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A DYNAMIC INVERSION BASED CONTROLLER FOR PATH FOLLOWING OF CAR-LIKE VEHICLES

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Abstract: This paper focuses on a special path following task arising from the needs of vision-based autonomous guidance: a given front point of a car-like vehicle that is within the look-ahead range of a stereo vision system, must follow a prespecified Cartesian path. Solution to this path following problem is provided by a new feedforward/feedback control strategy where the feedforward is determined by a dynamic generator based on exact dynamic inversion over the nominal vehicle model and the feedback is mainly issued by correcting terms proportional to the tangential and normal errors determined with respect to the vehicle's ideal trajectory. A convergence analysis of the resulting dynamic inversion based controller is established versus a vehicle's uncertain model defined via equation errors. A simulation example highlighting the controller's performances is included.

Keywords: Autonomous vehicles, Path following, Dynamic inversion, Nonlinear control systems, Feedforward control, Feedback control methods, Robust performance.

1. INTRODUCTION

Path following of car-like vehicles has been treated using various approaches in the literature. Focusing on motion planning, i.e. on methods to derive open-loop controls to steer the vehicle on desired Cartesian paths, a particularly relevant method is the differential flatness approach of Fliess and coworkers (Fliess *et al.*, 1995; Rouchon *et al.*, 1993). On the other hand, path following can be approached with feedback control and in many cases the feedback strategy is derived by reduction from a trajectory tracking methodology (Sampei *et al.*, 1991) or by extension from a point stabilization task (Sørdalen and de Wit, 1993). Another possible approach could be feedback receding horizon scheme to maneuver regulation as proposed in (Hauser and Jadbabaie, 2000). When better performances are sought an integrated feedforward/feedback design is in order as pointed out in (Al-Hiddabi *et al.*, 1999; Luca *et al.*, 1998).

This paper focuses on a special path following task arising from the needs of vision-based autonomous guidance (Broggi *et al.*, 1999*b*; Piazzi and Guarino Lo Bianco, 2000): a given front point of a car-like vehicle that is within the look-ahead range of a stereo vision system must follow a prespecified Cartesian path. A purely feedforward solution to this problem has been delineated in (Consolini *et al.*, 2001) by means of an exact dynamic inversion procedure. In this paper, building upon this result, a new feedforward/feedback control strategy is proposed. The feedforward control is determined by a dynamic inversion generator based on the nominal vehicle model and the feedback is mainly issued by inserting, in the generator equations, correcting terms proportional to the tangential and normal errors determined with respect to the vehicle's desired trajectory. A distinguished feature of the proposed approach is the explicit use of a vehicle's uncertain model defined via equation errors. Overall, the resulting dynamic inversion based controller can provide robust performance with guaranteed bounds in the path following task of the autonomous vehicle.

Paper's organization. Section 2 recalls the main results (Theorem 1 and Theorem 2) for the motion planning of the vehicle using a dynamic inversion approach. In particular, the explicit equations that generate the open-loop steering control are given in (5). Section 3 introduces the uncertain vehicle model (6), the dynamic inversion based controller (9), and the main result of the paper (Theorem 3) that gives sufficient conditions, involving the model error bounds and the curvature of the desired path (see (11)), for which arbitrarily good robust path following may be achieved. A simulation example is included in Section 4 and final remarks are presented in Section 5.

