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ROBUST MODEL PREDICTIVE CONTROL UNDER OUTPUT CONSTRAINTS

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Abstract: Control of a linear time-varying uncertain dynamical system with delayed inputs is considered. The model parameters, disturbance inputs and model structure errors are unknown but bounded, and the parameter value can abruptly change. The objective is to keep the system output within prescribed limits regardless of the uncertainty scenario. A model predictive type of controller is designed that utilises a set bounded model of the uncertainty and employs safety zones modifying the original constraints so that the control input feasibility can be guaranteed. The controller is applied to quality control in a benchmark Drinking Water Distribution System, and its performance is validated by simulation. *Copyright © 2002 IFAC*

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1. INTRODUCTION

Model Predictive Control (MPC) is an effective tool to deal with time-delay and hard constraints in a systematical way. The issues of optimisation feasibility, stability and performance for linear systems have been investigated to a sufficient maturity in the literatures (Mayne, 2000). The situation gets highly complicated when the output constraints are present and there are model-reality differences. Several properties of MPC have to be rechecked, such as the performance, stability and constraint fulfilling (Beporad and Morari, 1999). The feasible controller action may violate the constraints when applied to the physical system. Much of the work on the MPC that was carried out in rent years addressed this issue. In the papers of Bemporad and Garulli (2000) and Chisci (2001) feasible MPC algorithm was developed that uses a set bounded state estimation and an extra constraint added to the optimisation problem in order to improve the computational efficiency.

In this paper, as in (Brdys, *et al*, 1998), a method of fulfilling output constraint by introducing so called safety zones into the original constraints under the uncertainty in the systems is proposed. For the modified set of constraints a feasible region in a control space is shrank comparing to the original one. Hence, with suitably chosen safety zones the model feasible control input also satisfies constraints of a physical system if the uncertainty radius is not too large. The safety zones are validated and redesigned at each step of MPC, if necessary, by applying a robust plant response prediction based on a set-membership bounded uncertainty model with the proposed control action and an estimate of a system state at a present time instant.

The paper is organised as follows. The problem is formulated in section 2. The robust MPC is performed in section 3, and its design is described in section 4. The controller is applied to quality control in a benchmark Drinking Water Distribution System (DWDS), and its performance is validated by simulation. The results are presented in section 5. The quality control in DWDS is at present extremely important application field still waiting for practically sound solution (Polycarpou, *et al*, 2001).

2. PROBLEM FORMULATION

2.1. Time-varying Bounded Model and Control Objective

The discrete time-varying SISO system is described by the input-output model:

$$y(t) = \sum_{i=1}^{q} b_i(t) y(t-i) + \sum_{i=i_l}^{i=i_u} a_i(t) u(t-i) + \varepsilon_0(t) \qquad (1)$$

where $y(t)$, $u(t)$, $b_i(t)$, $a_i(t)$ and $\varepsilon_0(t) \in R$, $t \in Z$, $y(t)$ is the output, $u(t)$ is the plant input, $b_i(t)$

and $a_i(t)$ are time-varying model parameters, $\varepsilon_0(t)$ is the composite error of modelling and disturbance input, the integer i_l and i_u define range of delays in the control input, and Z denotes set of integers.

Let $\theta(t) = [\theta_1(t) \ \cdots \ \theta_n(t)]$

$= [b_1(t) \ \cdots \ b_q(t) \ \ a_{i_l}(t) \ \cdots \ a_{i_u}(t)]$

where $n = q + i_u - i_l + 1$

$$\varphi(t-1) = [y(t-1)\cdots y(t-q) \quad u(t-i_l)\cdots u(t-i_u)]^T$$

Then equation (1) can be compactly written as:

