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$\mathcal{L}_2$-GAIN OF DOUBLE INTEGRATORS WITH SATURATION NONLINEARITY

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Abstract: This paper uses quadratic surface Lyapunov functions to efficiently check if a double integrator in feedback with a saturation nonlinearity has $\mathcal{L}_2$-gain less than $\gamma > 0$. We show that for many of such systems, the $\mathcal{L}_2$-gain is non-conservative in the sense that they are approximately equal to the lowerbound obtained by replacing the saturation with a constant gain of 1. These results allow the use of classical analysis tools like μ-analysis or IQCs to analyze systems with double integrators and saturations, including servo systems like some mechanical systems, satellites, hard-disks, CD players, etc.

Keywords: Robust Analysis, Saturations, Double Integrator, Quadratic Surface Lyapunov Functions

1. INTRODUCTION

There are many control applications that can be modeled as a plant with a single integrator, a saturation nonlinearity, and a PI controller as shown in figure 1. One of the most simple one is the position control of a body with a PI controller and a power limit actuator. In this case, the force $F = m\ddot{x} + k\dot{x}$, where m and k represents the mass of the body and the coefficient of friction, respectively. Typically if the position $x(t)$ is to track some reference command $u(t)$, a PI controller is used. In this case, $P(s) = (ms + k)^{-1}$.

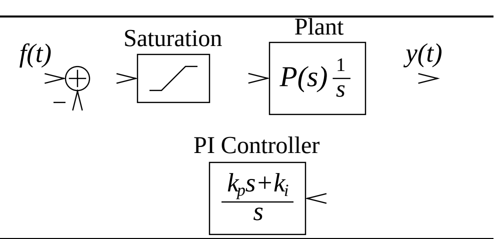

Fig. 1. PI position control system with power limited actuator

Not only systems satisfying the Newton's law $F = ma$ can be modeled as in figure 1. Many servo systems, including mechanical systems, are often modeled this way. A double integrator system may be used as a simple model for satellite control, modeling the relation between the angular position and velocity and the reaction jets. Other examples are the control of a hard-disk drive head, the laser beam of a CD, etc.

Analysis of saturation systems with double integrators has been done for many years. As explained in (Kao, 2001), in order to perform robustness analysis the system is typically transformed into one shown in figure 2, where the saturation is treated as an uncertainty. The problem with this approach is that it gives us a nominal plant that is marginally unstable, preventing us to apply some classical analysis tools such as the Popov criterion, μ-analysis, and Integral Quadratic Constraints (IQCs).

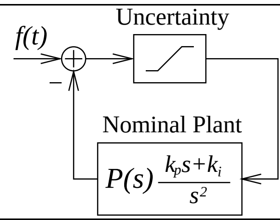

Fig. 2. Nominal system and uncertainty

An alternative is to encapsulate the unstable operator in an artificial feedback loop which defines a bounded operator. Robustness analysis can then be performed on the transformed system which consists of bounded operators. Assuming $P(s)$ is stable, this leaves us with the double integrator and the saturation to worry about. A possible loop transformation is shown in figure 3. In order to analyze the system, we mush first check if Δ is a bounded operator. In this case, Δ is a double integrator in feedback interconnection with a saturation nonlinearity, where the output consists of signals from both the first and second integrator. The question whether the system $\ddot{x} = sat(-c_1 x - c_2 \dot{x} + u)$ has finite $\mathcal{L}_2$-gain from u to x, $\dot{x}$, or $\ddot{x}$, has been posted as an open problem (Blondel *et al.*, 1999). It has been shown, meanwhile, that the $\mathcal{L}_2$-gain from u to x is infinite (Liu *et al.*, 1996), and the $\mathcal{L}_2$-gain from u to $\dot{x}$ is also infinite (Kao, 2001). The question as if the $\mathcal{L}_2$-gain from u to $\ddot{x}$ is finite is still open. This means the loop transformation in figure 3 results in an unstable operator Δ.