Notation. $\|\mathbf{P}\|$ and $\mathbf{P}^T$ will denote the Euclidean norm and the transpose of a vector $\mathbf{P}$ respectively. If $\mathcal{I} \subset \mathbb{R}$, given a function $f : \mathcal{I} \to \mathbb{R}$ we set $\|f\|_\infty = \sup_{t\in\mathcal{I}} |f(t)|$. Let a curve on the Cartesian $\{x, y\}$-plane be described by a parameterization $\boldsymbol{\gamma}(\lambda) = [\xi(\lambda)\ \eta(\lambda)]^T$ with real parameter $\lambda \in [0, a]$ where a is a finite real value; the associated "path", called Γ, is the image of $[0, a]$ under the vectorial function $\boldsymbol{\gamma}$, i.e. $\Gamma = \boldsymbol{\gamma}([0, a])$. We say that the curve $\boldsymbol{\gamma}$ is regular if there exists $\dot{\boldsymbol{\gamma}}(\lambda)$ and $\dot{\boldsymbol{\gamma}}(\lambda) \neq 0\ \forall\lambda \in [0, a]$. A curve $\boldsymbol{\gamma}$ is of class C^k if $\boldsymbol{\gamma} \in C^k([0, a], \mathbb{R}^2)$, i.e. both the coordinate functions ξ and η have continuous derivatives up to the kth-order. Associated to every point $\boldsymbol{\gamma}(\lambda)$ of a regular curve γ there is the orthonormal *moving* frame $\{\boldsymbol{\tau}(\lambda), \boldsymbol{\nu}(\lambda)\}$ where $\boldsymbol{\tau}(\lambda)$ is the unit tangent vector and $\boldsymbol{\nu}(\lambda)$ is the unit normal vector oriented in such a way that $\{\boldsymbol{\tau}(\lambda), \boldsymbol{\nu}(\lambda)\}$ is congruent to the $\{x, y\}$-plane. Let $\gamma \in C^1([0, a], \mathbb{R}^2)$, we say that γ has arc-length parameterization if $\|\dot\gamma(\gamma)\| = 1$ $\forall\lambda \in [0, a]$, therefore $\boldsymbol{\tau}(\lambda) = \dot{\boldsymbol{\gamma}}(\lambda)$ and, as known from Frenet formulae, $\dot{\boldsymbol{\tau}}(\lambda) = \kappa(\lambda)\boldsymbol{\nu}(\lambda)$ where $\kappa(\lambda)$ is the local curvature.

2. THE OPEN-LOOP DYNAMIC INVERSION BASED GENERATOR

Let the motion model of a car-like vehicle be given by the following simplified nonholonomic system:

$$\begin{cases} \dot x(t) &= v\cos\theta(t) \\ \dot y(t) &= v\sin\theta(t) \\ \dot\theta(t) &= \dfrac{v}{l}\tan\delta(t) \end{cases} \quad (1)$$

where (see Fig. 1) x and y are the coordinates of the middle point $\mathbf{P}$ of the rear axle, v is the constant velocity of $\mathbf{P}$ (i.e. $\|\dot{\mathbf{P}}(t)\| = v$ for any t), θ is the vehicle's heading angle, l is the inter-axle distance and δ, the front wheel angle, is the control variable to steer the vehicle. Let the initial conditions of the above model be given by $x(0) = x_0$, $y(0) = y_0$, and $\theta(0) = \theta_0$.

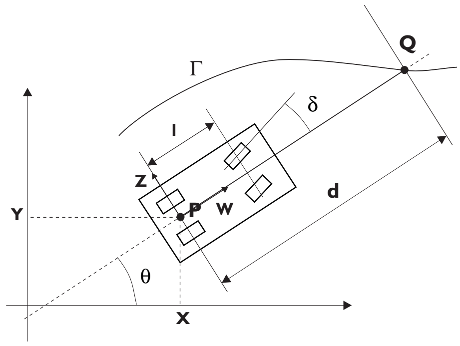

Fig. 1. The car-like vehicle with front point $\mathbf{Q}$.

$\mathbf{Q}$, called the "front point" of the vehicle, is a distinguished point of the model; it belongs to the vehicle's symmetry axis at a fixed distance d from $\mathbf{P}$ ahead of the vehicle. This point could be a physical point of the vehicle or a virtual one belonging to the road scene as viewed in the look-ahead range by the vehicle's vision system.

Introduce the orthonormal frame $\{\mathbf{w}(\theta), \mathbf{z}(\theta)\}$ as a function of the vehicle's heading angle:

$$\mathbf{w}(\theta) := \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}\ ,\ \mathbf{z}(\theta) := \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}\ . \quad (2)$$

This frame can be thought as attached to the vehicle's body and is congruent to the Cartesian $\{x, y\}$ plane; then the coordinates of $\mathbf{Q}(t)$ are given by

$$\mathbf{Q}(t) = [x_{\mathbf{Q}}(t)\ y_{\mathbf{Q}}(t)]^T = \mathbf{P}(t) + d\mathbf{w}(\theta(t)) \quad (3)$$

and its motion is governed by the following system:

$$\begin{cases} \dot x_{\mathbf{Q}}(t) = v\cos\theta(t) - \dfrac{dv}{l}\sin\theta(t)\tan\delta(t) \\ \dot y_{\mathbf{Q}}(t) = v\sin\theta(t) + \dfrac{dv}{l}\cos\theta(t)\tan\delta(t) \\ \dot\theta(t) = \dfrac{v}{l}\tan\delta(t)\ . \end{cases} \quad (4)$$

A pertinent motion planning problem for the the d-ahead point $\mathbf{Q}$ can be introduced as it follows: given a sufficiently smooth Cartesian curve $\boldsymbol{\gamma}$, find a continuous steering control δ and initial conditions of model (1) in such a way that the motion path of the front point $\mathbf{Q}$ matches the path Γ exactly. For the degenerate case $d = 0$, i.e. $\mathbf{Q} = \mathbf{P}$, this problem has been solved in (Broggi *et al.*, 1999*b*) by means of an elegant closed-form solution exploiting the curvature function along the curve $\boldsymbol{\gamma}$. For the nondegenerate case $d > 0$ we can state the following two results that can easily deduced from (Consolini *et al.*, 2001).

