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CONSIDERATIONS ON THE ZERO-DYNAMICS OF PORT HAMILTONIAN SYSTEMS AND APPLICATION TO PASSIVE IMPLEMENTATION OF SLIDING-MODE CONTROL

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Abstract: In this paper, a passive control scheme for port Hamiltonian systems with dissipation (PHD) is presented. The control scheme is able to *conserve* the PHD structure of the system when constrained on a sub-manifold of the state space. The idea is to modify both the interconnection and damping structures of the system and to add a proper dynamical extension in such a way that the constraint can be related to some dynamical invariants of the resulting closed-loop system. Since part of the structure of this dynamical extension can be arbitrarily chosen, it is also possible to drive the state of the system on the constraint and to obtain the dynamical behavior defined by the constraint. For example, if a proper variable structure dynamical extension is chosen, it is possible to achieve a sliding-mode behavior that can be interpreted by means of energetic considerations.

Keywords: Hamiltonian systems, Casimir functions, constrained dynamics, sliding mode

1. INTRODUCTION

This paper is divided into two distinct parts. In the first, some considerations about the zero-dynamics of port Hamiltonian systems with dissipation (PHD) (Maschke and van der Schaft, 1992), (van der Schaft, 1999) are presented. In (Nijmeijer and van der Schaft, 1991), a detailed analysis of the zero-dynamics of Hamiltonian system is described, assuming that no dissipation effects are present and that the input signals modulate Hamiltonian vector fields that are strictly related to the constraints on which the zero-dynamics is calculated. Moreover, in (van der Schaft, 1999), the reduced dynamics of a mechanical system expressed in PHD form and subject to holonomic constraints can be found. The starting point is a PHD system and a generic constraint defined on the state space. In the first part of this paper, a passive controller guaranteeing that the closed-loop PHD dynamics satisfying the constraints is still representable in PHD formalism is introduced. This can be achieved by relating the constraints to some dynamical invariants (Casimir functions) of the closed-loop system.

Since the constraints can be seen as a sliding surface in the framework of the sliding-mode control technique, in the second part of this paper the development of an *energy-based* variable-structure control characterized by a sliding-mode behavior is presented (Utkin, 1978). Given a non-linear system in *classical* state space representation and a constraint defined on the state variables, in (Sira-Ramirez, 1999) it is shown that the action of the drift vector with respect to the constraint is strictly connected to the energetic structure of the system. In particular, the fact that the system spontaneously moves toward the constraints can be interpreted as the results of dissipation effects or, on the other hand, the fact that the system diverges can be related to the presence of regenerative effects. In this work, it is pointed out that a proper modification of the *structure* of the plant (in particular of the dissipative part) is needed for reaching the sliding surface and therefore obtaining a PHD constrained dynamics.

In Sec. 2, some considerations about the constrained dynamics of PHD systems are discussed, and a control scheme guaranteeing that the constrained dynamics is still representable within the PHD formalism is illustrated. Moreover, it is shown how to obtain the attractiveness of the constraint by using a passive controller. In Sec. 3, an application of these ideas is presented. In particular, a passive implementation of sliding-mode controller is illustrated. Sec. 4 concludes with final remarks.

2. DYNAMICS OF A PHD UNDER CONSTRAINTS

Let us consider the port Hamiltonian system with dissipation (PHD system)

$$\begin{cases} \dot{x} = [J(x) - R(x)]\,\dfrac{\partial H}{\partial x} + G(x)u \\ y = G^T(x)\dfrac{\partial H}{\partial x} \end{cases} \tag{1}$$

where $x \in \mathcal{X}$, $u \in \mathcal{U} \subset \mathcal{R}^m$, $y \in \mathcal{Y} \equiv \mathcal{U}^*$, being $\mathcal{U}^*$ the dual space of $\mathcal{U}$, and $H : \mathcal{X} \to \mathcal{R}$ the energy function, bounded from below. Moreover, $J(x) = -J^T(x)$ and $R(x) = R^T(x) \geq 0$, $G(x) = [g_1(x) \cdots g_m(x)]$, with $\text{rank}[G(x)] = m$, $\forall x \in \mathcal{X}$. Finally, it is assumed that $\dim(\mathcal{X}) = n$ and $m \leq n$.