$$y(t) = \theta(t)\varphi(t-1) + \varepsilon_0(t) \qquad (2)$$

The $\theta(\cdot)$ and $\varepsilon_0(\cdot)$ constitute the model uncertainty. Let $y^p_{[t_0,t]}(\cdot)$ denote the plant response to input $u_{[t_0,t]}(\cdot)$ over time period $[t_0, t]$. Clearly there exist the trajectories of $\theta(\cdot)$ and $\varepsilon_0(\cdot)$ so that with these scenarios of the model uncertainty the model response equals to the plant response, that is $y_{[t_0,t]}(\cdot) = y^p_{[t_0,t]}(\cdot)$. It is assumed that the control input is valued on a compact set so that the trajectories of $\theta(\cdot)$ and $\varepsilon_0(\cdot)$ can be bounded above and below over a time interval $[t_0, t_0 + T_m]$ by the bounded envelopes $\theta^{u,0}(\cdot)$, $\theta^{l,0}(\cdot)$ and $\varepsilon_0^{u,0}(\cdot)$, $\varepsilon_0^{l,0}(\cdot)$ respectively, where t_0 is the initial time and T_m is the modelling horizon. The tightest envelopes are not known. It is assumed that at t_0 a priori envelopes $\theta^u(\cdot)$, $\theta^l(\cdot)$ and $\varepsilon_0^u(\cdot)$, $\varepsilon_0^l(\cdot)$ sufficiently well approximate the unknown envelopes so that no bounding modifications will be considered during control design. Hence the following holds:

$$\theta(t) \in \Theta(t)\,,\ \Theta(t) \overset{\Delta}{=} \left\{\theta \in R^n : \theta^l(t) \le \theta \le \theta^u(t)\right\} \qquad (3)$$

$$\varepsilon_0(t) \in \mathrm{E}_0(t)\,,\mathrm{E}_0(t) \overset{\Delta}{=} \left\{\varepsilon_0 \in R : \varepsilon_0^l(t) \le \varepsilon_0 \le \varepsilon_0^u(t)\right\} \ (4)$$

The envelopes known at t_0 covers the period over control horizon and the uncertainty radius must be sufficiently small so that the control objectives can be robustly achieved. Although there is a modelling error, there is no one set of true parameter values matching the model and plant for all inputs. The model structure reflects well real plant dynamics so that the parameter and modelling error bounds are reasonable.

The control aims at keeping the plant output $y^p(\cdot)$ within the output constraints described by the lower-upper bounds:

$$y^{\min}(t) \le y^p(t) \le y^{\max}(t) \qquad (5)$$

over the control horizon $t \in [t_0, t_0 + T_c]$, and there are constraints on the control input to be satisfied:

$$u^{\min}(t) \le u(t) \le u^{\max}(t)$$
$$|\, u(t) - u(t-1) | \le \Delta u^{\max} \qquad (6)$$

As the control problem is under constraints and the inputs are delayed the Model Predictive Control (MPC) will be applied to design the controller. Hence, it is assumed that $T_c < T_m$.

2.2 Parameter Piece-Wise Constant Bounded Model

Parameter $\theta(t)$ can abruptly change their values at time instant $t_j \in [t_0, t_0 + T_m]$, $j = 1,2,\cdots N_p$ and $t_{N_p} = t_0 + T_m$. Within the time interval $[t_{j-1}, t_j]$ the changes can be assumed slow and also, only certain parameters are active, that is their values are nonzero. The instants t_j are assumed known. Under these assumptions the whole model horizon T_m can be partitioned into N_p time slots, defined as:

$$S_j \overset{\Delta}{=} \left\{t \in Z : t_{j-1} \le t < t_j\right\},\ j = 1,2,\cdots,N_p \qquad (7)$$

Let I_j define set of active parameter indices over S_j. Hence, $\theta^j = \{\theta_{i_j}\}_{i_j \in I_j}$ is the active parameter sub-vector over S_j. Hence, $\theta(t) = \theta^j$ for $t \in S_j$, and θ^j is treated as constant. The corresponding bounds on θ^j are calculated as follows:

$$\theta^j_{\min} = \{\min_{t \in S_j}\left(\theta^l_{i_j}(t)\right)\}_{i_j \in I_j} \ ,\ \ \theta^j_{\max} = \{\max_{t \in S_j}\left(\theta^u_{i_j}(t)\right)\}_{i_j \in I_j} \ (8)$$

Hence, the parameter bounding orthotope Θ_j over S_j can be calculated as:

$$\Theta_j \overset{\Delta}{=} \left\{\theta^j \in R^{\dim(I_j)} : \theta^j_{\min} \le \theta^j \le \theta^j_{\max}, t \in S_j\right\} \ \ (9)$$

