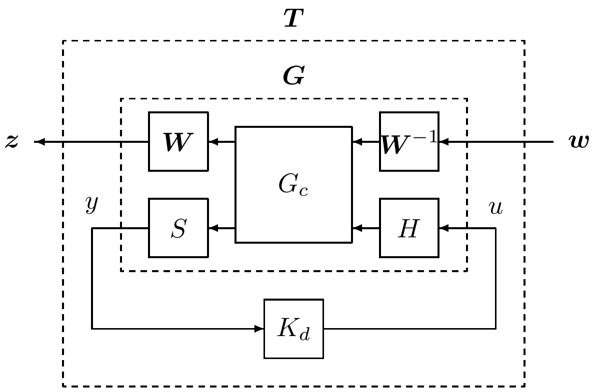

Fig. 2. Lifted System

$$(\bar{\boldsymbol{C}}\chi)(\tau) = \bar{C}_c\, \mathrm{e}^{\bar{A}_c \tau}\, \chi, \quad \bar{\boldsymbol{C}} := \begin{bmatrix} \boldsymbol{C}_1 & \boldsymbol{D}_{12} \end{bmatrix}$$

where $\tau \in [0,\, h)$ and

$$\bar{C}_c := \begin{bmatrix} C_{c1} & D_{c12} \end{bmatrix}, \quad \bar{A}_c := \begin{bmatrix} A_c & B_{c2} \\ 0 & 0 \end{bmatrix}.$$

We then get a state-space representation of $\boldsymbol{T}$:

$$\boldsymbol{T} : \begin{bmatrix} x[k+1] \\ x_d[k+1] \\ \hline \boldsymbol{z}[k] \end{bmatrix} = \begin{bmatrix} A_{c\ell} & \boldsymbol{B}_{c\ell} \\ \boldsymbol{C}_{c\ell} & \boldsymbol{D}_{11} \end{bmatrix} \begin{bmatrix} x[k] \\ x_d[k] \\ \hline \boldsymbol{w}[k] \end{bmatrix}$$

where

$$A_{c\ell} := \begin{bmatrix} A + B_2 D_d C_2 & B_2 C_d \\ B_d C_2 & A_d \end{bmatrix},$$

$$\boldsymbol{B}_{c\ell} := \begin{bmatrix} I \\ 0 \end{bmatrix} \boldsymbol{B}_1, \quad \boldsymbol{C}_{c\ell} := \bar{\boldsymbol{C}} \begin{bmatrix} I & 0 \\ D_d C_2 & C_d \end{bmatrix}.$$

Throughout of this paper, we assume the stability of T:

Assumption 1. The sampled-data system T is stable, i. e., all eigenvalues of $A_{c\ell}$ lie inside the unit circle.

The operator $\hat{\boldsymbol{T}}[z]$ acting on $\mathbf{L}_2[0,\, h]$ is called the transfer operator of the sampled-data system (Bamieh and Pearson, 1992; Yamamoto, 1994), where

$$\hat{\boldsymbol{T}}[z] := \boldsymbol{C}_{c\ell}(zI - A_{c\ell})^{-1} \boldsymbol{B}_{c\ell} + \boldsymbol{D}_{11}$$
$$= \bar{\boldsymbol{C}} \Phi[z] \boldsymbol{B}_1 + \boldsymbol{D}_{11},$$
$$\Phi[z] := \begin{bmatrix} I & 0 \\ D_K C_2 & C_K \end{bmatrix} (zI - A_{c\ell})^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix}.$$

Finally, the frequency response of sampled-data system is defined (Yamamoto and Khargonekar, 1996):

Definition 1. The frequency response operator of the sampled-data system is the operator $\hat{\boldsymbol{T}}[\mathrm{e}^{j\omega h}]$: $\mathbf{L}_2[0,\, h] \to \mathbf{L}_2[0,\, h]$ regarded as a function of $\omega \in [0,\, 2\pi/h)$. Its gain at ω is defined to be

$$\left\| \hat{\boldsymbol{T}}[\mathrm{e}^{j\omega h}] \right\| := \sup_{v \in \mathbf{L}_2[0,\, h]} \frac{\left\| \hat{\boldsymbol{T}}[\mathrm{e}^{j\omega h}] v \right\|_2}{\|v\|_2} \tag{2}$$

Remark 1. Another definition of frequency response gain in (Araki *et al.*, 1996) is equivalent to Definition 1 (Yamamoto and Araki, 1994).