Assume that $S_i : \mathcal{X} \to \mathcal{R}$, $i = 1, \ldots, m$ is a set of function defined on the state manifold and, for each S_i, let us define the following state sub-manifolds:

$$\begin{aligned} \mathcal{S}_{i,0} &\triangleq \{x \in \mathcal{X} \,|\, S_i(x) = 0\} \\ \mathcal{S}_{i,+} &\triangleq \{x \in \mathcal{X} \,|\, S_i(x) > 0\} \\ \mathcal{S}_{i,-} &\triangleq \{x \in \mathcal{X} \,|\, S_i(x) < 0\} \end{aligned} \tag{2}$$

Moreover, let us define

$$S(x) \triangleq \left[\, S_1(x) \;\cdots\; S_m(x) \,\right]^T$$

If

$$\text{rank}\left[\frac{\partial S}{\partial x}\right] = \text{rank}\left[\frac{\partial S_1}{\partial x} \cdots \frac{\partial S_m}{\partial x}\right] = m$$

$\forall x \in \mathcal{X}$, then the state sub-manifold $S(x) = 0$, that is

$$\mathcal{S}_0 \triangleq \bigcap_{i=1}^{m} \mathcal{S}_{i,0}$$

is $(n-m)$-dimensional.

By introducing a state-feedback law, that properly modifies the interconnection and damping matrices of system (1), and by interconnecting the resulting system with another PHD system in a power-conserving manner, it is possible to obtain a *new* system such that, *if* constrained on $\mathcal{S}_0$, it can be still described in the PHD formalism.

In the following, it will be assumed that the *transversality condition* (Sira-Ramirez, 1988)

$$\text{rank}\left(\frac{\partial^T S}{\partial x} G\right) = m \tag{3}$$

holds $\forall x \in \mathcal{X}$. From a sliding mode point of view, this means that $\mathcal{S}_0$ has locally relative degree one in $\mathcal{X}$.

Consider the following PHD system (controller)

$$\begin{cases} \dot{\xi} = [J_c(\xi) - R_c(\xi)]\dfrac{\partial H_c}{\partial \xi} + G_c(\xi)u_c \\ y_c = G_c^T(\xi)\dfrac{\partial H_c}{\partial \xi} \end{cases} \tag{4}$$

with $H_c(\xi)$ the *arbitrary* energy function, $\xi \in \mathcal{X}_c$, $u_c \in \mathcal{U}_c \equiv \mathcal{Y}$ and $y_c \in \mathcal{Y}_c \equiv \mathcal{U}$. This system has to be connected to system (1) via power-conserving feedback interconnection.

We see in (van der Schaft, 1999) that the resulting system is still PHD and that, if some conditions on the interconnection and damping matrices of (1) and (4) are satisfied, it is possible to relate the state variables of the controller to the state variables of the plant by using *Casimir functions*. In particular, we want that $\xi_i - S_i(x)$, $i = 1, \ldots, m$ is a set of Casimir function for the closed-loop system. Clearly, it is enough to choose $\dim(\mathcal{X}_c) = m$ since we have m sliding surfaces.

