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ROBUSTNESS ANALYSIS OF A RELAY OSCILLATOR

Subbarao Varigonda

United Technologies Research Center, East Hartford, CT, USA.
Email: varigos@utrc.utc.com

Abstract: Relaxation oscillators typically consist of a hysteresis element and a dynamic feedback. The robustness of oscillations in such an interconnection is of interest in many applications. In particular, we seek the robustness margin on the component perturbations in the gap metric so that oscillatory behavior of the closed loop is preserved. A new distance measure for oscillatory signals utilizing time scaling is introduced for this purpose. The robustness of a system consisting of an ideal relay with hysteresis and an integrator has been studied in (Georgiou and Smith 2000). Here analogous results are derived for a system consisting of an integrator and a relay whose branches have non-zero slope. This system resembles the classical van der Pol oscillator more closely than the ideal relay system.

Keywords: Relaxation oscillation, relays, hysteresis, robustness

1. INTRODUCTION

Oscillatory systems are omnipresent in nature and have been a subject of intense investigation (Jackson 1991, Guckenheimer and Holmes 1983, Strogatz and Stewart 1993). Yet, there remain several questions to be answered. Here, we are interested in the robustness of an autonomous oscillator to internal changes in the system. There have been numerous studies on how variation of parameters in a model of an oscillator can affect the behavior. However, we seek a more general approach that relies on the internal structure found in most oscillators. A large class of oscillators, especially relaxation oscillators, can be represented as an interconnection of a hysteretic subsystem with a dynamic feedback (Varigonda 2001). Often, a feedback connection of a static hystresis and a linear subsystem yields an oscillator sufficient to model complex mathematical as well as natural phenomena (Jackson 1991, Grasman 1984, Scott 1994, Varigonda 2001). The geometric robustness theory of feedback control systems provides a qualitatively and quantitatively elegant characterization of robustness (Foias *et al.* 1993, Georgiou and Smith 1990, Georgiou and Smith 1997). This theory also elucidates the equivalence of robustness to system model perturbations and robustness to external disturbances.

Considering oscillators as feedback systems enables the tools of robustness analysis from control theory to be employed. A first effort in this direction has been made in (Georgiou and Smith 2000) where the robustness margin (*i.e.,* the maximum size of the perturbation in a subsystem that can be tolerated) of a relaxation oscillator consisting of an ideal (on–off) relay hysteresis and a negative integral feedback was computed.

This work extends the analogy to a more complex relay oscillator, we call, the *sloping relay oscillator*. One of the main issues with an on-off relay oscillator, as pointed out in (Georgiou and Smith 2000), is that the response of the system can be unbounded even when the disturbances are bounded. The parallel projection operator from