The uncertainty in $\varepsilon_0(\cdot)$ is tackled similarly. Finally, the piecewise constant parameter model is given as:

Model $\Re(\cdot)$: $\quad y(t) = \theta(t)\varphi(t-1) + \varepsilon_0(t) \qquad (10)$

$$\theta(t) = \theta^j\ ,\ \text{for } t \in S_j \qquad (11)$$

$$\theta^j \in \Theta_j \quad \varepsilon_0(t) \in E_j \qquad (12)$$

It is understood that the vector $\varphi(\cdot)$ changes its structure following changes of the parameter vector. Model $\Re(\cdot)$ of (10)-(12) is called the parameter piece-wise constant bounding model that will be applied for MPC design. Piecewise constant model is preferred for practical reason of the computational and model identification efficiency.

3. CONTROLLER STRUCTURE

3.1 MPC Based on Nominal Model and Modified Constraints

The nominal model *N(·)* is defined by the nominal scenario of uncertainty. Let us take some kind of centre of Θ_j as the nominal parameter value for $t \in S_j$, denoted as $\hat{\theta}^j$. For example:

$$\hat{\theta}^j = (\theta^j_{\min} + \theta^j_{\max})/2 \qquad (13)$$

Hence, the nominal parameter trajectories satisfy $\hat{\theta}(t) = \hat{\theta}^j$ for $t \in S_j$. It is assumed without any loss of generality that $\hat{\varepsilon}_0^j = 0$. The optimisation problem of MPC at time instant *t* can be formulated as:

$$J(\hat{U}) = \hat{U}^T Q \hat{U} + \mu[y(t + H_p \,|\, t) - y_r]^2 \qquad (14)$$

$$\hat{Y} = \left[y(t+1\,|\,t) \ \ \cdots \ \ y(t+H_p\,|\,t)\right]^T$$

$$\hat{U} = \left[u(t\,|\,t) \ \ \cdots \ \ u(t+H_m-1\,|\,t)\right]^T$$

$$y(t+k\,|\,t) = \hat{\theta}(t+k)\varphi(t+k-1),\quad k = 1,\cdots,H_p \ \ (15)$$

where H_m, H_p are the control dimension and the prediction horizon respectively, $H_m \le H_p$, and Q is a positive-definite matrix. The terminal output of the finite predictive horizon is penalised to the reference value y_r by applying a tuning knob μ. The delays imply that at least $i_l < H_m$. However, preferably $i_u < H_m$. Hence, availability of sufficiently tight uncertainty bounds needed for achieving the control objectives and the delay range may be faced.

Notice that the deterministic nominal model allows to quickly generate control sequence that is optimal for the selected scenario of the uncertainty. However, it is suboptimal, if feasible, for the real plant. The optimality robustness can be improved by formulating the optimisation problem as the min-max one. This however is not further pursued in the paper as solving constrained min-max problem would greatly increase the computational burden even for linear-quadratic problem at hand. The feasibility problem however, needs to be addressed. Minimisation of the performance index giving by (14) over the constraints described by (5) and (6) where the nominal model is used to replace unknown plant mapping $y^p(\cdot)$ yields the solution that may not be feasible when applied to the plant. In order to guarantee the feasibility the nominal model based constraints are modified by introducing so called safety zones. The modified constraints define narrower set for the control actions and this is the price to be paid for model-reality differences.

The output constraints over the prediction horizon described by the upper and lower limits are:

$$Y^{\min} = \left[y^{\min}(t+1) \quad \cdots \quad y^{\min}(t+H_p)\right]^T \qquad (16)$$

$$Y^{\max} = \left[y^{\max}(t+1) \quad \cdots \quad y^{\max}(t+H_p)\right]^T \qquad (17)$$