Theorem 1. Let a curve $\boldsymbol{\gamma} : [0, a] \to \mathbb{R}^2$ of class C^2 be given with arc-length parameterization. Assume that the initial state of model (1) satisfies the conditions:

$\mathbf{Q}(0) = \boldsymbol{\gamma}(0)$ and $\dot{\boldsymbol{\gamma}}(0)^T \mathbf{w}(\theta_0) > 0$.

a) Then there exist a sufficiently small $\bar{t} \in \mathbb{R}^+$ and a steering function $\delta \in C^0([0, \bar{t}\,], \mathbb{R})$ such that the motion of the front point $\mathbf{Q}$ follows $\Gamma = \boldsymbol{\gamma}([0, a])$, i.e. $\mathbf{Q}(t) \in \Gamma\ \forall t \in [0, \bar{t}\,]$.

b) Moreover, if the curvature $\kappa(\lambda)$ of $\boldsymbol{\gamma}(\lambda)$ satisfies the following condition

$$|\kappa(\lambda)| < 1/d\ \forall \lambda \in [0, a]\ ,$$

then there exist $t_f \in \mathbb{R}^+$ and a steering function $\delta \in C^0([0, t_f], \mathbb{R})$ such that the point $\mathbf{Q}$ exactly covers the entire path Γ, i.e. $\mathbf{Q}([0, t_f]) = \Gamma$.

The invertibility conditions appearing in the above theorem have a simple geometrical interpretation. In addition to the obvious necessary condition of $\mathbf{Q}(0)$ to be equal to the starting point of $\boldsymbol{\gamma}$, the exact motion planning is possible, at least for a while, if at the initial time the angle between the vehicle's direction and the tangent on the curve has absolute value less than $\pi/2$. Moreover, if the maximum absolute value of the curvature along $\boldsymbol{\gamma}$ is less than $1/d$, then the entire path Γ can be followed and the suitable steering input can be determined by the dynamic inversion based generator exposed below, see equations (5). The proof of Theorem 1 can be found in (Consolini *et al.*, 2001) where a more general condition on the curvature is also given. A constructive procedure to solve the posed motion planning problem is exposed below.

Theorem 2. Let a curve $\boldsymbol{\gamma} \in C^2([0, +\infty[, \mathbb{R}^2)$ be given with arc-length parameterization and such that $|\kappa(\lambda)| < 1/d$, $\forall \lambda \in [0, +\infty[$.If the following initial state conditions are satisfied:

$\mathbf{Q}(0) = \boldsymbol{\gamma}(0), \quad \dot{\boldsymbol{\gamma}}(0)^T \mathbf{w}(\theta_0) > 0,$

there exists a steering function $\delta \in C^1([0, +\infty[, \mathbb{R})$, given by the following "open-loop generator":

$$\begin{cases} \dot{\mu}(t) = v \dfrac{1}{\dot{\xi}(\mu(t))\cos(\sigma(t)) + \dot{\eta}(\mu(t))\sin(\sigma(t))} \\ \dot{\sigma}(t) = \dfrac{v}{d} \dfrac{\dot{\eta}(\mu(t))\cos(\sigma(t)) - \dot{\xi}(\mu(t))\sin(\sigma(t))}{\dot{\xi}(\mu(t))\cos(\sigma(t)) + \dot{\eta}(\mu(t))\sin(\sigma(t))} \\ \delta(t) = \arctan(\dfrac{l}{v}\dot{\sigma}(t)) \\ \mu(0) = 0\ , \sigma(0) = \theta_0 \end{cases} \tag{5}$$

such that

$$\mathbf{Q}(t) = \boldsymbol{\gamma}(\mu(t))\ \forall t \geq 0\ , \quad \mathbf{Q}([0, +\infty[) = \boldsymbol{\gamma}([0, +\infty[)\ ;$$

in other words, the front point $\mathbf{Q}$ follows the entire desired path exactly and indefinitely.