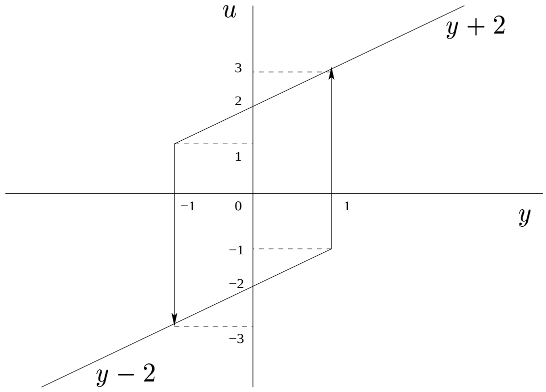

Fig. 1. The input-output characteristic of a modified relay hysteresis

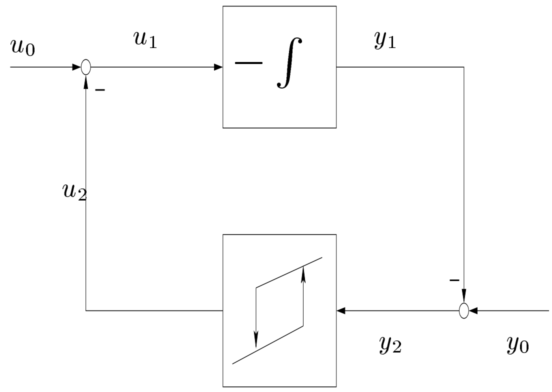

Fig. 2. The sloping relay oscillator

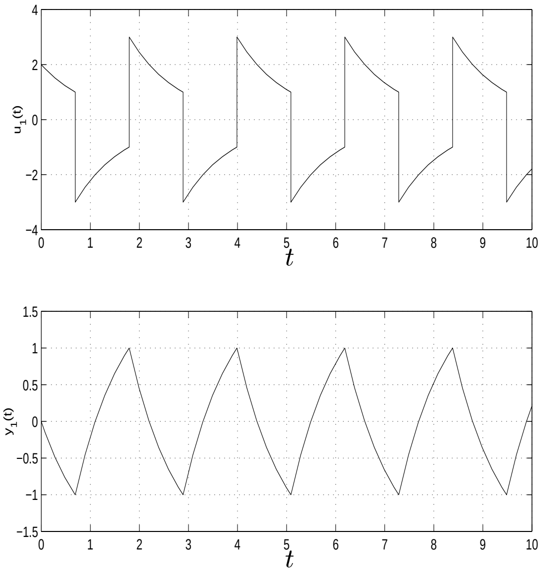

Fig. 3. Autonomous response ($u_0 = y_0 = 0$) of the sloping relay oscillator

the disturbances to the internal signals becomes unbounded which makes it difficult to estimate the effect of perturbations. To circumvent this problem, in (Georgiou and Smith 2000), the relay element was modified so that the branches of the relay characteristic have a positive slope *after* the absolute value of the input exceeds a certain threshold.

An alternative modification of the ideal relay that will result in a bounded projection operator involves considering the relay branches with a constant positive slope for all inputs as shown in Fig 1. We take the relay on-off points at $y = \pm 1$, the input levels at switching points as $u = \pm 3$ and the slope of the branches to be unity. That is, the branches of the relay are given by

$$u_h(y) = y + 2$$
$$u_l(y) = y - 2.$$

The oscillator obtained by connecting a negative integrator to this relay is shown in Fig 2. We refer to this oscillator as the *sloping relay oscillator* and it represents a piecewise linear analog of the van der Pol oscillator (Guckenheimer 1980) which, unlike the ideal relay oscillator, also retains the property of bounded response to bounded disturbances u_0 and y_0. The autonomous reponse (corresponding to zero external disturbances u_0 and y_0) of the sloping relay oscillator is shown in Fig 3 where the initial condition chosen is $y_1(0) = 0$. The integrator input u_1 varies in $[-3, 3]$ and the output y_1 varies in $[-1, 1]$. The first switching occurs at $t_1 = \ln 2$ and the subsequent switchings occur according to $t_{k+1} - t_k = \ln 3$. The entrainment behavior in such an oscillator has been well studied in the literature (Levi 1981, Guckenheimer and Holmes 1983, Jackson 1991).

The dynamics of relay feedback systems (RFS) is itself an active area of research and RFS are encountered in many practical applications like process identification and auto-tuning of controllers (Aström *et al.* 1995, Aström 1995, Johansson *et al.* 1999, Gonçalves *et al.* 2001, Varigonda and Georgiou 2001). We shall concern ourselves with only the robustness aspect here.

The main issue in extending the robustness analysis of feedback systems to oscillators is to identify the suitable signal spaces and the norms. The hysteresis element requires continuous, piece-wise monotone inputs. The linear or nonlinear feedback element takes discontinuous signals. Thus, the L_∞ signal space where the norm of a signal $x(t), t \in [0, \infty)$ is defined as

$$\|x\|_\infty = \sup_{t \in [0,\infty)} x(t)$$

seems suitable. For a system of the kind shown in Fig 2, (Georgiou and Smith 2000) showed that the class of linear systems that satisfies the above constraint on the input-output signals consists of those systems that have a piece-wise Lipschitz convolution kernel.

To analyze the robustness of the oscillator, we need a suitable notion of the distance between two oscillatory signals. Most of the metrics used traditionally in the analysis of feedback systems are suitable for steady state operating points or trajectories for which time is topologically linear. However, for oscillatory systems, one would like to consider time in a circular topology. To illustrate this further, consider two sinusoidal signals x and y with the same amplitude but slightly different phase or frequency. The distance $d_\infty(x,y) := \|x-y\|_\infty$ based on the L_∞-norm does not satisfactorily account for the oscillatory nature of x and y since $d_\infty(x,y)$ is of the order of the signal amplitude, no matter how small the difference in the frequencies is. Thus the d_∞ metric is overly conservative. We suggest an alternative notion of distance for oscillatory signals that allows a time scaling, *i.e.,* orientation preserving homeomorphisms of the interval $[0,\infty)$, to be used before taking the L_∞ metric. This idea is motivated by the concept of Zhukovskii stability which is defined as follows.