First of all, it is necessary to properly modify the interconnection and damping matrices of the system (1) by means of a state feedback action $u = \beta(x)$. In particular, as suggested in (Ortega *et al.*, 2000) for the IDA-PBC design technique, assume that there exist matrices $J_a(x) = -J_a^T(x)$

and $R_a(x) = R_a^T(x) \geq 0$ and two function $H_a : \mathcal{X} \to \mathcal{R}$ and $\beta : \mathcal{X} \to \mathcal{U}$, such that the following PDE holds

$$\begin{aligned} & [J + J_a - (R + R_a)] \frac{\partial H_a}{\partial x} = \\ = \quad & -[J_a - R_a] \frac{\partial H}{\partial x} + G\beta \end{aligned} \tag{5}$$

By now, define $H_d(x) \triangleq H(x) + H_a(x)$, $J_d(x) \triangleq J(x) + J_a(x)$ and $R_d(x) \triangleq R(x) + R_a(x)$; then the matrices J_a and R_a together with the parameters of the controller (4) will be chosen in such a way that the following conditions hold, see (van der Schaft, 1999):

$$\frac{\partial^T S}{\partial x}(x) J_d(x) \frac{\partial S}{\partial x}(x) = J_c(\xi) \tag{6a}$$

$$R_d(x) \frac{\partial S}{\partial x}(x) = 0 \tag{6b}$$

$$R_c(\xi) = 0 \tag{6c}$$

$$\frac{\partial^T S}{\partial x}(x) J_d(x) = G_c(\xi) G(x)^T \tag{6d}$$

If in (1) we impose $u = \beta(x) + u'$, from (5) we obtain the following PHD system with interconnection and damping matrices that satisfies conditions (6):

$$\begin{cases} \dot{x} = [J_d(x) - R_d(x)] \dfrac{\partial H_d}{\partial x}(x) + G(x) u' \\ y' = G^T(x) \dfrac{\partial H_d}{\partial x}(x) \end{cases} \tag{7}$$

Given the systems (4) and (7) and one of the two possible power-conserving feedback interconnections (Stramigioli, 2001), for example the following:

$$\begin{cases} u' = -y_c + e \\ u_c = y' + e_c \end{cases}$$

with $e_c \in \mathcal{Y}$ and $e \in \mathcal{U}$ external signals, it is possible to show that each $\xi_i - S_i(x)$, $i = 1, \ldots, m$ is a Casimir function for the closed loop system, see again (van der Schaft, 1999). In particular, the reduced dynamics on the foliation induced by the Casimir functions will be given (if $e_c = 0$) by

$$\begin{aligned} \dot{x} = {} & (J_d - R_d) \frac{\partial}{\partial x}(H + H_a) \\ & + (J_d - R_d) \frac{\partial}{\partial x} H_c(S_1, \ldots, S_m) + Ge \end{aligned} \tag{8}$$

where the energy function $H_c : \mathcal{X}_c \to \mathcal{R}$ of the controller (4) can be arbitrary. In particular, it can be chosen in such a way that the sub-manifold $\mathcal{S}_0$ can be reached. Clearly, the power conjugated output will be given by

$$y'' = G^T \frac{\partial}{\partial x}[H + H_a + H_c(S_1, \ldots, S_m)] \tag{9}$$

The system (8) naturally evolves in such a way that $\xi_i - S_i(x) = \text{cons.}$, $\forall i = 1, \ldots, m$. Moreover, if a damping action is injected by imposing $e = -Ky''$, with $K = K^T > 0$, from (6b) the resulting dynamics is given by:

$$\begin{aligned} \dot{x} = {} & [J_d - (R_d + GKG^T)] \frac{\partial}{\partial x}(H + H_a) \\ & + (J_d - GKG^T) \frac{\partial S}{\partial x} \frac{\partial H_c}{\partial S} \end{aligned} \tag{10}$$

where

$$\frac{\partial H_c}{\partial S} = \begin{bmatrix} \dfrac{\partial H_c}{\partial S_1} & \cdots & \dfrac{\partial H_c}{\partial S_m} \end{bmatrix}^T$$

It is easy to prove that, with a proper choice of $H_c(\cdot)$, the state can be brought on $\mathcal{S}_0$. First of all, consider the vector function (1-form) $v : \mathcal{R}^m \to \mathcal{R}^m$ and rewrite (10) as:

$$\begin{aligned} \dot{x} = {} & [J_d - (R_d + GKG^T)] \frac{\partial}{\partial x}(H + H_a) \\ & + (J_d - GKG^T) \frac{\partial S}{\partial x} v \end{aligned} \tag{11}$$