The purpose of this paper is to give an answer to the following problem:

Problem 1. For given $\hat{\boldsymbol{T}}[z]$, determine if

$$\left\|\hat{\boldsymbol{T}}[\mathrm{e}^{j\omega h}]\right\| < 1 \tag{3}$$

hold.

We can construct a bisection algorithm to determine $\left\|\hat{\boldsymbol{T}}[\mathrm{e}^{j\omega h}]\right\|$ to any degree of accuracy if we can solve Problem 1 with normilization.

Remark 2. We do not assume any conditions on $\hat{T}[\mathrm{e}^{j\omega h}]$ in Problem 1. Existing results require $\|\boldsymbol{D}_{11}\| < 1$ (Hara *et al.*, 1995) or $\|\boldsymbol{D}_{11}\| \neq 1$ (Ito *et al.*, 2001). Note also that condition in (Ito *et al.*, 2001) needs the exact value of $\|\boldsymbol{D}_{11}\|$.

We know that

$$\left\|\hat{\boldsymbol{T}}[\mathrm{e}^{j\omega h}]\right\| \geq \|D_{c11}\|$$

and hence (3) does not hold if $\|D_{c11}\| \geq 1$, where $\|\cdot\|$ denotes the maximum singular value. Hence we will consider the case where the following condition holds in the sequel:

Assumption 2. The matrix D_{c11} satisfies

$$\|D_{c11}\| < 1.$$

3. MAIN RESULTS

In this section, we will given an answer to Problem 1 as a matrix positivity condition. The basic idea to solve Problem 1 in this paper is closely related to (Dullerud, 1999), where a computation method to determine if $\|\boldsymbol{D}_{11}\| < 1$ is provided. In fact, $\boldsymbol{D}_{11}$ is a special case of $\hat{\boldsymbol{T}}[\mathrm{e}^{j\omega h}]$.

We first note that (3) is equivalent to

$$\left((I - \hat{\boldsymbol{T}}^*[\mathrm{e}^{j\omega h}]\hat{\boldsymbol{T}}[\mathrm{e}^{j\omega h}])v,\, v\right) > 0 \tag{4}$$

for any $v \in \mathbf{L}_2[0,\, h]$, $v \neq 0$. In the sequel, we write

$$I - \hat{\boldsymbol{T}}^*[\mathrm{e}^{j\omega h}]\hat{\boldsymbol{T}}[\mathrm{e}^{j\omega h}] > 0 \tag{5}$$

instead of (4).

It is easy to see that

$$\hat{\boldsymbol{T}}^*[\mathrm{e}^{j\omega h}]\hat{\boldsymbol{T}}[\mathrm{e}^{j\omega h}] = \boldsymbol{D}_{11}^*\boldsymbol{D}_{11} + \begin{bmatrix} \bar{\boldsymbol{C}}^*\boldsymbol{D}_{11} \\ \boldsymbol{B}_1 \end{bmatrix}^* Q_1 \begin{bmatrix} \bar{\boldsymbol{C}}^*\boldsymbol{D}_{11} \\ \boldsymbol{B}_1 \end{bmatrix}$$

$$Q_1 := \begin{bmatrix} 0 & \Phi[\mathrm{e}^{j\omega h}] \\ \Phi^*[\mathrm{e}^{j\omega h}] & \Phi^*[\mathrm{e}^{j\omega h}]M\Phi[\mathrm{e}^{j\omega h}] \end{bmatrix},$$

$$M := \bar{\boldsymbol{C}}^*\bar{\boldsymbol{C}} = \int_0^h \mathrm{e}^{\bar{A}_c' t}\,\bar{C}_c'\bar{C}_c\,\mathrm{e}^{\bar{A}_c t}\;\mathrm{d}t.$$