Definition 1. (Fradkov and Pogromsky 1998, pp 147) The limit set M containing the solution $\bar{x}(t)$ of the system $\dot{x} = f(x)$ with initial condition $\bar{x}(0)$ is

i) strongly orbitally stable if for each $\epsilon > 0$, there is $\delta > 0$ such that

$$|x(0) - \bar{x}(0)| < \delta \Rightarrow \forall t \geq 0, \quad |x(h_1(t)) - \bar{x}(h_2(t))| < \epsilon$$

for some time scalings h_1, h_2.

ii) asymptotically strongly orbitally stable if it is stable and δ can be chosen such that

$$|x(0) - \bar{x}(0)| < \delta \Rightarrow \lim_{t\to\infty} |x(h_1(t)) - \bar{x}(h_2(t))| = 0$$

for some time scalings h_1, h_2.

Without loss of generality, the two time scalings in the above definition can be replaced by a single one. Let $\mathcal{K}_\infty$ denote the set of time scalings. We define the distance between two oscillatory signals $x, y \in L_\infty$ as

$$d_z(x,y) = \inf_{\sigma\in\mathcal{K}_\infty} \{\|x(t) - y(\sigma(t))\|_\infty + \sup_{t\in[0,\infty)} |\sigma(t) - t|/t\ \}.$$

Such a notion of distance basically allows a scaling of time to match the two signals x, y but penalizes for the "amount" of scaling. The definition of $d_z(x,y)$ does not qualify to be a metric as the triangle inequality is not verified. But it may be possible to combine the idea of using a suitable time scaling with penalty along with the L_∞ metric to define a valid metric. We believe that such a metric is more suitable to define the operator norms of systems receiving and sending oscillatory signals and it enables the extension of the gap metric robustness results to oscillatory systems.

Using the above distance notion, (Georgiou and Smith 2000) studied the gap metric robustness of a relaxation oscillator consisting of an on-off relay and an integrator. They showed that if the linear element (the integrator) is perturbed slightly in the gap, the behavior still remains oscillatory and *close to* the nominal behavior. In the following section, we present a similar robustness study for the sloping relay oscillator.

2. ROBUSTNESS OF A PIECEWISE LINEAR OSCILLATOR

In this section, we derive a quantitative robustness margin for the sloping relay oscillator. Our procedure parallels the one used in (Georgiou and Smith 2000). The goal is to demonstrate that the behavior of this oscillator is robust to small perturbations in the linear component, namely, the integrator which we will sometimes refer to as "the plant". The perturbations of the plant are measured in the gap metric and the change in the system behavior is measured using the scaled L_∞ norm on the internal signals. Since the analysis is lengthy and closely follows the one in (Georgiou and Smith 2000), we shall only state the result and briefly outline the key steps in the proof.

Basic notions of gap metric robustness analysis can be found in (Georgiou and Smith 1997). Briefly, the graph of a system refers to the collection of all compatible input and output signals. The feedback connection is stable if the space of external disturbances is coordinatized by the graphs of the two subsystems *i.e.,* any external disturbance has a unique decomposition into two points, one each on the graphs of the subsystems. The robustness margin is related to the norm of the parallel projection operator that maps external disturbances to signals inside the loop. A suitable choice of the signal spaces is the space of bounded functions $L_\infty[0,\infty)$ for the plant inputs $\mathcal{U}$, and for the outputs $\mathcal{Y}$, we consider the subspace $C[0,\infty)$ of $L_\infty[0,\infty)$ which consists of continuous functions. For the plant output, we take the initial condition $y_1(0) = 0$. Recall that $w_0 = (u_0, y_0)$ denotes the external disturbance. For the nominal system, we have $w_0 = 0$.