If $V(x) \triangleq \frac{1}{2} S^T(x) S(x)$ is considered as a *Lyapunov function* that measures the state *distance* from the sub-manifold $\mathcal{S}_0$, it is sufficient to prove that $\dot{V}(x) < 0$ on the trajectory of the closed-loop system (11) for a proper choice of v. Since $\dot{V}(x) = S^T(x) \dfrac{\partial^T S}{\partial x} \dot{x}$, from (10), (6a) and (6b) we have

$$\begin{aligned} \dot{V} = S^T \Big\{ & \frac{\partial^T S}{\partial x} \left[J_d - GKG^T \right] \frac{\partial}{\partial x} (H + H_a) \\ & + \left[J_c - \frac{\partial^T S}{\partial x} GKG^T \frac{\partial S}{\partial x} \right] v \Big\} \end{aligned}$$

From (3), and since the damping matrix K is positive definite, it is possible to show that

$$\text{rank} \; \frac{\partial^T S}{\partial x} GKG^T \frac{\partial S}{\partial x} = m$$

and that

$$\text{rank} \; \left[J_c - \frac{\partial^T S}{\partial x} GKG^T \frac{\partial S}{\partial x} \right] = m \tag{12}$$

because J_c is skew-symmetric. As a consequence, by properly choosing the vector function v on the (controller) state space $\mathcal{X}_c$, the inequality $\dot{V}(x) < 0$ can be satisfied.

The presence of *non-singular* damping injection is sufficient to assure that the state will tend to $\mathcal{S}_0$. If the 1-form v is closed, or, equivalently, if it is a gradient of a scalar function, that is

$$\frac{\partial^T v}{\partial S} = \frac{\partial v}{\partial S}$$

and the set $\{s = \mathcal{S}(x) \,|\, x \in \mathcal{X}\} \subseteq \mathcal{R}^m$ is a contractile manifold, then an energy function $H_c(\cdot)$ for the controller (4) can be found such that

$$\frac{\partial H_c}{\partial S} = v(S)$$

guaranteeing the reaching of the sub-manifold $\mathcal{S}_0$.

In Fig. 1, a visual description of the behavior of this control scheme is presented. The state of the

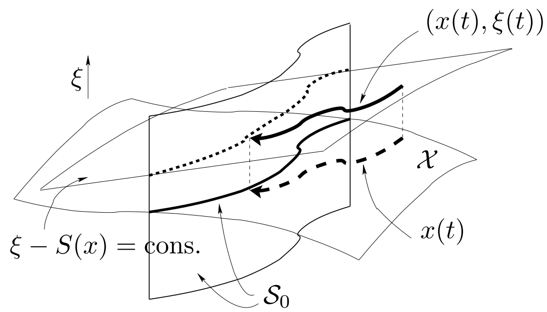

Fig. 1. Behavior of the proposed control scheme in the case of dim$\mathcal{X} = 2$ and $m = 1$.

closed-loop system $(x, \xi) \in \mathcal{X} \times \mathcal{X}_c$, evolves on the sub-manifold defined by the Casimir function $\xi - S(x) = $ cons.. A proper choice of the energy function $H_c(\cdot)$ of the controller and the action of the damping injection will bring the state on $\mathcal{S}_0 \subset \mathcal{X} \times \mathcal{X}_c$, that is the state x of the plant will be constrained on $S(x) = 0$.

Moreover, it is possible to prove that the dynamics of the system (10) constrained on S_0 is described by a PHD system. This fact can be seen as a generalization of (Nijmeijer and van der Schaft, 1991), where no dissipation term is present in the Hamiltonian model of the plant.