For a fixed $\theta \in (-\pi,\, \pi]$, define an operator $\boldsymbol{\Psi}$: $\ell_2 \to \mathbf{L}_2[0,\, h]$ by

$$(\boldsymbol{\Psi}p)(\tau) = \sum_{k=0}^{\infty} \psi_k(\tau)p[k], \quad \psi_k(\tau) := h^{-\frac{1}{2}}\,\mathrm{e}^{j\varphi_k\tau}$$

where $\tau \in [0,\, h]$ and

$$\varphi_k := \frac{2\pi v_k + \theta}{h}, \quad v_k := \{0,\, 1,\, -1,\, 2,\, -2,\, \ldots\}.$$

Noting that $\{\psi_k\}$ is a complete orthonormal basis of $\mathbf{L}_2[0,\, h]$, $\boldsymbol{\Psi}$ is a unitary operator, namely

$$\boldsymbol{\Psi}^*\boldsymbol{\Psi} = I, \quad \boldsymbol{\Psi}\boldsymbol{\Psi}^* = I$$

hold where

$$(\boldsymbol{\Psi}^*q)[k] = \int_0^h \psi_k^*(t)q(t)\,\mathrm{d}t.$$

Consequently, (5) is equivalent to

$$I - \left(\boldsymbol{\Psi}^*\boldsymbol{D}_{11}^*\boldsymbol{D}_{11}\boldsymbol{\Psi} + \bar{\boldsymbol{P}}^*Q_1\bar{\boldsymbol{P}}\right) > 0 \tag{6}$$

where

$$\bar{\boldsymbol{P}} := \begin{bmatrix} \bar{\boldsymbol{C}}^*\boldsymbol{D}_{11} \\ \boldsymbol{B}_1 \end{bmatrix}\boldsymbol{\Psi}. \tag{7}$$

The operator $\bar{\boldsymbol{P}}$ has the following representation:

Lemma 1. Suppose that $\mathrm{e}^{j\theta} I - A$ is invertible and $\theta \neq 0$. Then one has

$$\bar{\boldsymbol{P}}v = \sum_{k=0}^{\infty} C_P(j\varphi_k I - A_P)^{-1}B_P v[k].$$

$$A_P := \begin{bmatrix} A_c & 0 \\ -\bar{C}_c'C_{c1} & -\bar{A}_c \end{bmatrix}, \quad B_P := \begin{bmatrix} B_{c1} \\ -\bar{C}_c'D_{c11} \end{bmatrix}$$

$$\bar{C}_P := \begin{bmatrix} -h^{-\frac{1}{2}}M_1 & -\mathrm{e}^{-j\theta}\,\bar{V}^* \\ V & 0 \end{bmatrix}, \quad M_1 := M\begin{bmatrix} I \\ 0 \end{bmatrix},$$

$$V := h^{-\frac{1}{2}}(\mathrm{e}^{j\theta} - A), \quad \bar{V} := h^{-\frac{1}{2}}(\mathrm{e}^{j\theta} - \bar{A}), \quad \bar{A} := \mathrm{e}^{\bar{A}_c h}.$$

Proof: The proof is done by straightforward computation of $\bar{P}\psi_k$.