Consider a perturbation of the graph of the plant from $\mathcal{M}$ to $\mathcal{M}_1$. From the definition of gap, we have a bijective map Φ from $\mathcal{M}$ to $\mathcal{M}_1$. We would like to compare the difference in the response of the two feedback systems for a common external disturbance $w_0 = 0$. Let $w_0 = m_1 + n$ be the unique decomposition of w_0 induced by the perturbed feedback system. Thus,

$$m_1 = \Pi_{\mathcal{M}_1||\mathcal{N}} w_0 \quad \text{and} \quad n = \Pi_{\mathcal{N}||\mathcal{M}_1} w_0.$$

Define m, x_0 by $m_1 = \Phi m$ and $x_0 = m + n$. That is, x_0 represents the disturbance to be applied to the nominal system to produce an internal response that is *close* to the internal response of the system with the perturbed plant.

With some algebraic manipulation, it can be shown that (Georgiou and Smith 2000),

$$w_0 = \left[I + (\Phi - I)\Pi_{\mathcal{M}||\mathcal{N}}\right] x_0 \tag{1}$$

$$(\sigma\Pi_{\mathcal{M}||\mathcal{N}} - \Pi_{\mathcal{M}_1||\mathcal{N}})w_0 = (I - \Phi)\Pi_{\mathcal{M}||\mathcal{N}} x_0 + \sigma\Pi_{\mathcal{M}||\mathcal{N}} w_0 - \Pi_{\mathcal{M}||\mathcal{N}} x_0 \tag{2}$$

where σ is any time scaling. Given the size of the plant perturbation $\|I - \Phi\|$, (1) helps in bounding the equivalent disturbance, x_0. This step makes use of the global boundedness of the nominal system, namely, the bound on $\|\Pi_{\mathcal{M}||\mathcal{N}}\|$. The main robustness result is obtained using (2) which essentially relates effect of plant perturbation (the left hand side expression) to the effect of external disturbances (the last two terms on the right hand side). The result is an equivalent of (Georgiou and Smith 2000, Theorem 1) and is stated below.

Proposition 2. Consider the sloping relay oscilator shown in Fig 2. Let $\mathbf{G}$ be the negative integrator, $\mathbf{G}_1$ be a perturbation of $\mathbf{G}$ and $\mathbf{H}$ be the sloping relay shown in Fig 1. Let $\mathcal{M}, \mathcal{M}_1$ and $\mathcal{N}$ be their graphs respectively. If $\delta(\mathbf{G}, \mathbf{G}_1) \le \epsilon < \frac{1}{6}$, or equivalently, if there exists a bijective map $\Phi : \mathcal{M} \to \mathcal{M}_1$ such that

$$\|I - \Phi\| \le \epsilon < \frac{1}{6} \tag{3}$$

then there exists a time scaling *i.e.,* an orientation preserving homeomorphism of the positive real axis σ such that

$$\sup_t \frac{|\sigma(t) - t|}{t} \le \alpha(\epsilon) := \frac{\ln \dfrac{3 - 10\epsilon}{3(1 - 6\epsilon)}}{\ln 2(1 - 2\epsilon)} \tag{4}$$

and the response of the two feedback systems $[\mathbf{G}, \mathbf{H}]$ and $[\mathbf{G}_1, \mathbf{H}]$ with zero external disturbances ($w_0 = 0$) satisfy

$$\|\sigma\Pi_{\mathcal{M}||\mathcal{N}} 0 - \Pi_{\mathcal{M}_1||\mathcal{N}} 0\|_\infty \le \frac{4\epsilon}{1 - 3\epsilon}. \tag{5}$$

Proof: We provide only a sketch of the proof here. As described in (Georgiou and Smith 2000) for the case of the ideal relay oscillator, the proof involves two main steps. The first step involves getting a bound r on the norm of the equivalent disturbance x_0 corresponding to a given plant perturbation bound ϵ, as in (3). This step involves computing a bound on $\|\Pi_{\mathcal{M}||\mathcal{N}}\|$ by analyzing the nominal system and using the relation (1). The second step involves bounding the distance (in the sense defined in Sec 1) between the nominal response $\Pi_{\mathcal{M}||\mathcal{N}} 0$ and the response $\Pi_{\mathcal{M}||\mathcal{N}} x_0$ when the system is subjected to external disturbance $x_0 = (u_0, y_0)$ bounded by r. This step can be further broken into two steps: first, bounding the L_∞ distance

$$\|\sigma\Pi_{\mathcal{M}||\mathcal{N}} 0 - \Pi_{\mathcal{M}||\mathcal{N}} x_0\|_\infty$$

for a suitably chosen time scaling σ and second, bounding the size of σ which is defined as