It is important to notice that the (*orthogonality*) condition (6b) imposes some constraints on the structure of the dissipation term in (1), in relation to the sub-manifold $\mathcal{S}_0$. In particular, the damping matrix $R(x)$ in (1) is *compatible* with the sub-manifold $\mathcal{S}_0$ if and only if it is possible to find a linear, symmetric and positive semi-definite operator $R_a(x)$ such that $R_d(x) \triangleq R(x) + R_a(x)$ satisfies condition (6b).

The constrained dynamics of (10), or equivalently of (11), on $\mathcal{S}_0$ can be seen as a zero-dynamics on $S(x) = 0$. The *equivalent* control input v_{equiv} that constrains the system (11) on $\mathcal{S}_0$ can be calculated by imposing that $\frac{\partial^T S}{\partial x}\dot{x} = 0$ and by substituting the x dynamics given by (11). Since (6b) and (12) hold, it follows that

$$\begin{aligned} v_{equiv} = &-\left[\frac{\partial^T S}{\partial x}\left(J_d - GKG^T\right)\frac{\partial S}{\partial x}\right]^{-1} \cdot \\ &\cdot \frac{\partial^T S}{\partial x}\left(J_d - GKG^T\right)\frac{\partial}{\partial x}\left(H + H_a\right) \end{aligned} \tag{13}$$

From (6d), we can write

$$\begin{aligned} &\left[\frac{\partial^T S}{\partial x}\left(J_d - GKG^T\right)\frac{\partial S}{\partial x}\right]^{-1} = \\ &= \left(G^T\frac{\partial S}{\partial x}\right)^{-1}\left(G_c - \frac{\partial^T S}{\partial x}GK\right)^{-1} \end{aligned}$$

and

$$\frac{\partial^T S}{\partial x}\left(J_d - GKG^T\right) = \left(G_c - \frac{\partial^T S}{\partial x}GK\right)G^T$$

that, if substituted in (13), give the following expression for the *equivalent* control:

$$v_{equiv} = \left(G^T\frac{\partial S}{\partial x}\right)^{-1} G^T \frac{\partial}{\partial x}(H + H_a) \tag{14}$$

Finally, from (14) and (11), the constrained dynamics expression can be deduced:

$$\begin{aligned} \dot{x} = &\, (J_d - R_d)\frac{\partial}{\partial x}(H + H_a) \\ &- J_d\frac{\partial S}{\partial x}\left(G^T\frac{\partial S}{\partial x}\right)^{-1} G^T\frac{\partial}{\partial x}(H + H_a) \end{aligned} \tag{15}$$

It is easy to prove that the matrix

$$-J_d\frac{\partial S}{\partial x}\left(G^T\frac{\partial S}{\partial x}\right)^{-1} G^T$$

is skew-symmetric. From (6a) and (6d) we have that

$$\begin{aligned} -J_d\frac{\partial S}{\partial x} &= GG_c^T \\ G_c^T &= -\left(G^T\frac{\partial S}{\partial x}\right)^{-T} J_c \end{aligned}$$

and consequently

$$\begin{aligned} &-J_d\frac{\partial S}{\partial x}\left(G^T\frac{\partial S}{\partial x}\right)^{-1} G^T = \\ &= -G^T\left(G^T\frac{\partial S}{\partial x}\right)^{-T} J_c \left(G^T\frac{\partial S}{\partial x}\right)^{-1} G^T \triangleq \tilde{J} \end{aligned}$$

with $\tilde{J} = -\tilde{J}^T$, since $J_c = -J_c^T$.