Noting that

$$\int_0^h \mathrm{e}^{A_0 t}\;dt = A_0^{-1}(\mathrm{e}^{A_0 h} - I),$$

for a given invertible matrix A_0, we have

$$\boldsymbol{B}_1\psi_k = V(j\varphi_k I - A_c)^{-1}B_{c1},$$

$$\bar{\boldsymbol{C}}^* \boldsymbol{D}_{11} \psi_k = \left[\, 0 \;\; -\mathrm{e}^{-j\theta}\, \bar{V}^* \,\right] (j\varphi_k I - A_P)^{-1} B_P - h^{-\frac{1}{2}} M_1 (j\varphi_k I - A_c)^{-1} B_{c1} = \left[\, -h^{-\frac{1}{2}} M_1 \;\; -\mathrm{e}^{-j\theta}\, \bar{V}^* \,\right] (j\varphi_k I - A_P)^{-1} B_P.$$

Consequently we have

$$\begin{bmatrix} \boldsymbol{B}_1 \\ \bar{\boldsymbol{C}}^* \boldsymbol{D}_{11} \end{bmatrix} \psi_k = \bar{C}_P (j\varphi_k I - A_P)^{-1} B_P.$$

This completes the proof.

Similarly we manipulate $\boldsymbol{\Psi}^* \boldsymbol{D}_{11}^* \boldsymbol{D}_{11} \boldsymbol{\Psi}$ to get the following lemma. The proof is similar to that of Lemma 1, so it is omitted:

Lemma 2. Suppose that $\mathrm{e}^{j\theta} I - A$ is invertible. Then one has

$$\boldsymbol{\Psi}^* \boldsymbol{D}_{11}^* \boldsymbol{D}_{11} \boldsymbol{\Psi} = \boldsymbol{J} + \check{\boldsymbol{P}}^* Q_0 \check{\boldsymbol{P}}$$

where

$$\boldsymbol{J} : \ell_2 \to \ell_2; \quad (\boldsymbol{J} \upsilon)[k] := \hat{G}_{c11}^*(j\varphi_k) \hat{G}_{c11}(j\varphi_k) \upsilon[k],$$

$$\hat{G}_{c11}(s) := C_{c1}(sI - A)^{-1} B_{c1} + D_{c11},$$

$$Q_0 := \begin{bmatrix} -M_{11} & \mathrm{e}^{j\omega h} I - A \\ (\mathrm{e}^{j\omega h} I - A)^* & 0 \end{bmatrix}^{-1},$$

$$M_{11} := \boldsymbol{C}_1^* \boldsymbol{C}_1 = \int_0^h \mathrm{e}^{A_c' t}\, C_{c1}' C_{c1}\, \mathrm{e}^{A_c t}\, \mathrm{d}t = \left[\, I \;\; 0 \,\right] M_1,$$

$$\check{\boldsymbol{P}} \upsilon = \sum_{k=0}^{\infty} \check{C}_P (j\varphi_k I - \check{A}_P)^{-1} \check{B}_P \upsilon[k],$$

$$\check{A}_P := \begin{bmatrix} A_c & 0 \\ -C_c' C_{c1} & -A_c \end{bmatrix}, \quad \check{B}_P := \begin{bmatrix} B_{c1} \\ -C_c' D_{c11} \end{bmatrix},$$

$$\check{C}_P := \begin{bmatrix} 0 & -\mathrm{e}^{-j\theta}\, V^* \\ V & 0 \end{bmatrix}.$$

Remark 3. Lemma 2 is closely related to (Dullerud, 1999, Eq. (3)) where M_{11} is replaced by the sum of a certain matrix sequence. The formula there for computing the sum includes manipulation of complex numbers. In contrast, computation of M_{11} is simple.