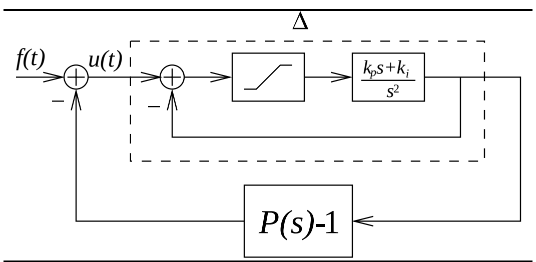

Fig. 3. Loop transformation with an *unstable* operator Δ

In this paper we propose the loop decomposition shown in figure 4, where k_1, k_2, and $G(s)$ are functions of k_p, k_i, and $P(s)$, and $G(s)$ is stable. The loops of both systems in figures 1 and 4 are identical and analysis properties can be inferred from one to another and vice versa. The low-pass filter is used to exclude high frequency content from the feedback loop, as expected from real applications. In order to perform analysis on the transformed system in figure 4, we need first to show that Δ is a stable operator. Thus, the goal of this paper is to give sufficient conditions to check if the $\mathcal{L}_2$-gain of Δ is finite for given $k_1 > 0$, $k_2 \geq 0$, and $\alpha > 0$. We show that our method is not conservative for many values of k_1, k_2, and α since we able to find upper bounds of the $\mathcal{L}_2$-gain of Δ that were approximately equal to lower bounds obtained when the saturation is replaced by a unity constant gain. The method is based on constructing quadratic Lyapunov functions on the switching surface associated with the saturation system. The construction of such Lyapunov functions is done by solving a set of LMIs.

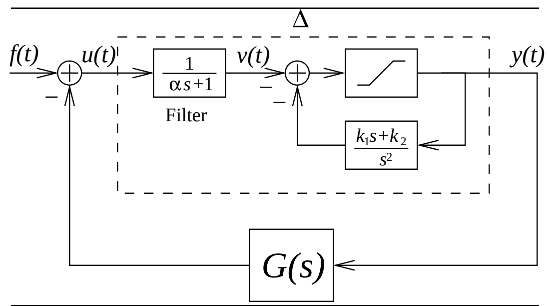

Fig. 4. Loop transformation with *stable* operators

This paper is organized as follows. The following section gives some preliminary definitions and results. Section 3 contains the main result of the paper. There, conditions in the form of LMIs are given to check if γ is an upper bound of the $\mathcal{L}_2$-gain of Δ in figure 4. Section 4 contains several illustrative examples and, finally, section 5 gives conclusions. Due to space limitations, this paper excludes many important details, including the proof of the main result. These can all be found at (Gonçalves, 2001).

2. PRELIMINARIES

Let $\mathcal{L}_2$ denote the space of all functions $f : [0, \infty) \rightarrow \mathbb{R}$ which are square summable, i.e.,

$$\|f\|^2 = \int_0^\infty f^2(t)dt < \infty$$

The extended space $\mathcal{L}_{2e}$ consists of all functions $f(t)$ which satisfy $P_T f(t) \in \mathcal{L}_2$, for all $T \geq 0$, where P_T is a truncation operator defined as $(P_T f)(t) = f(t)$ if $t \leq T$ and $(P_T f)(t) = 0$ otherwise.

We say that the $\mathcal{L}_2$-gain from input u to output y of some system is less than $\gamma \geq 0$ if

$$\int_0^T y^2(t)dt \leq \gamma \int_0^T u^2(t)dt \qquad (1)$$

for all $T \geq 0$, and all $u \in \mathcal{L}_{2e}$. The $\mathcal{L}_2$-gain γ^* of the system from u to y is the infimum over all γ such that (1) is satisfied.

Consider the operator Δ in figure 4. For given k_1, k_2, α, we are interested in finding an upper bound of the $\mathcal{L}_2$-gain of Δ. The following proposition gives an easy way to find a lower bound of the $\mathcal{L}_2$-gain of Δ. The proof, based on the fact that the saturation behaves linearly for small inputs, can be found in appendix B in (Gonçalves, 2001).