As a consequence, if we define $J_{d,0}(x) \triangleq J_d(x) + \tilde{J}(x)$ and $R_{d,0}(x) \triangleq R_d(x)$, the constrained dynamics can be written in the following PHD system form

$$\dot{x} = \left[J_{d,0}(x) - R_{d,0}(x)\right]\frac{\partial}{\partial x}\left[H(x) + H_a(x)\right] \tag{16}$$

Remark. The zero-dynamics (16) is given in terms of the evolution of an n-dimensional state. Since the system is constrained on the $n - m$ dimensional sub-manifold $\mathcal{S}_0$, the achieved dynamics can be described by a system of order $n - m$. In other words, a proper state coordinate transformation that explicitly show the $n - m$ order constrained dynamics can be defined. If

$$z = \Phi(x) \triangleq \begin{bmatrix} S(x) \\ T(x) \end{bmatrix} \text{ with rank}\left(\frac{\partial \Phi}{\partial x}\right) = n$$

is a well-defined coordinate transformation, the system dynamics (16) can be expressed in the *new* coordinates as

$$\dot{z} = \left[\bar{J}_{d,0}(z) - \bar{R}_{d,0}(z)\right]\frac{\partial}{\partial z}\left[\bar{H}(z) + \bar{H}_a(z)\right]$$

where

$$\begin{aligned} \bar{J}_{d,0} &= \frac{\partial^T \Phi}{\partial x} J_{d,0}\frac{\partial \Phi}{\partial x} \\ \bar{R}_{d,0} &= \frac{\partial^T \Phi}{\partial x} R_{d,0}\frac{\partial \Phi}{\partial x} \\ H(x) &= \bar{H}[\Phi(x)] \\ H_a(x) &= \bar{H}_a[\Phi(x)] \end{aligned}$$

Since $\frac{\partial \Phi}{\partial x} = \left[\frac{\partial S}{\partial x}, \frac{\partial T}{\partial x}\right]$, it follows

$$\bar{J}_{d,0} = \begin{bmatrix} \bar{J}_{d,0}^{(1,1)} & \bar{J}_{d,0}^{(1,2)} \\ -\bar{J}_{d,0}^{(1,2)\,T} & \bar{J}_{d,0}^{(2,2)} \end{bmatrix}$$

$$\bar{R}_{d,0} = \begin{bmatrix} \bar{R}_{d,0}^{(1,1)} & \bar{R}_{d,0}^{(1,2)} \\ \bar{R}_{d,0}^{(1,2)\,T} & \bar{R}_{d,0}^{(2,2)} \end{bmatrix}$$

with, in particular,

$$\begin{aligned} \bar{J}_{d,0}^{(1,1)} &= \frac{\partial^T S}{\partial x} J_d \frac{\partial S}{\partial x} - J_c = 0 \\ \bar{J}_{d,0}^{(1,2)} &= \left[\frac{\partial^T S}{\partial x} J_d - J_c \left(G^T \frac{\partial S}{\partial x}\right)^{-1} G^T\right] \frac{\partial T}{\partial x} \\ &= \left[G_c G^T - G_c G^T \frac{\partial S}{\partial x}\left(G^T \frac{\partial S}{\partial x}\right)^{-1} G^T\right] \frac{\partial T}{\partial x} \\ &= 0 \\ \bar{R}_{d,0}^{(1,1)} &= \frac{\partial^T S}{\partial x} R_d \frac{\partial S}{\partial x} = 0 \\ \bar{R}_{d,0}^{(1,2)} &= \frac{\partial^T S}{\partial x} R_d \frac{\partial T}{\partial x} = 0 \end{aligned}$$

as can be deduced from (6). It follows that

$$\bar{J}_{d,0} = \begin{bmatrix} 0_{m\times m} & 0_{m\times(n-m)} \\ 0_{(n-m)\times m} & \star\star \end{bmatrix}$$

$$\bar{R}_{d,0} = \begin{bmatrix} 0_{m\times m} & 0_{m\times(n-m)} \\ 0_{(n-m)\times m} & \star\star \end{bmatrix}$$

where '$\star\star$' indicates a quantity (in general) different from 0. These interconnection and damping matrices clearly define a dynamics of order $n-m$.