$$\sup_t \frac{|\sigma(t) - t|}{t}.$$

From the evolution equation of y_1 for the nominal system $[\mathbf{G}, \mathbf{H}]$ with external disturbance x_0 such that $\|x_0\| \le r$, it can be shown that, for $r < 3$,

$$\|y_1\|_\infty \le 1 + r$$
$$\|u_1\|_\infty \le 3(1 + r)$$

and hence

$$\|\Pi_{\mathcal{M}||\mathcal{N}} x_0\| \le 3(1 + r).$$

Using (1), we can deduce that whenever $\|I - \Phi\| < \epsilon < \dfrac{3}{10}$, we have $r \le \dfrac{\epsilon}{1 - 3\epsilon}$. Notice that the the upper bound on ϵ considered in the statement of the proposition is below $\dfrac{3}{10}$. This decrease comes from the analysis in the second step.

In the second step of the proof, we need to obtain a bound on $\|\sigma\Pi_{\mathcal{M}||\mathcal{N}} 0 - \Pi_{\mathcal{M}||\mathcal{N}} x_0\|_\infty$. Let u_1, y_1 denote the signals on the graph of the plant for the nominal system ($w_0 = 0$) and let u_1', y_1' denote the same signals when an external disturbance $w_0 = x_0$ is applied. Denote by t_k ($k \ge 0, t_0 := 0$) the switching times of the autonomous system and denote by t_k' ($k \ge 0, t_0' := 0$), the switching times of the system with disturbance x_0. Observe from Fig 3 that $y_1(t)$ is monotonic in each switching interval $[t_k, t_{k+1}]$. For sufficiently small x_0, $y_1'(t)$ also remains monotonic in each switching interval $[t_k', t_{k+1}']$. In fact, for $r \le 1$, $y_1'(t)$ can be guaranteed to be monotonic. We restrict our attention to this regime only and this further reduces the bound on the plant perturbations to $\epsilon \le 1/4$.

Since $y_1'(t)$ is monotonic, we can define a time scaling $\sigma(t)$ given by

$$\sigma(t) = t_k - \ln\left[1 - \frac{2}{3}\frac{y_1'(t) - y_1'(t_k')}{y_1'(t_{k+1}) - y_1'(t_k')}\right]$$

in each interval $[t_k', t_{k+1}']$. Notice that $\sigma(t_k') = t_k$ for all k. Using this time scaling, it can easily be shown that

$$|\sigma y_1(t_k') - y_1'(t_k')| \leq r \qquad \text{for all } k.$$

The function $\sigma y_1(t) - y_1'(t)$ is monotonic in t since by our choice of σ, it turns out to be a linear in $y_1(t)$ which in turn in monotonic in t. A continuous monotone function bounded at both the ends of the interval $[t_k', t_{k+1}']$ by r must be bounded everywhere in the interval by r. Thus we have

$$|\sigma y_1(t) - y_1'(t)| \leq r.$$

From this, it can be shown that

$$|\sigma u_1(t) - u_1'(t)| \leq 3r$$

which in turn implies that

$$\|\sigma \Pi_{\mathcal{M}||\mathcal{N}} 0 - \Pi_{\mathcal{M}||\mathcal{N}} x_0\|_\infty \leq 3r. \qquad (6)$$

The final task is to bound the size of σ. From the monotonicity of σ in each interval $[t_k', t_{k+1}']$, it follows that

$$\sup_t \frac{|\sigma(t) - t|}{t} = \sup_{k>0} \frac{|\sigma(t_k') - t_k|}{t_k'}.$$

By analyzing the scenarios for the slowest and fastest switchings, we find that r must be below 1/3 to guarantee switching. This gives rise to the bound 1/6 on ϵ as stated in the proposition. We also obtain

$$\ln\frac{2(1+r)}{1+3r} \leq \quad t_1' \quad \leq \ln\frac{2(1-r)}{1-3r}$$
$$\ln\frac{3+r}{1+3r} \leq t_{k+1}' - t_k' \leq \ln\frac{3-r}{1-3r} \quad \text{for } k \geq 1.$$

Adding up the above equations, we get

$$\ln\frac{2(1+r)(3+r)^{k-1}}{(1+3r)^k} \leq \quad t_k' \leq \quad \ln\frac{2(1-r)(3-r)^{k-1}}{(1-3r)^k}.$$