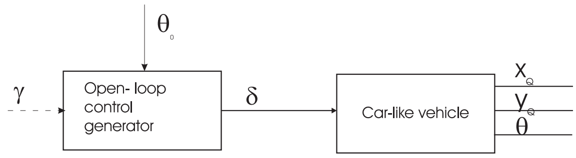

Fig. 2. The dynamic inversion based generator.

It is worth noting that the internal states of the generator (5) have a special meaning: $\mu(t)$ is the distance covered at time t by point $\mathbf{Q}$ along path $\boldsymbol{\gamma}$ and $\sigma(t) \equiv \theta(t)\ \forall t > 0$. In such a way, the given generator can be interpreted as a control law based on a special open-loop observer of model (4).

3. THE DYNAMIC INVERSION BASED CONTROLLER

Consider the desired path be given by a curve $\boldsymbol{\gamma} \in C^2([0, +\infty[, \mathbb{R}^2)$ with arc-length parameterization and denote by

$\mathbf{E} = \mathbf{E}(\mathbf{Q}, \boldsymbol{\gamma}(\lambda)) = \mathbf{Q} - \boldsymbol{\gamma}(\lambda)$

the "error" vector of $\mathbf{Q}$ with respect to the curve point $\boldsymbol{\gamma}(\lambda)$; then it is natural to name "absolute error" the distance $\mathcal{E}(\mathbf{Q})$ between $\mathbf{Q}$ and the path $\Gamma(= \boldsymbol{\gamma}([0, +\infty[)$, i.e.

$\mathcal{E}(\mathbf{Q}) := \inf_{\lambda \in [0, +\infty[} \|\mathbf{E}(\mathbf{Q}, \boldsymbol{\gamma}(\lambda))\|.$

Now introduce the following uncertain model for the car-like vehicle:

$$\begin{cases} \dot{x}(t) = v\cos\theta(t) + e_x(t) \\ \dot{y}(t) = v\sin\theta(t) + e_y(t) \\ \dot{\theta}(t) = \dfrac{v}{l}\tan\delta(t) + e_\theta(t) \end{cases} \tag{6}$$

with initial conditions $x(0) = x_0$, $y(0) = y_0$, $\theta(0) = \theta_0$, where the functions $e_x, e_y, e_\theta \in C^1([0, +\infty[)$ and satisfy the following bounds:

$$\|e_x\|_\infty \leq M_x,\ \|e_y\|_\infty \leq M_y,\ \|e_\theta\|_\infty \leq M_\theta. \tag{7}$$

Since the coordinates of the front point $\mathbf{Q}$ are given by (3), the perturbed motion of $\mathbf{Q} = [x_{\mathbf{Q}} \quad y_{\mathbf{Q}}]^T$ is governed by the following system:

$$\begin{cases} \dot{x}_{\mathbf{Q}} = v\cos\theta - \dfrac{dv}{l}\sin\theta\tan\delta + e_x - d(\sin\theta)e_\theta \\ \dot{y}_{\mathbf{Q}} = v\sin\theta + \dfrac{dv}{l}\cos\theta\tan\delta + e_y + d(\cos\theta)e_\theta \\ \dot{\theta} = \dfrac{v}{l}\tan\delta + e_\theta \end{cases} \tag{8}$$

Using the open-loop control strategy of the previous section, due to the modeling errors introduced in (6), the actual position $\mathbf{Q}(t)$ of the front point at time t may be different from the *estimate position* $\boldsymbol{\gamma}(\mu(t))$ as shown in Figure 3. Therefore, in order to decrease the distance of $\mathbf{Q}(t)$ from $\boldsymbol{\gamma}(\mu(t))$, we modify the equations of the generator (5) by

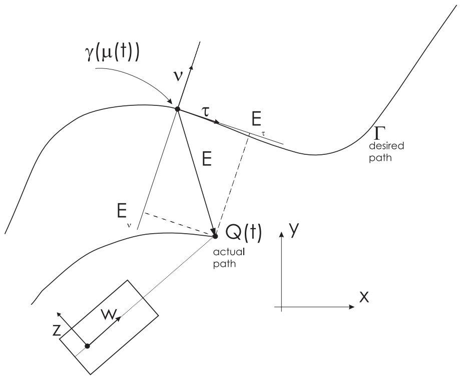

Fig. 3. The error vector and its components.