Remark. Suppose that $m = n$, that is the submanifold $\mathcal{S}_0$ reduces to a point. We have seen that

$$J_{d,0} = J_d \left[\mathcal{I}_{n\times n} - \frac{\partial S}{\partial x}\left(G^T \frac{\partial S}{\partial x}\right)^{-1} G^T\right]$$

where

$$\frac{\partial S}{\partial x}\left(G^T \frac{\partial S}{\partial x}\right)^{-1} G^T = \mathcal{I}_{n\times n}$$

since it is idempotent and non-singular, and then $J_{d,0} = 0$. Moreover, from (6b) and from the independence of the functions $S_1, \dots, S_m$, also $R_{d,0}(x) = 0$. Therefore, the constrained dynamics becomes $\dot{x} = 0$, that is coherent with the fact that the control action tries to keep the state in a specific point.

3. ENERGY-BASED APPROACH TO SLIDING-MODE

With the dynamical extension (4), the system (1) is characterized by a set of Casimir functions which are strictly related to the sliding surfaces $S_1(x), \dots, S_m(x)$. By properly choosing the energy function $H_c(S_1, \dots, S_m)$ of the controller (4) it is possible to constrain the state of the system (1) on $\mathcal{S}_0$ in a passive way.

The same behavior of *classical* sliding-mode controllers (and clearly the same robustness properties) can be achieved if the function $H_c(\cdot)$ is characterized by a variable structure. Since this function is also an energy function, it *must* be at least of $\mathcal{C}^0$ class. Moreover, the energy function of the controller $H_c(\cdot)$ has to be chosen in such a way that the state of the plant (1) reaches $\mathcal{S}_0$, after a (finite) *reaching* phase, and that a sliding regime is possible on it. This means that the sliding surface has to become *attractive* for the closed-loop system.

In the case of several sliding surfaces, the problem is more complex, also without introducing energy constraints into the controller design. This is due, in particular, to the couplings that are, in general, present in the dynamics of the system (Sira-Ramirez, 1988).
If only one sliding surface is given, it is easy to find explicit solutions for control laws that assure the completion of the reaching phase and the *stability* of the sliding mode, also within the framework presented in this paper.

An explicit solution in terms of a set of $2m$ inequalities that involve $\partial H_c/\partial S_i$, $i = 1, \dots, m$, can be found under some hypotheses. It is interesting to point out that, under these hypothesis, the dynamical system (4) will have a *nice* physical interpretation. First of all, suppose that

$$\begin{gathered} \forall x \in \mathcal{X},\ \mathrm{L}_{g_j} S_i(x) = 0 \text{ if } i \neq j \text{ and} \\ \mathrm{L}_{g_i} S_i(x) \neq 0,\ \forall i, j = 1, \dots, m \end{gathered} \tag{17}$$

Clearly, this assumption is compatible with the hypothesis (3). In this case we have that

$$\frac{\partial^T S}{\partial x} G(x) = \mathrm{diag}\left[\mathrm{L}_{g_1} S_1(x), \dots, \mathrm{L}_{g_m} S_m(x)\right]$$

is non singular. Condition (17) requires a decoupling between input signals and sliding surfaces: this means that each input signal can drive the state variable only on one sliding surface. Moreover, in (10) we choose $K = \mathrm{diag}[k_1, \dots, k_m]$, with $k_i > 0$, $i = 1, \dots, m$ and finally in (4-6a) we suppose that is possible to choose $J_c(\xi) = 0$.

Since we want to design a controller that is able to bring the state of the system on the intersection of m independent sliding surfaces, it is important to determine what is the behavior of the controlled system in relation to each sliding surface $S_i(x) = 0$. Given the closed-loop dynamics (10) and under the previous hypotheses, we have that

$$\mathrm{L}_{\dot{x}} S_i = \frac{\partial^T S_i}{\partial x} \dot{x} = \\ = \left[\frac{\partial^T S_i}{\partial x} J_d - k_i \mathrm{L}_{g_i} S_i g_i^T \right] \frac{\partial}{\partial x}(H + H_a) \\ -k_i \left(\mathrm{L}_{g_i} S_i\right)^2 \frac{\partial H_c}{\partial S_i} \tag{18}$$