They are modified by the safety zones σ^l, σ^u as:

$$Y_s^{\min} = Y^{\min} + \sigma^l \qquad Y_s^{\max} = Y^{\max} - \sigma^u$$

$$\sigma^l = [\sigma_1^l \quad \cdots \quad \sigma_{H_p}^l]^T \qquad \sigma^u = [\sigma_1^u \quad \cdots \quad \sigma_{H_p}^u]^T$$

$$[\sigma^l \quad \sigma^u] \in \Sigma$$

$$\Sigma \overset{\Delta}{=} \{[\sigma^l \quad \sigma^u] : \sigma_i^l \ge 0, \sigma_i^u \ge 0, \mathit{for} \quad i = 1 \cdots H_p \ \mathit{and}$$
$$y^{\min}(t+i) + \sigma_i^l < y^{\max}(t+i) - \sigma_i^u\} \qquad (18)$$

The modified model based constraints read:

$$Y_s^{\min} \le \hat{Y} \le Y_s^{\max} \qquad (19)$$

The condition (18) is needed in order to make sure that the modified constraints define nonempty set. The input constraints are treated as hard constraints in the optimisation formulation. The actuator error ε_e will also be considered in implementing a control command. Hence, the input constraints are modified in order to cater for the actuator error as:

$$U^{\min} \le \hat{U} \le U^{\max} \ \text{and} \ -\Delta U^{\max} \le \Delta\hat{U} \le \Delta U^{\max} \qquad (20)$$

$$U^{\min} = \left[u^{\min}(t) + \varepsilon_e \quad \cdots \quad u^{\min}(t+H_m-1) + \varepsilon_e\right]^T$$

$$U^{\max} = \left[u^{\max}(t) - \varepsilon_e \quad \cdots \quad u^{\max}(t+H_m-1) - \varepsilon_e\right]^T$$

$$\Delta U^{\max} = \left[\Delta u^{\min}(t) - 2\varepsilon_e \quad \cdots \quad \Delta u^{\min}(t+H_m-1) - 2\varepsilon_e\right]^T$$

Finally, the MPC optimisation task at *t* reads:

$$\hat{U}(\sigma^l, \sigma^u) = \begin{matrix} \arg & \min\limits_{\hat{U}} (J(\hat{U})) \\ \mathit{subject} & \mathit{to} \quad \hat{U} \in \Omega_s \end{matrix} \qquad (21)$$

$$\Omega_s \overset{\Delta}{=} \left\{ \hat{U} \in R^{H_m} : \quad c_s(\hat{U}, \hat{Y}) \le 0, \hat{Y} = \hat{U} \overset{N(\bullet)}{\rightarrow} \hat{Y} \right\}$$

where $c_s(\cdot)$ is the mapping describing the modified input and output constraints (20) and (19).

The feasibility is assessed by performing at *t* a robust prediction of the plant response to $\hat{U}(\sigma^l, \sigma^u)$. If $\hat{U}(\sigma^l, \sigma^u)$ is feasible for the plant then $u(t \mid t)$ is applied, else σ^l, σ^u are redesigned and new control actions are generated as before. This repeats till suitable safety zones are determined.

3.2 Assessment of Feasibility by Robust Prediction

Set bounded model of uncertainty that is used in this paper enable us to calculate upper and lower envelopes bounding real plant response to a specific input. Comparing these envelopes against the bounds defining the plant output constraints allows to assess the input feasibility. An algorithm for the envelope calculating shall now be presented. The errors in the plant output measurements and in executions of the control inputs (actuator error) are bounded as:

$$| y^m(t) - y^p(t) | \le \varepsilon_m \ \text{and} \ | u^c(t) - u(t) | \le \varepsilon_e \qquad (22)$$

where $y^m(t)$ is the plant output measurement, $u^c(t)$ is the controller output or the actuator input and ε_m and ε_e are the error bounds.

The plant output prediction at *t* over $[t, t+H_p]$ is performed based on a priori information contained in the past inputs and output measurements, future inputs, input-output model equations and uncertainty bounds. This a priori information has been described in a form of equalities and inequalities constraining outputs over $[t, t+H_p]$. Any output trajectory satisfying these constraints can be the plant response. The robust output prediction provides the intervals:

$$Y_p^l = [y_p^l(t+1 \mid t) \quad \cdots \quad y_p^l(t+H_p \mid t)]^T$$

$$Y_p^u = [y_p^u(t+1 \mid t) \quad \cdots \quad y_p^u(t+H_p \mid t)]^T$$

over $[t, t+H_p]$ bounding the plant output values over $[t, t+H_p]$. Hence,

$$y_p^l(t+k \mid t) \le y^p(t+k) \le y_p^u(t+k \mid t) \ \ k = 1 \cdots H_p \qquad (23)$$