Proposition 2.1. Consider the system Δ in figure 4. The $\mathcal{L}_2$-gain γ_L of the same system but with the saturation replaced by a constant gain of 1 is a lower bound of the $\mathcal{L}_2$-gain of Δ, i.e., $0 \leq \gamma_L \leq \gamma^*$.

Note that when the saturation is replaced by a constant gain of 1, the system becomes linear. Thus, γ_L is simply the square of the $\mathcal{H}_\infty$ of the linear system

$$\frac{Y(s)}{U(s)} = \frac{s^2}{(\alpha s + 1)(s^2 + k_1 s + k_2)}$$

From this expression we immediately see that it is necessary $k_1 > 0$, $k_2 \geq 0$, and $\alpha > 0$, or otherwise $\gamma_L = \infty$. When $k_2 = 0$ the original system is reduced to a single integrator which was studied in (Jönsson and Megretski, 2000; Megretski, 2001). The proof of the following proposition can also be found in appendix B in (Gonçalves, 2001).

Proposition 2.2. Consider the system Δ in figure 4. If there exists an $\alpha = \alpha_1 > 0$ such that the $\mathcal{L}_2$-gain of Δ is finite then the $\mathcal{L}_2$-gain is finite for any $\alpha > 0$.

A state-space representation of system Δ in figure 4 is

$$\begin{cases} \dot{x}_1 &= k_2 x_2 \\ \dot{x}_2 &= y \\ \dot{v} &= -\dfrac{1}{\alpha} v + \dfrac{1}{\alpha} u \\ y &= sat(-x_1 - k_1 x_2 - v) \end{cases} \qquad (2)$$

Let $x = [x_1 \;\; x_2 \;\; v]'$ and $C = [1 \;\; k_1 \;\; 1]$. In the state-space, the system can be seen as piecewise linear system, with 3 cells and two switching surfaces (see figure 5). The switching surfaces are

$$S = \{x \in \mathbb{R}^n : \; Cx = 1\}$$

and $\underline{S} = -S$. When $Cx \geq 1$, $\dot{x}_2 = -1$, when $Cx \leq -1$, $\dot{x}_2 = 1$, and, finally, when $-1 \leq Cx \leq 1$, $\dot{x}_2 = -Cx$.

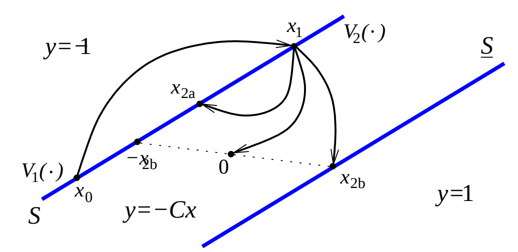

Fig. 5. Possible trajectories in the state-space

3. DOUBLE INTEGRATOR

Assume that $k_2 > 0$. The following matrices will be needed in the main result. For some $T > 0$, let

$$W_a(T) = \begin{pmatrix} \frac{k_1}{k_2} - \frac{T}{2} \\ 0 \\ \frac{k_1}{k_2} + \frac{T}{2} \\ 0 \end{pmatrix}, W_b(T) = \begin{pmatrix} -\frac{1}{k_2 T} & \frac{1}{k_2 T} \\ 1 & 0 \\ -\frac{1}{k_2 T} & \frac{1}{k_2 T} \\ 0 & 1 \end{pmatrix}$$

and

$$W_j(T) = \frac{2\gamma\alpha}{1 - e^{-\frac{2T}{\alpha}}} \begin{pmatrix} 1 \\ -e^{-\frac{T}{\alpha}} \end{pmatrix} \begin{pmatrix} 1 & -e^{-\frac{T}{\alpha}} \end{pmatrix}$$