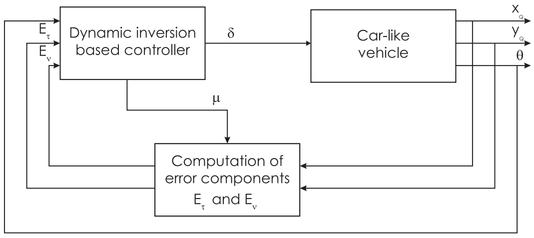

Fig. 4. Control architecture.

means of correcting feedback terms which use the error components $E_\tau(t)$ and $E_\nu(t)$ defined with respect to the moving frame $\{\boldsymbol{\tau}(\mu(t)), \boldsymbol{\nu}(\mu(t))\}$:

$$E_\tau = E_\tau(t) = \mathbf{E}(\mathbf{Q}(t), \boldsymbol{\gamma}(\mu(t)))^T \boldsymbol{\tau}(\mu(t))$$

$$E_\nu = E_\nu(t) = \mathbf{E}(\mathbf{Q}(t), \boldsymbol{\gamma}(\mu(t)))^T \boldsymbol{\nu}(\mu(t))$$

The overall feedback strategy is then given by the following dynamic inversion based controller:

$$\begin{cases} \dot{\mu} = \dfrac{v}{\boldsymbol{\tau}(\mu)^T \boldsymbol{w}(\sigma)} + K_\tau E_\tau \\ \dot{\sigma} = \dfrac{v}{d} \dfrac{\boldsymbol{\tau}(\mu)^T \boldsymbol{z}(\sigma)}{\boldsymbol{\tau}(\mu)^T \boldsymbol{w}(\sigma)} - K_\nu E_\nu + K_\theta(\theta - \sigma) \\ \delta = \arctan(\dfrac{l}{v}(\dfrac{v}{d} \dfrac{\boldsymbol{\tau}(\mu)^T \boldsymbol{z}(\sigma)}{\boldsymbol{\tau}(\mu)^T \boldsymbol{w}(\sigma)} - K_\nu E_\nu)) \end{cases} \tag{9}$$

with initial conditions $\mu(0) = 0$, $\sigma(0) = \theta_0$, where K_τ, K_ν, K_θ are positive feedback-gain constants. The corresponding control architecture is depicted in Figure 4. The closed-loop system equations are then given by:

$$\begin{cases} \dot{x}_{\mathbf{Q}} = v \cos\theta - d(\sin\theta)u + e_x - d(\sin\theta)e_\theta \\ \dot{y}_{\mathbf{Q}} = v \sin\theta + d(\cos\theta)u + e_y + d(\cos\theta)e_\theta \\ \dot{\theta} = u + e_\theta \\ \dot{\mu} = \dfrac{v}{\boldsymbol{\tau}(\mu)^T \boldsymbol{w}(\sigma)} + K_\tau E_\tau \\ \dot{\sigma} = u + K_\theta(\theta - \sigma) \\ u = \dfrac{v}{d} \dfrac{\boldsymbol{\tau}(\mu)^T \boldsymbol{z}(\sigma)}{\boldsymbol{\tau}(\mu)^T \boldsymbol{w}(\sigma)} - K_\nu E_\nu \end{cases} \tag{10}$$

with initial condition $x_{\mathbf{Q}}(0) = x_0 + d\cos\theta_0$, $y_{\mathbf{Q}}(0) = y_0 + d\sin\theta_0$, $\theta(0) = \theta_0$, $\sigma(0) = \theta_0$, $\mu(0) = 0$.

Set $\bar{\kappa} = \max_{\lambda \in [0,+\infty[}\{|\kappa(\lambda)|\}$ the maximum absolute value of the curvature of $\boldsymbol{\gamma}$ and $M = \|[M_x, M_y]^T\|$.

Theorem 3. In the previous hypotheses and notations, suppose that x_0,y_0, and θ_0 are such that:

$$\mathbf{Q}(0) = \boldsymbol{\gamma}(0) \quad \text{and} \quad \dot{\boldsymbol{\gamma}}(0)^T \mathbf{w}(\theta_0) > 0.$$

If the following inequalities hold

$$\begin{aligned} &2(M_\theta d + M) < v, \\ &d\bar{\kappa} + \frac{3(M_\theta d + M) + M_\theta d}{v - 2(M_\theta d + M)} < 1\,, \end{aligned} \tag{11}$$

then we can find a suitable constant $\bar{K}_\theta > 0$ such that $\forall K_\theta > \bar{K}_\theta$, $\forall K_\tau > 0$ and $\forall K_\nu > 0$ there exists one and only one solution defined on $[0, +\infty[$ of the closed-loop system (10) and with the time-varying errors satisfying (7), i.e. the feedback control strategy (9) is well posed for the entire family of uncertain models (6).