Noting that

$$\begin{bmatrix} I & 0 \\ 0 & \left[\, I \;\; 0 \,\right] \end{bmatrix} (j\varphi_k I - A_P)^{-1} B_P = (j\varphi_k I - \check{A}_P)^{-1} \check{B}_P,$$

(6) is equivalent to

$$I - \boldsymbol{J} - \boldsymbol{P}^* Q \boldsymbol{P} > 0 \qquad (8)$$

when $\mathrm{e}^{j\theta} I - A$ is invertible and $\theta \neq 0$, where

$$\boldsymbol{P} \upsilon := \sum_{k=0}^{\infty} (j\varphi_k I - A_P)^{-1} B_P \upsilon[k],$$

$$Q := C_P^* \begin{bmatrix} Q_0 & 0 \\ 0 & Q_1 \end{bmatrix} C_P, \quad C_P := \begin{bmatrix} \check{C}_P \begin{bmatrix} I & 0 \\ 0 & \left[\, I \;\; 0 \,\right] \end{bmatrix} \\ \bar{C}_P \end{bmatrix}.$$

Now we will reduce the function space condition (8) into a finite dimensional condition: Let N be an integer such that

$$\left\| \hat{G}_c(j\varphi_k) \right\| < 1 \quad \text{for any} \quad k > N, \qquad (9)$$

where $\|\cdot\|$ denotes the maximum singular value. Such N exists if $\|D_{c11}\| < 1$ as we have assumed above, and this is a necessary condition for $\left\| \hat{\boldsymbol{T}}[\mathrm{e}^{j\omega h}] \right\| < 1$. A systematic search method for such N is found in (Dullerud, 1999, Section 4).

Define two projection operators $\boldsymbol{\Pi}$ and $\boldsymbol{\Pi}^\perp$ by:

$$(\boldsymbol{\Pi} \upsilon)[k] = \begin{cases} \upsilon[k]; & k = 0, 1, \ldots, N, \\ 0; & k = N+1, N+2, \ldots \end{cases},$$

$$(\boldsymbol{\Pi}^\perp \upsilon)[k] = \begin{cases} 0; & k = 0, 1, \ldots, N, \\ \upsilon[k]; & k = N+1, N+2, \ldots \end{cases}$$

It is trivial that

$$(\boldsymbol{\Pi} + \boldsymbol{\Pi}^\perp) \boldsymbol{J} (\boldsymbol{\Pi} + \boldsymbol{\Pi}^\perp) = \boldsymbol{J}_0 + \boldsymbol{J}_1,$$

$$\boldsymbol{J}_0 := \boldsymbol{\Pi} \boldsymbol{J} \boldsymbol{\Pi}, \quad \boldsymbol{J}_1 := \boldsymbol{\Pi}^\perp \boldsymbol{J} \boldsymbol{\Pi}^\perp.$$

Then (8) is equivalent to

$$I - \boldsymbol{\Xi} \left(\boldsymbol{J}_0 + \boldsymbol{P}^* Q \boldsymbol{P} \right) \boldsymbol{\Xi} > 0, \quad \boldsymbol{\Xi} := (I - \boldsymbol{J}_1)^{-\frac{1}{2}},$$

where (9) guarantees that the $\boldsymbol{\Xi}$ is well-defined. In fact $\boldsymbol{\Xi} > 0$ holds. We further transform it to

$$I - \left[\, \boldsymbol{\Pi} \;\; \boldsymbol{\Xi} \boldsymbol{P}_1^* \,\right] \begin{bmatrix} \boldsymbol{J}_0 + \boldsymbol{P}_0^* Q \boldsymbol{P}_0 & \boldsymbol{P}_0^* Q \\ Q \boldsymbol{P}_0 & Q \end{bmatrix} \begin{bmatrix} \boldsymbol{\Pi} \\ \boldsymbol{P}_1 \boldsymbol{\Xi} \end{bmatrix} > 0 \qquad (10)$$

where

$$\boldsymbol{P}_0 := \boldsymbol{P} \boldsymbol{\Pi}, \quad \boldsymbol{P}_1 := \boldsymbol{P} \boldsymbol{\Pi}^\perp.$$

Now we state one of the main results of this paper which provides a finite dimensional condition to check (3):