Based on model $\Re(\cdot)$ of (10)-(12), the plant output bounding constraints at *t+k* can be summarised as:

$$y(t+m \mid t) - \theta(t+m)\varphi(t+m-1) \in \mathrm{E}(t+m)$$

$$\theta(t+m) \in \Theta(t+m)$$

$$| y^m(t+k-m) - y^p(t+k-m) | \le \varepsilon_m, \mathit{for} \quad k-m \le 0,$$

$$| u^c(t+k-m) - u(t+k-m) | \le \varepsilon_e, \ m = 1, \cdots, k$$

where $E(t) = E_j, \Theta(t) = \Theta_j \ \mathit{for} \quad t \in S_j$. Let P^k denote the set of all $y(t+k \mid t)$ satisfying the above constraints. Hence, the $k-th$ *step* robust output prediction at time instant *t* can be defined as:

$$y_p^l(t+k \mid t) = \begin{matrix} \min\left[y(t+k \mid t)\right] \\ \mathit{subject\,to} \quad y(t+k \mid t) \in P^k \end{matrix} \qquad (24)$$

$$y_p^u(t+k\mid t)= \begin{matrix}\max[y(t+k\mid t)] \\ subject\ to \quad y(t+k\mid t)\in P^k\end{matrix} \quad (25)$$

If the predicted plant output satisfy:

$$Y_p^l \ge Y^{\min} \quad \text{and} \quad Y_p^u \le Y^{\max} \quad (26)$$

then clearly, the assessed control sequences $\hat{U}$ is guaranteed to be feasible. In other words it is robustly feasible.

3.3 Operation of Robust MPC Controller

In general, determining suitable safety zones requires a number of iterations to be performed. The algorithm for iterative calculation of the safety zone will be described in the next section. The overall robust MPC controller is of iterative type and it operates as follows:

(i) Let $[\sigma^l \quad \sigma^u]=0$, solve $\hat{U}$ using (21);

(ii) Calculate Y_p^l, Y_p^u using (24),(25) respectively;
If (26) is satisfied go to (iv)
Else go to (iii)

(iii) Redesign $[\sigma^l \quad \sigma^u]$, and calculate $\hat{U}$ based on this safety-zone design, then go to (ii)

(iv) Let $u^c(t)=u(t\mid t)$

4. ROBUST PREDICTIVE CONTROLLER DESIGN

In the previous section a structure of robust MPC controller has been proposed. An implementation of this structure needs dedicated algorithms for solving variety of problems. Firstly, calculating $\hat{U}(\sigma^l,\sigma^u)$ requires solving constrained linear quadratic optimisation problem. A number of efficient solvers exist to perform this task. Secondly, performing the robust plant output prediction requires solving the nonlinear and non-convex optimisation problems (24) and (25). An approximated solving approach is proposed by piece-wise linearisation, with the linearisation error included in the modelling error. The final problem to be solved is a linear mixed-integer programming (MIP) problem, and it can be solved by using a standard solver. Thirdly, in order to calculate suitable safety zones a penalty function as in (Teo, et al, 1991) related to the constraints (26) is employed. Let us define:

$$C(\sigma^l,\sigma^u)=[f(V_1)\cdots \quad f(V_{2H_p})]^T \quad (27)$$

$$V=[V_1\cdots V_{2H_p}]^T=\left[(Y^{\min}-Y_p^l)^T \quad (Y_p^u-Y^{\max})^T\right]^T$$

$$f(x)=\begin{cases} x & if \quad x>\varepsilon^+ \\ (x+\varepsilon^+)^2/4\varepsilon^+ & if \quad -\varepsilon^+\le x\le\varepsilon^+ \\ 0 & if \quad x<-\varepsilon^+\end{cases} \quad (28)$$

where ε^+ is small positive number. Notice, that if $C(\sigma^l,\sigma^u)=0$ holds then (26) holds as well. Moreover, the constraint $C(\sigma^l,\sigma^u)=0$ can get arbitrarily close to (26) by setting ε^+ sufficiently small. The multiplier type of penalty function associated with (27) is defined as (Fletcher, 1987):