Moreover, for any given $\epsilon > 0$, there exist suitable positive constants $\bar{K}_\tau$ and $\bar{K}_\nu$ such that, if $K_\tau > \bar{K}_\tau$, $K_\nu > \bar{K}_\nu$, and $K_\theta > \bar{K}_\theta$ then

$$\sup_{t>0} \mathcal{E}(\mathbf{Q}(t)) < \epsilon \tag{12}$$

Remark 1. The above theorem guarantees a robust path following stability of the proposed controller. Indeed, the front point $\mathbf{Q}$ remains arbitrarily near to the desired path Γ provided that the feedback gains are sufficiently high and the modeling errors obey to conditions (11).

Remark 2. It is interesting to note that the proposed inversion based control architecture combines the (feedforward) dynamic inversion and the feedback action in a novel manner. Indeed, the proposed control strategy uses feedback corrections on the equations of the dynamic inversion generator directly, whereas in analogous control scheme known in the literature (Devasia *et al.*, 1996; Hunt and Meyer, 1997) the inversion based generator is not affected by the feedback.

In the proof of Theorem 3 we use the following lemma whose proof is omitted for the sake of brevity.

Lemma 1. Let it be given a solution of system (10) defined in a closed interval $[0, \bar{t}\,]$, with the error functions satisfying bounds (7) and R a positive constant such that $\boldsymbol{\tau}(\mu(t))^T \boldsymbol{w}(\theta(t)) \geq R$ $\forall t \in [0, \bar{t}\,]$. Provided that $K_\theta > M_\theta / R$ it follows that $\forall t \in [0, \bar{t}\,]$:

$$\begin{aligned} |E_\tau(t)| &\leq \frac{1}{K_\tau}(\frac{vM_\theta}{K_\theta}(1 + \frac{1}{R}) + M_\theta d + \\ &\quad +M)(1 + \frac{\frac{dM_\theta}{K_\theta} + 1}{R - \frac{M_\theta}{K_\theta}}) \\ |E_\nu(t)| &\leq \frac{\frac{vM_\theta}{K_\theta}(1 + \frac{1}{R}) + M_\theta d + M}{dK_\nu(R - \frac{M_\theta}{K_\theta})} \end{aligned} \tag{13}$$

Proof of Theorem 3: By the given initial conditions, it is easy to see that there exists a unique local solution of system (10) and let $[0, \bar{t}\ [$ be its maximum interval of existence; clearly, if we show that $\bar{t} = +\infty$, the first part of the Theorem is proved. This holds if we obtain that $\exists \bar{K}_\theta \geq 0$, such that $\forall K_\theta > \bar{K}_\theta$, $\forall K_\tau > 0$, $\forall K_\nu > 0$, we have that $\inf_{0 \leq t < \bar{t}}\{\boldsymbol{\tau}(\mu(t))^T \mathbf{w}(\theta(t))\} = \inf_{0 \leq \lambda < \bar{\lambda}}\{\boldsymbol{\tau}(\lambda)^T \mathbf{w}(\theta(\mu^{-1}(\lambda)))\ \} > 0$, where $\bar{\lambda} = \sup \mu([0, \bar{t}[)$.

To this aim, set $\alpha(\lambda) = \theta(\mu^{-1}(\lambda)) - \beta(\lambda)$, $\beta(\lambda) = \arg(\boldsymbol{\tau}(\lambda))$, $\alpha_M = \max\{|\alpha(0)|, \arcsin(d\bar{\kappa} + \frac{3(M_\theta d + M) + M_\theta d}{v - 2(M_\theta d + M)})\ \}$, remark that $0 \leq \alpha_M < \frac{\pi}{2}$ since $\boldsymbol{\tau}(0)^T \mathbf{w}(\theta_0) > 0$ and by (11). Set $\lambda' = \sup\{s < \bar{\lambda} | \cos(\alpha(\lambda)) \geq \frac{\cos(\alpha_M)}{2},\ \forall \lambda \in [0, s]\}$, clearly $\lambda' > 0$; we have to show that $\lambda' = \bar{\lambda}$. Suppose, by contradiction, that $\lambda' < \bar{\lambda}$, then it must be:

$$
\begin{aligned}
R &= \inf_{0 \leq \lambda < \lambda'}\{\boldsymbol{\tau}(\lambda)^T \mathbf{w}(\theta(\mu^{-1}(\lambda)))\ \} = \\
&= \inf_{0 \leq \lambda < \lambda'}\{\cos(\alpha(\lambda))\ \} = \frac{\cos \alpha_M}{2}
\end{aligned} \tag{14}
$$