Define also

$$A = \begin{pmatrix} 0 & k_2 & 0 \\ -1 & -k_1 & -1 \\ 0 & 0 & -\frac{1}{\alpha} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\alpha} \end{pmatrix}$$

and

$$H = \begin{pmatrix} A & BB'/\gamma \\ -C'C & -A' \end{pmatrix} \qquad (3)$$

Let $e_{ij}(T)$ be the square matrices of dimension 3 by 3 obtained from e^{HT} and

$$W_t(T) = \begin{pmatrix} E_1(T) & E_3(T) \\ E_3'(T) & E_2(T) \end{pmatrix}$$

where $E_1(T) = e_{22}e_{12}^{-1}$, $E_2(T) = e_{12}^{-1}e_{11}$, and $E_3(T) = \left(e_{21} - e_{22}e_{12}^{-1}e_{11} - (e_{12}^{-1})'\right)/2$, where the notation $e_{ij} = e_{ij}(T)$ was used for simplification. Finally, define

$$W_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \; W_2 = \begin{pmatrix} -k_1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -k_1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $W_3 = (\,-1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0\,)'$.

We are now ready for the main result of the paper. In this result, we sometimes drop the argument (T) for simplification.

Theorem 3.1. Consider the system Δ in figure 4. Given $k_1, k_2, \alpha > 0$, let $\gamma \geq \gamma_L$. Let also $p > 0$ be a 2 by 2 diagonal matrix and $g \in \mathbb{R}^2$. Define

$$P = \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix}, \quad G = \begin{pmatrix} g \\ -g \end{pmatrix}, \quad \bar{G} = \begin{pmatrix} -g \\ -g \end{pmatrix}$$

and $r_{12}(T) = -T - W_a'PW_a - 2W_a'G$, $r_{13}(T) = -W_b'(PW_a + G)$, $r_{23}(T) = W_2'W_tW_1 - G$, and $r_{33}(T) = W_2'W_tW_3 - \bar{G}$. If

$$R_1(T) \stackrel{\text{def}}{=} \begin{pmatrix} W_j - W_b'PW_b & r_{13}(T) \\ r_{13}'(T) & r_{12}(T) \end{pmatrix} > 0 \quad (4)$$

$$R_{2a}(T) \stackrel{\text{def}}{=} \begin{pmatrix} W_2'W_tW_2 - P & r_{23}(T) \\ r_{23}'(T) & W_1'W_tW_1 \end{pmatrix} > 0 \quad (5)$$

$$R_{2b}(T) \stackrel{\text{def}}{=} \begin{pmatrix} W_2'W_tW_2 - P & r_{33}(T) \\ r_{33}'(T) & W_3'W_tW_3 \end{pmatrix} > 0 \quad (6)$$

for all $T > 0$ then the $\mathcal{L}_2$-gain of Δ is less or equal than γ.

The last theorem gives us a set of infinite dimensional LMIs that, when satisfied, guarantee that Δ not only has finite $\mathcal{L}_2$-gain, but also that this is upper bounded by γ. This allows us to write an IQC of the form

$$\int_0^T y^2(t)dt \leq \gamma \int_0^T u^2(t)dt \qquad (7)$$

which, in turn, allows us to perform robustness and performance analysis on the system in figure 4 or, equivalently, on the original system in figure 1.

The method of proof is as follows. First, inequality (7) is satisfied if for every $u \in \mathcal{L}_{2e}$ there exists a Lyapunov function $V(\cdot)$ such that the solution $x(t)$ from the initial state $x(0) = 0$ satisfies

$$\int_{T_i}^{T_f} \left[\gamma u^2(t) - y^2(t)\right] dt \geq V(x(T_f)) - V(x(T_i)) \quad (8)$$

for all $0 \leq T_i \leq T_f$. To see this, let $T_i = 0$. Then, $V(x(0)) = 0$ and $V(x(T_f)) \geq 0$, since V is a Lyapunov function.

Figure 5 shows possible trajectories of (2) starting at S. Depending on the control input u, a trajectory may enter the region where $y = -1$. Since $u \in \mathcal{L}_{2e}$, a switch must eventually occur at some point $x_1 \in S$. The control u may also be such that the trajectory enters the linear region where $y = -Cx$. In this case, there are three possibilities: the trajectory does not switch again and goes to zero as $t \to \infty$, it returns to S, or it intersects $\underline{S}$. Since the system is symmetric around the origin, for analysis purposes, any other trajectories can be reduced the ones just described.