$$\phi(\sigma^l,\sigma^u,\Psi,\Lambda)=\frac{1}{2}[C(\sigma^l,\sigma^u)]^T\Psi[C(\sigma^l,\sigma^u)] - \Lambda C(\sigma^l,\sigma^u)$$

where $\Psi=diag_{2H_p}\psi_i$, $\Lambda=[\lambda_1 \quad \cdots \quad \lambda_{2H_p}]$, λ_i are the multipliers. Under rather mild conditions there exists such value Λ^* of the multiplier Λ that the safety zones can be calculated by solving the following problem (Fletcher, 1987):

$$[\sigma^l \quad \sigma^u]=\underset{[\sigma^l\sigma^u]}{\arg\min}\ \phi(\sigma^l,\sigma^u,\Psi,\Lambda^*) \quad (29)$$
$$subject \quad to \quad [\sigma^l\sigma^u]\in\Sigma$$

If the optimum value of the penalty function is equal to zero then the wanted safety zones are found. If it is nonzero then it means that the MPC controller is not able to produce control that feasibility can be robustly assessed. In this situation, assuming good controllability of the plant, the uncertainty radius must be reduced in order to regain robustly feasible operation of the MPC controller. Noticing that solution for (29) is not unique, finding the smallest safety zones remains an open problem. Following (Powell, 1969) an algorithm for simultaneous solving (29) and finding Λ^* shall be derived. A key assumption needed is second order Fréchet differentiability of the mapping $C(\cdot,\cdot)$. This mapping is a composition of the mappings describing the nominal model based MPC generation (21), robust plant output prediction (24-25) and $f(\cdot)$ (28). Clearly, the later one is smooth. The former ones are defined by a solution of a constrained optimisation problem parameterised by σ^l,σ^u and by $\hat{U}(\sigma^l,\sigma^u)$ respectively. As the parameters enter the constraints the elegant sufficient conditions for differentiability of the solution do not exist (Hager, 1979; Clarke, 1983). However, for broad class of problems the differentiability holds (Findeisen, et al, 1980). Hence, an existence of all derivatives needed is assumed in the sequel. The algorithm I can now be stated as:

Algorithm I:

(i) Set $k=0,\ \Lambda=\Lambda^{(1)}, \Psi=\Psi^{(1)}, \|C^{(0)}\|_\infty=\infty$;

(ii) Find the minimizer of (29) $x(\Psi,\Lambda)$;

(iii) If $\|C\|_\infty>\frac{1}{4}\|C^{(k)}\|_\infty$, $\forall i:|C_i|>\frac{1}{4}\|C^{(k)}\|_\infty$ set $\psi_i=10\psi_i$, and go to (ii);

(iv) Set $k=k+1, \Lambda^{(k)}=\Lambda, \Psi^{(k)}=\Psi, C^{(k)}=C$;

(v) Set $\Lambda=\Lambda^{(k)}-\Psi^{(k)}C^{(k)}$ and go to (ii) until the constraint is fulfilled with desired accuracy, where $x=[\sigma^l,\sigma^u], C=C(x)$.

The algorithm can ensure global convergence with the convergent rate of 0.25 by online tuning in step (iii). If Λ^* is known, (29) can be attempted by Newton methods. Applying the Newton algorithm we obtain that:

$$\nabla\phi(x)=A\Psi C(x)-A\Lambda^* \quad \text{where} \quad A=\nabla C(x)$$

$$\nabla^2\phi(x)=\nabla^2 C(x)[\Psi C(x)-\Lambda^*]+A\Psi A^T=W_x$$

For large values of ψ_i, the following approximation holds (Fletcher, 1987):

$$[A^T W_x^{-1} A]^{-1} \approx \Psi \quad (30)$$

Hence, assuming the inverse of A exist,

$$W_x^{-1} \approx (A^T)^{-1} \Psi^{-1} A^{-1}$$

and Newton method of solving (29) yields:

$$x^{(k+1)} = x^{(k)} - (A^T)^{-1}[C(x^{(k)}) - \Psi^{-1}\Lambda^*] \quad (31)$$

It is a special property of our problem that A is square matrix so that existence of A^{-1} is not unusual. As Λ^* does not depend on Ψ then $\Psi^{-1}\Lambda^* \approx 0$ for large values of ψ_i. Finally, a greatly simplified algorithm is obtained as:

$$x^{(k+1)} = x^{(k)} - (A^T)^{-1} C(x^{(k)}) \quad (32)$$

Notice that the iteration (32) does not require Λ^* any longer. It is surprising at the first glance. However, the formula (32) has a very appealing form. Namely, new safety zones are calculated by a correction of the present ones using the extend of constraint violation. As calculating $(A^T)^{-1}$ may be still computationally demanding further simplification is proposed that consist in replacing $(A^T)^{-1}$ by constant scalar gain. The resulting algorithm becomes of standard relaxation type:

Algorithm II:

(i) Set $x = [\sigma^l \quad \sigma^u] = 0$;

(ii) Solve $\hat{U}$ using (21), if $C(x) = 0$ then go to (iv);

(iii) Using $x^{(k+1)} = x^{(k)} - \nu C(x^{(k)})$ find new safety-zone $[\sigma^l \quad \sigma^u]$, go to (ii);

(iv) Set $u^c(t) = u(t \mid t)$.

where ν is the relaxation gain and its possible choice is $\nu = \max(diag[\nabla C(0)])$.

5. SIMULATION RESULT

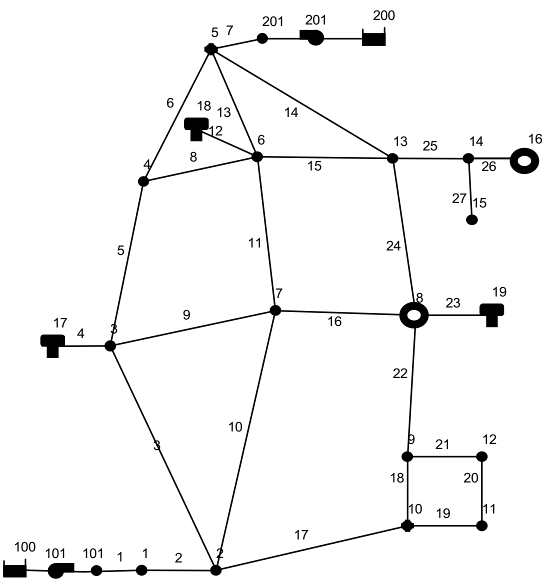

Fig. 1 A Drinking Water Distribution Network

The network structure in Fig.1 was investigated. The chlorine concentrations at node 16 and 8 are the two outputs y_1 and y_2. Two booster stations are installed at the nodes 5 and 10. The chlorine concentrations at the injection nodes are the inputs u_1 and u_2. It is a 2-input 2-output system. The chlorine concentrations at the monitored nodes should be maintained within 0.20[*mg/l*]-0.30[*mg/l*] limits. Hence, $y^{\max} = 0.30[mg/l]$ and $y^{\min} = 0.20[mg/l]$. As quality controller is the lower layer in a hierarchical integrated quantity and quality control structure (Brdys, *et al*, 2001), quantity information is available before run quality control. The same as for the quantity control a 24-hour control horizon is considered. The sampling period is 5 minutes yielding totally 288 steps for the whole control horizon.The water network was implemented using EPANET2.0(Rossman, 2000) in order to simulate the plant responses. The chlorine concentration can be described by an input-output model as in (Polycarpou, 2001):

$$y_n(t) = \sum_{i=1}^{2} b_{ni}(t) y_n(t-i) + \sum_{m=1}^{2} \sum_{i=6}^{i=24} a_{m,n,i}(t) u_m(t-i)$$

where $n = 1,2$, $a_{m.n.i}$ describes an impact of the injection input *m* that is delayed by *i* steps on the output *n*. The delay range is 6-24, yielding 76 parameters. Fig.2 shows the envelopes bounding the parameters $a_{1,1,7}$ over a whole horizon of 288 steps. Notice that the parameters are active only over certain time periods. With these parameter estimation results, the modelling error was $\pm 4\%$ of the plant output value.