Theorem 1. Suppose that $\mathrm{e}^{j\theta} I - A$ is invertible and $\theta \neq 0$. Then (3) holds if and only if

$$I - \begin{bmatrix} J_0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} P_0^* \\ F' \end{bmatrix} Q \left[\, P_0 \;\; F \,\right] > 0 \qquad (11)$$

where J_0 and P_0 are defined by

$$J_0 := \mathrm{diag}(\hat{G}_{c11}(j\varphi_0), \hat{G}_{c11}(j\varphi_1), \ldots, \hat{G}_{c11}(j\varphi_N)),$$

$$P_0 := \left[\, P_{00} \;\; P_{01} \;\; \cdots \;\; P_{0N} \,\right], \quad P_{0k} := (j\varphi_k I - A_P)^{-1} B_P$$

respectively. F is any matrix satisfying

$$F F' = \boldsymbol{P}_1 (I - \boldsymbol{J}_1)^{-1} \boldsymbol{P}_1^*.$$

Proof: The procedure is essentially the same that in (Dullerud, 1999).

The condition (10) is equivalent to

$$I - \boldsymbol{E}^* \begin{bmatrix} \boldsymbol{J}_0 + \boldsymbol{P}_0^* Q \boldsymbol{P}_0 & \boldsymbol{P}_0^* Q \\ Q\boldsymbol{P}_0 & Q \end{bmatrix} \boldsymbol{E} > 0$$

if and only if

$$\begin{bmatrix} \boldsymbol{\Pi} \\ \boldsymbol{P}_1 \boldsymbol{\Xi} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Pi} & \boldsymbol{\Xi} \boldsymbol{P}_1^* \end{bmatrix} = \boldsymbol{E}\boldsymbol{E}^*$$

holds. It is obvious that such $\boldsymbol{E}$ is given by

$$\boldsymbol{E} = \begin{bmatrix} \boldsymbol{\Pi} & 0 \\ 0 & F \end{bmatrix}$$

and hence (10) is equivalent to

$$I - \left(\begin{bmatrix} \boldsymbol{J}_0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \boldsymbol{P}_0^* \\ F' \end{bmatrix} Q \begin{bmatrix} \boldsymbol{P}_0 & F \end{bmatrix} \right) > 0.$$

Finally, we see that matrix expressions of $\boldsymbol{J}_0$ and $\boldsymbol{P}_0$ are given by

$$\mathrm{diag}(J_0,\, 0,\, 0,\, \ldots), \quad \begin{bmatrix} P_0 & 0 & 0 & \cdots \end{bmatrix}$$

respectively, and this implies (11).

Theorem 1 reduces (3) to a finite dimensional condition. The rest of the task is to show how to compute F. In fact, computations of other elements in (11), J_0, P_0, and Q, are easily done by their definitions.

The next theorem gives a computational formula for $\boldsymbol{P}_1(I - \boldsymbol{J}_1)^{-1}\boldsymbol{P}_1$. We can compute F by using the formula:

Theorem 2. Suppose that $\mathrm{e}^{j\theta} I - A_J$ is invertible and $\theta \neq 0$ where

$$A_J := \check{A}_P + \check{B}_P C_J, \quad C_J := R^{-1} \begin{bmatrix} D_{c11}' C_{c1} & B_{c1}' \end{bmatrix},$$
$$R := I - D_{c11}' D_{c11}.$$

Then one has

$$\boldsymbol{P}_1(I - \boldsymbol{J}_1)^{-1}\boldsymbol{P}_1^* = h\,\mathrm{e}^{j\theta} \begin{bmatrix} I & 0 \end{bmatrix} (\mathrm{e}^{j\theta} - \mathrm{e}^{A_F h})^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} - \begin{bmatrix} I & 0 \end{bmatrix} \sum_{k=0}^{N} (j\varphi_k I - A_F)^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix}$$

where

$$A_F := \begin{bmatrix} A_P + B_P \begin{bmatrix} C_J & 0 \end{bmatrix} & -B_P R^{-1} B_P' \\ 0 & -A_P' \end{bmatrix}.$$