Subtracting the autonomous switching time $t_k = \ln 2 + (k-1)\ln 3$, we obtain

$$\ln\frac{(1+r)(1+r/3)^{k-1}}{(1+3r)^k} \leq \quad t_k' - t_k \leq \quad \ln\frac{(1-r)(1-r/3)^{k-1}}{(1-3r)^k}.$$

In the range of interest, namely, for $r < 1/3$, the upper bound becomes the bound on $|t_k' - t_k|$. Thus, we have

$$|t_k' - t_k| \leq \ln\frac{(1-r)(1-r/3)^{k-1}}{(1-3r)^k}.$$

Adding the positive quantity $\ln\frac{1-r/3}{1-r}$ to the bound, we get

$$|t_k' - t_k| \leq k\ln\frac{1-r/3}{1-3r}.$$

Using the relation

$$t_k' \geq \ln\frac{2(1+r)(3+r)^{k-1}}{(1+3r)^k} = k\ln\frac{3+r}{1+3r} - \ln\frac{3+r}{2(1+r)},$$

we obtain

$$\frac{|t_k' - t_k|}{t_k'} \leq \frac{k\ln\frac{1-r/3}{1-3r}}{k\ln\frac{3+r}{1+3r} - \ln\frac{3+r}{2(1+r)}}.$$

Taking the supremum over $k > 0$ which occurs for $k = 1$, we obtain

$$\sup_t \frac{|\sigma(t) - t|}{t} \leq \frac{\ln\frac{1-r/3}{1-3r}}{\ln\frac{2(1+r)}{1+3r}}.$$

We can now relate the effect of plant perturbation in the gap to the effect of disturbances using (2) which implies

$$\|(\sigma\Pi_{\mathcal{M}||\mathcal{N}} - \Pi_{\mathcal{M}_1||\mathcal{N}})0\|_\infty \leq \|(I - \Phi)\Pi_{\mathcal{M}||\mathcal{N}} x_0\|_\infty + \|\sigma\Pi_{\mathcal{M}||\mathcal{N}} 0 - \Pi_{\mathcal{M}||\mathcal{N}} x_0\|_\infty.$$

Observe from (1) that $(I - \Phi)\Pi_{\mathcal{M}||\mathcal{N}} x_0 = x_0 - w_0 = x_0$. Thus, $\|(I - \Phi)\Pi_{\mathcal{M}||\mathcal{N}} x_0\|_\infty \leq r$ and together with (6), this implies that

$$\|(\sigma\Pi_{\mathcal{M}||\mathcal{N}} - \Pi_{\mathcal{M}_1||\mathcal{N}})0\|_\infty \leq 4r.$$

All the bounds we have obtained are monotonic in r and can be represented in terms of ϵ by substituting $r = \frac{\epsilon}{1-3\epsilon}$ since, from the first step of the proof, this is the maximum possible r for a given plant perturbation of size ϵ. Thus we have the relations (3)–(5) stated in the proposition. ■

This study demonstrates that the feedback paradigm for relaxation oscillators indeed facilitates an unstructured robustness analysis. We strengthen the case made in (Georgiou and Smith 2000) by providing a robustness result that parallels Theorem 1 of (Georgiou and Smith 2000) for another piecewise linear oscillator which resembles the van der Pol system. The next challenge is a similar analysis for the case of fully nonlinear oscillators like the van der Pol system.

3. CONCLUSION

The theory of feedback control provides an elegant geometric characterization of robustness. This theory considers robustness in a very general sense that is independent of the way one chooses to represent of the system. A feedback mechanism driving a bistable (hysteretic) subsystem has been observed in several oscillator systems. Thus one is naturally interested in an extension of the robustness analysis of feedback systems to oscillators that can be represented as feedback systems. The robustness analysis of oscillators requires a suitable measure of distance between two oscillatory signals. We proposed a distance notion based on Zhukovskii stability and this distance measure has been employed in the robustness analysis of an ideal on-off relay relaxation oscillator consisting of an integrator in (Georgiou and Smith 2000). We presented here a similar study for another piecewise linear system with a qualitatively different time profiles for the internal signals. This analysis further strengthens the view that a feedback paradigm for oscillators facilitates unstructured robustness analysis. Such an analysis for the case of higher dimensional relay oscillators and other nonlinear oscillators is challenging and can provide a valuable insight into the robustness of natural oscillators.

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