In the following presentation of the simulation results, the chlorine concentration was scaled by the factor of 0.25*mg/l*, so the upper and lower output limits are converted into 0.8 and 1.2 respectively, and the output reference is $y_r = 1.0$. The measurement and actuator error were $\pm 2.5\%$ of the measurement value and controller output value respectively, and the corresponding error bounds of $\varepsilon_m, \varepsilon_e$ were obtained. The MPC described in section 2 was designed using *algorithm II*, where $H_m = H_p = 36$. The controller starts with zero safety zones. Its operation over a whole horizon and the output constraint violation are illustrated in Fig. 3. The violation is about 5% above the output limit around steps *t=140 and t=230*. The operation of the controller over the same time period but with the modified output constrains by safety zones is shown in Fig.4, hence achieving the feasibility. The safety zones generated at step *t=204* are illustrated in Fig. 5. The relaxation gain used was $\upsilon = 1.0$.

6. CONCLUSION

A robust MPC controller has been developed for keeping an output of a linear time varying systems under uncertainty within prescribed limits with delayed inputs. The uncertainty in: time varying parameters, measurement and actuator errors and modelling error has been modelled applying set-bounded models. The safety zones have been introduced to modify the model-based output constraints so that control input feasibility can be robustly achieved. The safety zones have been iteratively designed at each generation time instant of the MPC based on the envelopes bounding the predicted plant output responses. Algorithms for generating the safety zones based on the constraint violations extend over the MPC prediction horizon have been derived and their convergence has been analysed. An efficient simple relaxation scheme has been designed to reduce the computational burden. The controller has been applied to a DWDS to maintain chlorine concentration at a monitored demand node within prescribed limits by controlling injection of the chlorine at the booster station nodes. The simulation results have illustrated good performance of the controller.

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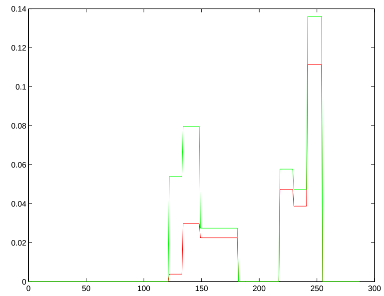

Fig.2 Piece-wise Constant Model:Parameter Example

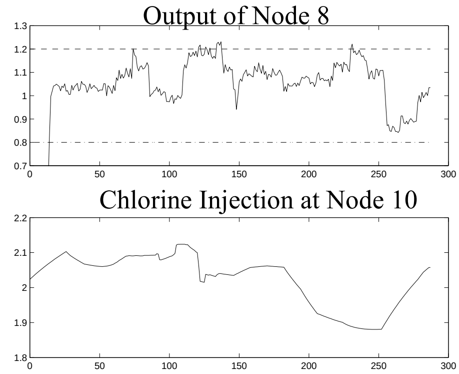

Fig.3 Controller Performance with Zero Safety Zones

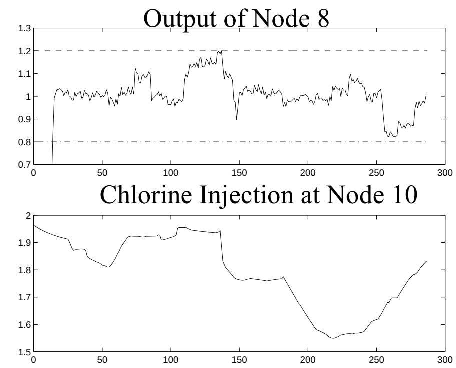

Fig.4 Controller Performance with Safety Zone (Dashed: Upper Limit, Dash-Dot: Lower Limit)

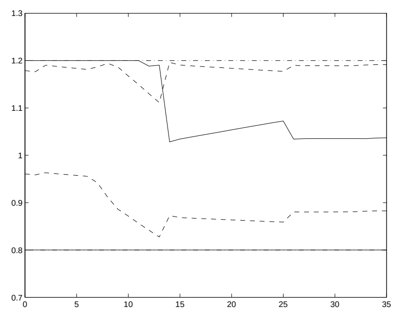

Fig.5 An Example of Safety Zone Design (Solid: Modified Output Constraints Dashed: Robust Output Prediction Envelope Dash-Dot: Original Output Constraints)