Now we find the expression for $\frac{d\alpha}{d\lambda}$:

$$
\frac{d\alpha}{d\lambda} = \frac{d\theta}{dt}\frac{d\mu^{-1}}{d\lambda} - \frac{d\beta}{d\lambda} = \frac{-\frac{v}{d}\tan(\alpha) - k_\nu E_\nu + e_\theta}{\frac{v}{\cos(\alpha)} + K_\tau E_\tau} - \kappa(\lambda) = -\frac{1}{d}\sin(\alpha) + \cos(\alpha)\frac{\sin(\alpha) K_\tau E_\tau - d K_\nu E_\nu + d e_\theta}{dv + K_\tau E_\tau d \cos(\alpha)} - \kappa(\lambda)\ .
$$

We set $\frac{M_\theta}{\kappa_\theta} = hR$, thanks to (11), applying Lemma 2, the following inequality holds (remark that $R \leq 1$):

$$
\frac{d\alpha}{d\lambda} \leq -\frac{1}{d}\sin(\alpha) + \cos(\alpha) \cdot
$$

$$
\cdot \frac{(vh(R+1) + M_\theta d + M)(1 + \frac{dhR+2}{R-Rh}) + M_\theta d}{vd - d\cos(\alpha)(vh(R+1) + M_\theta d + M)(1 + \frac{dhR+1}{R-hR})} + \bar{\kappa}
$$

$$
\leq -\frac{1}{d}\sin(\alpha) + \cos(\alpha) \cdot
$$

$$
\cdot \frac{((2vh) + M_\theta d + M)(\frac{dh+3}{R-Rh}) + M_\theta d}{vd - d\cos(\alpha)(2vh + M_\theta d + M)(\frac{(1-h)+dh+1}{R-hR})} + \bar{\kappa},
$$

for every h sufficiently small, that is, for K_θ sufficiently big.

Now, applying Lemma 2 to be found at the end of the section, we obtain, unless of decreasing h, that $\alpha(\lambda) \leq \alpha_h, \forall \lambda \in [0, \lambda'[$ where:

$$
\alpha_h = \arcsin\left(\frac{(2vh + M_\theta d + M)(\frac{dh+3}{1-h}) + M_\theta d}{v - (2vh + M_\theta d + M)(\frac{(1-h)+dh+1}{1-h})} + d\,\bar{\kappa}\right).
$$

In the same way we can prove that $\alpha(\lambda) \geq -\alpha_h$, $\forall \lambda \in [0, \lambda'[$ provided that h is sufficiently small. Since $\lim_{h \to 0} \cos \alpha_h \geq \cos \alpha_M$, we can find an $\bar{h}$ such that $\forall h \in [0, \bar{h}[$, $\forall \lambda \in [0, \lambda'[$, $\cos \alpha(\lambda) \geq \cos \alpha_h \geq \cos \alpha_{\bar{h}} > \frac{\cos(\alpha_M)}{2}$, therefore $R > \frac{\cos(\alpha_M)}{2}$ which contradicts (14).

This concludes the the proof of the first part which implies the proof of (12) by Lemma 2 and (13).□

Remark 3. **(Setting of the feedback gains)** Relying on Theorem 3 and Lemma 1, the setting of the feedback gains can be obtained by the procedure below. This guarantees to keep the path following absolute error smaller than a given $\epsilon > 0$.

1) Find a sufficiently small h for which

$$
\frac{(2vh + M_\theta d + M)(\frac{h+3}{1-h}) + M_\theta d}{v - (2vh + M_\theta d + M)(\frac{dh+2}{1-h})} + d\,\bar{\kappa} < 1;
$$

the existence of such h is guaranteed provided that conditions (11) of Theorem 3 hold.

2) Set

$$
R = \sqrt{1 - \left(\frac{(2vh + M_\theta d + M)(\frac{dh+3}{1-h}) + M_\theta d}{v - (2vh + M_\theta d + M)(\frac{(1-h)+dh+1}{1-h})} + d\,\bar{\kappa}\right)^2}.
$$

3) Set the feedback gains according to:

$$
K_\tau \geq \frac{\sqrt{2}}{\epsilon}\left((vh(1+R) + M_\theta d + M)(1 + \frac{1+dhR}{R(1-h)})\right)
$$

$$
K_\nu \geq \frac{\sqrt{2}}{\epsilon}\left(\frac{vh(1+R) + M_\theta d + M}{dR(1-h)}\right)
$$

$$
K_\theta \geq \frac{M_\theta}{hR}.
$$

It is easy to prove the following Lemma.