Proof: Noting that

$$\left((I - \boldsymbol{J}_1)^{-1} v\right)[k] = \left(C_J (j\varphi_k I - A_J)^{-1} \check{B}_P + I\right) R^{-1} v[k],$$

we get

$$\boldsymbol{P}_1(I - \boldsymbol{J}_1)^{-1}\boldsymbol{P}_1^* = \begin{bmatrix} I & 0 \end{bmatrix} \sum_{k=N+1}^{\infty} (j\varphi_k I - A_F)^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

Invoking (Dullerud, 1999, Proposition 5), we have

$$\sum_{k=0}^{\infty} (j\varphi_k I - A_F)^{-1} = \frac{h}{2} (\mathrm{e}^{j\theta} - \mathrm{e}^{A_F h})^{-1} (\mathrm{e}^{j\theta} + \mathrm{e}^{A_F h}).$$

Some manipulation implies

$$\begin{aligned} &\begin{bmatrix} I & 0 \end{bmatrix} (\mathrm{e}^{j\theta} - \mathrm{e}^{A_F h})^{-1} (\mathrm{e}^{j\theta} + \mathrm{e}^{A_F h}) \begin{bmatrix} 0 \\ I \end{bmatrix} \\ &= 2\,\mathrm{e}^{j\theta} \begin{bmatrix} I & 0 \end{bmatrix} (\mathrm{e}^{j\theta} - \mathrm{e}^{A_F h})^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix}. \end{aligned}$$

This completes the proof.

Remark 4. Theorems 1 and 2 are a generalization of results in (Dullerud, 1999). In fact, Theorems reduces to formulas in (Dullerud, 1999) if $\hat{\boldsymbol{T}}[\mathrm{e}^{j\omega h}] = \boldsymbol{D}_{11}$. By this mean, results in this paper unifies the computation of the frequency response gain of sampled-data systems and $\|\boldsymbol{D}_{11}\|$.

Remark 5. Existing method in (Ito *et al.*, 2001) requires computation of $\|\boldsymbol{D}_{11}\|$ before computing $\left\|\hat{\boldsymbol{T}}[\mathrm{e}^{j\omega h}]\right\|$. In contrast, the proposed method requires no such information.

4. CONCLUDING REMARKS

In this paper, we have proposed a condition to determine the frequency response gain of sampled-data system is less than a given positive number. As a summary of the results, the frequency response gain is computed in the following step:

Step 0: Fix $\omega \in [0,\, 2\pi)$. Given a upper bound γ_u and a lower bound γ_ℓ of $\left\|\hat{\boldsymbol{T}}[\mathrm{e}^{j\omega h}]\right\|$. See, e. g., (Fujioka and Ito, 2001) to get γ_u and γ_ℓ.
Step 1: Let $\gamma := (\gamma_u + \gamma_\ell)/2$.
Step 2: Scale $\hat{\boldsymbol{T}}[\mathrm{e}^{j\omega h}]$ in order to normalize γ.
Step 3: Fix θ such that $\mathrm{e}^{j\theta} I - A_J$ is invertible and $\theta \neq 0$. Fix N such that (9) holds.
Step 4: Compute J_0, P_0, Q and F. If (11) holds, update $\gamma_u = \gamma$. If not, update $\gamma_\ell = \gamma$. Go to Step 1.

In contrast to existing results (Hara *et al.*, 1995; Ito *et al.*, 2001), the derived condition does not assume any conditions on γ. Moreover, it does not require no information of the system such as $\|\boldsymbol{D}_{11}\|$ except the state-space data.

We also emphasize that the derived condition unifies the computation of the frequency response gain of sampled-data systems and the norm of state-space compression operators $\|\boldsymbol{D}_{11}\|$.

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