The reaching phase can be completed and the sliding mode is possible on the sliding surface if

$$\begin{array}{l} \mathrm{L}_{\dot{x}} S_i(x) < 0, \ \forall x \in \mathcal{S}_{i,+} \\ \mathrm{L}_{\dot{x}} S_i(x) > 0, \ \forall x \in \mathcal{S}_{i,-} \end{array} \quad i = 1, \ldots, m \tag{19}$$

Then, define

$$H_c[S_1(x), \ldots, S_m(x)] \triangleq \sum_{i=1}^{m} H_{c,i}[S_i(x)] \tag{20}$$

with

$$H_{c,i}[S_i(x)] \triangleq \begin{cases} H_{c,i}^+[S_i(x)] \text{ if } x \in \mathcal{S}_{i,+} \cup \mathcal{S}_{i,0} \\ H_{c,i}^-[S_i(x)] \text{ if } x \in \mathcal{S}_{i,-} \end{cases}$$

such that $H_{c,i}^+(0) \equiv H_{c,i}^-(0)$, $i = 1, \ldots, m$ so that the energy function of the controller is continuous. From (19) and (18), it follows that the sliding mode is *possible* on $S(x) = 0$ if, $\forall i = 1, \ldots, m$:

$$\frac{\partial H_{c,i}^+}{\partial S_i} > \\ \frac{1}{k_i \left(\mathrm{L}_{g_i} S_i\right)^2} \left[\frac{\partial^T S_i}{\partial x} J_d - k_i \mathrm{L}_{g_i} S_i g_i^T \right] \frac{\partial}{\partial x}(H + H_a) \tag{21a}$$

$\forall x \in \mathcal{S}_{i,+}$, and if

$$\frac{\partial H_{c,i}^-}{\partial S_i} < \\ \frac{1}{k_i \left(\mathrm{L}_{g_i} S_i\right)^2} \left[\frac{\partial^T S_i}{\partial x} J_d - k_i \mathrm{L}_{g_i} S_i g_i^T \right] \frac{\partial}{\partial x}(H + H_a) \tag{21b}$$

$\forall x \in \mathcal{S}_{i,-}$.

If the set (21) of $2m$ independent inequalities in the $2m$ unknown functions

$$\frac{\partial H_{c,1}^+}{\partial x}, \ldots, \frac{\partial H_{c,m}^+}{\partial x}, \frac{\partial H_{c,1}^-}{\partial x}, \ldots, \frac{\partial H_{c,m}^-}{\partial x}$$

can be satisfied, then the system will reach the sub-manifold $\mathcal{S}_0$ and then evolve on it according to the dynamics defined by (16).

Remark. Since the dynamical extension (4) of the plant (1) has both the interconnection and damping matrices equal to zero, that is $J_c = R_c = 0$, and an energy function given by (20), it is easy to see that the action of this dynamical system can be interpreted as the action of a set of m nonlinear *springs*, each with center of stiffness on a corresponding sub-manifold $\mathcal{S}_{i,0}$.

4. CONCLUSIONS

In this paper, the problem of the constrained dynamics for PHD systems is discussed. A passive control scheme that assures that the resulting constrained dynamics for a PHD system is still representable in PHD form is presented. The proposed feedback law relates the given constraints to dynamical invariants of the closed-loop system. In this framework, the action of the constraints over the plant can be seen as a modification of its interconnection structure, as it is pointed out in (16). By exploiting the degrees of freedom available thanks to the dynamical extension, it is possible to satisfy the constraints on the system state in a passive way. Depending on the structure of this dynamical extension, it is possible to implement different control schemes. If we choose a variable structure implementation, a sliding mode behavior can be achieved, as pointed out in Sec. 3.

Future work will address both the implementation of this control technique and its extension to more general cases. In particular, it appears of interest to remove condition (6b) that limits practical implementations of the presented control method, since it imposes strong requirements on the dissipation structure of the plant or, equivalently, limits the class of constraints that can be considered.

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