Lemma 2. Let $f : [0, \delta[\rightarrow \mathbb{R}$ be a function with the following properties: there exist $\bar{y} \in [0, \delta[$ such that $f(y) \leq 0$, $\forall y \in [\bar{y}, \delta[$ and $y \in C^1(I, \mathbb{R})$ where I is a real interval. Suppose that $\dot{y} \leq f(y(t))$, $\forall t \in I$ such that $y(t) \in]\bar{y}, \delta[$. If there exists $t_0 \in I$ such that $y(t_0) \leq \bar{y}$ then $y(t) \leq \bar{y}$, $\forall t \in I$, with $t \geq t_0$.

4. A SIMULATION EXAMPLE

Consider the uncertain model given by equations (8) with the following parameters $v = 25$ m/s, $l = 2.67$ m, $d = 4$ m (data are taken from the ARGO car (Broggi *et al.*, 1999*a*)) and the perturbation functions are bounded as follows:

$\|e_x\|_\infty = 2\,\text{m/s} = M_x$, $\|e_y\|_\infty = 2\,\text{m/s} = M_y$, $\|e_\theta\|_\infty = 2\,\text{deg/s} = M_\theta$.

With simulations the example examines the performances of the proposed dynamic inversion based controller for the path following of a composite road path modelled by quintic G^2-splines (Piazzi and Guarino Lo Bianco, 2000) where the maximum absolute value of the curvature is $\bar{\kappa} = 0.15\ \text{m}^{-1}$ (see Figure 5). The maximum tolerable absolute error for the path following has been set to $\epsilon = 0.1$ m (see (12) of Theorem 3) and the feedback gains of the controller have been determined according to the procedure described in Remark 3: specifically $K_\tau := 19.4$, $K_\nu := 127$, $K_\theta := 5.6$, $h := 0.01$. Figures 6 and 7 report the results of the simulations by plotting the steering control and the absolute error of the front point $\mathbf{Q}$ with respect to the desired path respectively.

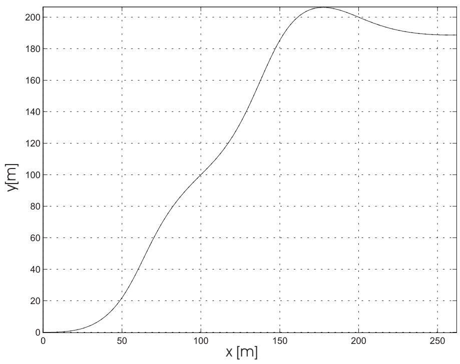

Fig. 5. The desired vehicle's path modelled with quintic G^2-splines.

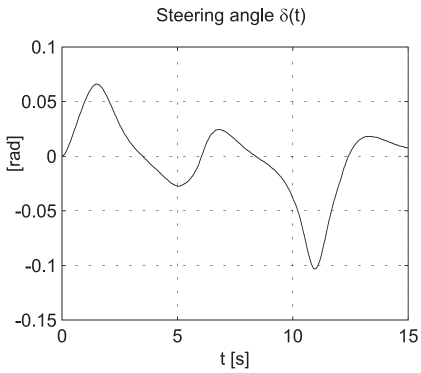

Fig. 6. The steering control for the example.

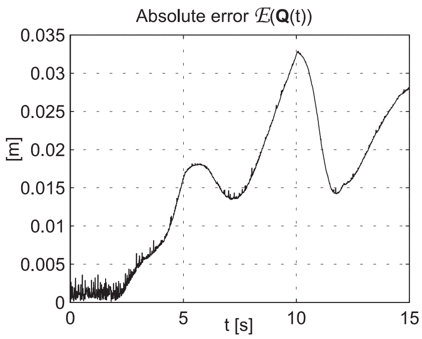

Fig. 7. The absolute path following error for the example.

5. CONCLUSIONS

Essentially, dynamic inversion is a control methodology to synthesize feedforward input signals to achieve desired output functions or output path planning. For the latter case, in the context of autonomous car-like vehicles, a path following controller has been devised by combining, in a new way, the action of a feedforward dynamic inversion with a feedback one. A key point of the paper is the convergence result (Theorem 3) ensuring the robust path following within a guaranteed bound for an entire family of vehicle's uncertain models.

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