Second, define two Lyapunov functions V_1 and V_2 on the switching surface S. Condition (8) is satisfied if

$$\int_0^{T_1} \left[\gamma u_1^2(t) - y^2(t)\right] dt \geq V_2(x_1) - V_1(x_0)$$

$$\int_0^{T_{2a}} \left[\gamma u_{2a}^2(t) - y^2(t)\right] dt \geq V_1(x_{2a}) - V_2(x_1)$$

$$\int_0^{T_{2b}} \left[\gamma u_{2b}^2(t) - y^2(t)\right] dt \geq V_1(-x_{2b}) - V_2(x_1)$$

for all $x_0, x_1, x_{2a}, -x_{2b} \in S$, and $T_1, T_{2a}, T_{2b} > 0$, and where $u_1(t) \in \mathcal{L}_2$ is such that a trajectory starting at x_0 satisfies $x_1 = x(T_1)$ and $y = -1$, $t \in [0, T_1]$, and $u_i(t) \in \mathcal{L}_2$, $i = 2a, 2b$ is such that a trajectory starting at x_1 satisfies $x_i = x(T_i)$ and $y = -Cx$, $t \in [0, T_i]$.

Finally, under certain assumptions, the inputs u_i, $i = 1, 2a, 2b$, that minimize the integrals on the left side of the above inequalities can be explicitly found. If the Lyapunov functions are chosen to be quadratic functions, the result are conditions (4)-(6). The complete proof can be found in section 3 of (Gonçalves, 2001).

4. EXAMPLES

In order to solve an infinite dimensional set of LMIs, there are some extra steps we need to take to make this solution computationally attractive. Obviously, it is not possible to solve the three quadratic inequalities for all $T > 0$. The idea is to find a finite sequence of times $\{T_i\}$ defined on some bounded set $\mathcal{T} = (0, T_+]$ such that it is sufficient (4)-(6) are satisfied in $\mathcal{T}$ to prove the desired result. It is then necessary to guarantee they are also satisfied in $T \in (T_+, \infty)$, and $T \in (T_i, T_{i+1})$ for all $T_i, T_{i+1} \in \mathcal{T}$. The latest can be guaranteed by estimating bounds on the derivative of each condition (4)-(6) between $T_i, T_{i+1} \in \mathcal{T}$. Conditions to guarantee that (4)-(6) are also satisfied in $T \in (T_+, \infty)$, for some $0 < T_+ < \infty$, are given in propositions C.1 and C.2 in (Gonçalves, 2001).

The following examples were processed in `matlab` code. The latest version of this software is available at (Gonçalves' web page, 2001). Before presenting the examples, we briefly explain the `matlab` function we developed. The user supplies $k_1 > 0$, $k_2 \geq 0$ (the case when $k_2 = 0$ results in the single integrator which will be dealt in the next section), and $\alpha > 0$. If all three conditions (4)-(6) are satisfied for all $T \in \mathcal{T}$, the function returns a graphic showing the minimum eigenvalues of each $R_i(T)$, which, obviously, must be positive for all $T \in \mathcal{T}$.

Example 4.1. Let $k_1 = 0.5$, $k_2 = 2$, and $\alpha = 2$. In this example, we find the smallest upper bound γ of the $\mathcal{L}_2$-gain of Δ in figure 4 using theorem 3.1. A lower bound can be found by computing the linear gain, i.e., the $\mathcal{L}_2$-gain of Δ when the saturation nonlinearity is replaced by a constant gain of 1. Here, this is $\gamma_L = 0.8892297$. Using the software described above, we found an upper bound of the $\mathcal{L}_2$-gain of Δ of $\gamma = 0.8892299$. Note that the difference between the upper and lower bound is smaller than 2×10^{-7}, i.e., the precision is less than $2.15 \times 10^{-5}\%$.

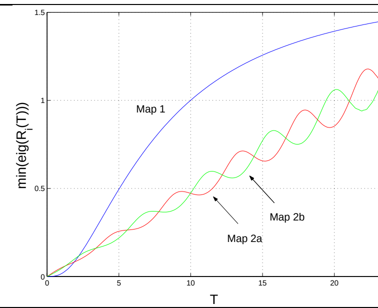

Fig. 6. Minimum eigenvalues of $R_i(T)$, $i = 1, 2a, 2b$

Figure 6 shows the minimum eigenvalues of $R_i(T)$, $i = 1, 2a, 2b$. For visualization purposes, the minimum eigenvalues of $R_{2a}(T)$ and $R_{2b}(T)$ were scaled by 2×10^6. ■

Example 4.2. Let $k_1 = k_2 = 1$. In this example we find the smallest upper bound γ of the $\mathcal{L}_2$-gain of Δ for different values of $\alpha > 0$. The left side of figure 7 shows the lower bound γ_L and the upper bound γ on the $\mathcal{L}_2$-gain of Δ using theorem 3.1. The right side of figure 7 plots $\gamma - \gamma_L$. Logarithmic scales were used for better visualization.

From this figure we can see that the difference between the upper and lower bound goes to zero as α goes to infinity. In fact, for $\alpha > 0.5$ the difference between γ and γ_L is less than 0.76%. For $\alpha > 5$ this difference is already smaller than 0.009% and less than $6 \times 10^{-8}\%$ for $\alpha > 100$.

If $\gamma \geq \gamma_L$ is chosen small enough, the Hamiltonian matrix H in (3) has pure imaginary

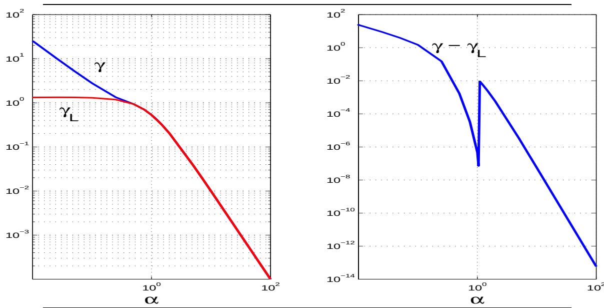

Fig. 7. γ and γ_L as a function of α (left) and $\gamma - \gamma_L$ (right)

eigenvalues. For $\alpha \geq 0.5$, it turns out that for *all* $\gamma > \gamma_L$ such that H has no pure imaginary eigenvalues, it was *always* possible to find p, g such that conditions (4)-(6) are satisfied. In other words, numerically we found that for $\alpha \geq 0.5$ conditions (4)-(6) are satisfied if and only if H has no pure imaginary eigenvalues. Thus, for $\alpha \geq 0.5$, figure 7 also shows the smallest γ such that H does not have pure imaginary eigenvalues. For $\alpha < 0.5$, however, we encountered several numerical problems and γ tended to be higher than smallest γ such that H has no pure imaginary eigenvalues.

Several questions can now be raised: is the gap between γ and γ_L increasing as α approaches zero due to numerical errors, conservatism of the method, or the fact that the $\mathcal{L}_2$-gain of the system is just larger than γ_L, and this gap increases as α approaches zero? Or is true that $\gamma = \gamma_L$ or $\gamma \approx \gamma_L$ for all $\alpha > 0$? Answers to such questions are currently under investigation.

For sure, this example shows that our method is *not conservative*, except maybe for small values of α, since the upper and lower bounds of the $\mathcal{L}_2$-gain of Δ are almost identical. ■

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5. CONCLUSIONS

This paper gives conditions in the form of LMIs that, when satisfied, guarantee a system with a double integrator in feedback with a saturation nonlinearity has finite $\mathcal{L}_2$-gain. Moreover, for a large class of such systems, we showed that the linear $\mathcal{L}_2$-gain of the system, i.e., the $\mathcal{L}_2$-gain of the same system but with the saturation nonlinearity replaced by a constant gain of 1, is approximately equal to the $\mathcal{L}_2$-gain of the original system. These results allow the use of classical analysis tools like μ-analysis or IQCs to analyze systems with double integrators and saturations, including servo systems like some mechanical systems, satellites, hard-disks, CD players, etc.

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