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STABILIZATION OF ST VENANT EQUATIONS USING RIEMANN INVARIANTS : APPLICATION TO WATERWAYS WITH MOBILE SPILLWAYS

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Abstract: This paper deals with the regulation of water flow and level in water-ways made up of channels separated by spillways. This control problem is solved by using boundary control based on Riemann invariants. The control performance is illustrated with an experiment for the Sambre river.

Keywords: Boundary control, Open channel, Riemann invariants, Regulation, Shallow water equations, Partial differential equations.

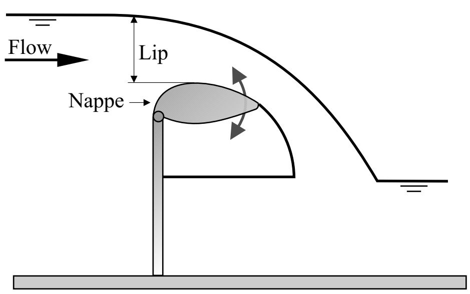

Fig. 1. A spillway.

1. INTRODUCTION

Canalised waterways are often made up of a cascade of reaches separated by hydraulic structures such as mobile spillways (see Fig. 1). When the water flow rate is slow enough, the water passes over the nappe and the discharge depends on the upstream lip but does not depend on the downstream. In order to guarantee the navigation, an important issue is to stabilize the water level in the reaches in spite of variations of the natural flow rate of the river. In this paper, the flow is modelled using the Saint Venant PDEs. The stabilization problem is solved by using boundary control based on Riemann invariants. The control performance is illustrated with a realistic simulation experiment for the Sambre river in Belgium. The paper is organized as follows:

In Section 2, we present the Saint Venant partial differential equations for the modelling of an horizontal reach. The steady-state of the system is calculated and the control objective is formulated. Section 3 is devoted to the stability analysis of the steady-states and the control design. The model is reformulated in terms of Riemann invariants which are more convenient for our purpose. The stability of the steady-states is analysed in Theorem 1 which is derived from a general result of (Greenberg and Li, 1984) concerning the stability of quasilinear wave equations. On this basis, a control law based on Riemann invariants is proposed. Its efficiency is illustrated with a simulation experiment.

Finally in Section 4, we investigate the applicability of this control approach to a part of the Sambre river (Belgium) composed of a cascade of 7 reaches for a total length of about 50 km.

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2. MODEL OF A HORIZONTAL REACH

2.1 *Saint Venant equations*

We consider a one-dimensional portion of canal as represented in Fig. 2. The dynamics of the system are described by Saint Venant equations (see e.g. (Malaterre, 1994; Chow, 1954; Graf, 1998; Gerbeau and Perthame, 2000)). We restrict our attention to the case of an horizontal reach with a prismatic section, without viscous friction terms and we suppose that the flow is sub-critical (3), so that the dynamical equations simplify as follows :

Continuity equation :

$$\partial_t H + \partial_x (VH) = 0. \tag{1}$$

Dynamical equation :

$$\partial_t V + \partial_x (gH + \frac{V^2}{2}) = 0, \tag{2}$$

where $x \in [0, L]$ is the space coordinate, $t \in [0, T]$ is time, ∂_x, ∂_t are the partial derivative w.r.t. x,t respectively, L is the reach length, $V(x,t)$ is the water velocity (at point x and time t), $H(x,t)$ is the water level (at point x and time t) and g is the gravity constant. The sub-critical flow condition is :

$$V < \sqrt{gH} \tag{3}$$

The water flow rate is defined as (we suppose an unitary width):

$$Q(x,t) = V(x,t)H(x,t) \tag{4}$$

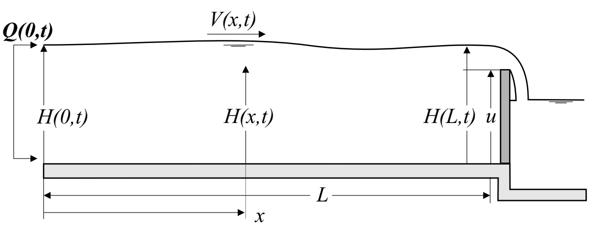

Fig. 2. The horizontal reach.

The inflow rate at $x = 0$ is therefore:

$$Q(0,t) = V(0,t)H(0,t) \tag{5}$$

The control action is provided by one weir gate located at the right end ($x = L$) of the reach (see Fig. 1). The gate opening is denoted u. A standard discharge relationship of weir gates (see e.g. (Graf, 1998, Chapter 4)) is as follows :

$$V(L,t)H(L,t) = k(H(L,t) - u)^m \tag{6}$$

where $k > 0$ and $m \in [1, 3/2]$ are constant parameters.

Equations (5) and (6) are the boundary conditions at $x = 0$ and $x = L$, associated with the PDEs (1)-(2).

2.2 *Steady-states*

For given constant opening $\bar{u}$ and constant inflow rate $\bar{Q}$, there exists a steady state solution $(\bar{V}, \bar{H})$ of equations (1), (2) which satisfies, from (5) and (6), the following relations :

$$\bar{H} = u + \left(\frac{\bar{Q}}{k}\right)^{1/m} \qquad \bar{V} = \frac{\bar{Q}}{\left(u + \frac{\bar{Q}}{k}\right)^{1/m}} \tag{7}$$

2.3 *Statement of the control problem*

The control objective is to stabilize the system (1), (2), (5), (6) around a set point $(\bar{H}, \bar{Q})$. The control action is the gate opening u. The water level $H(L,t)$ supposed to be measured online at each time instant t. The water level set point $\bar{H}$ is selected in order to satisfy navigability requirements. The flow rate set point $\bar{Q}$ is selected on the basis of meteorological forecasts.

3. STABILITY ANALYSIS FOR STEADY-STATES AND CONTROL DESIGN

3.1 *Characteristic velocities*

We can rewrite system (1)-(2) in a matrix-vector form :

$$\partial_t \begin{pmatrix} H \\ V \end{pmatrix} + A(H,V)\partial_x \begin{pmatrix} H \\ V \end{pmatrix} = 0 \tag{8}$$

with the *characteristic* matrix :

$$A(H,V) = \begin{pmatrix} V & H \\ g & V \end{pmatrix} \tag{9}$$

The eigenvalues of this matrix:

$$c_\alpha(H,V) = V + \sqrt{gH}, \quad c_\beta(H,V) = V - \sqrt{gH} \tag{10}$$

are called the characteristic velocities. The sub-critical flow hypothesis (3) implies :

$$c_\beta(H,V) < 0 < c_\alpha(H,V) \tag{11}$$

3.2 *Model in terms of Riemann invariants*

Let us now consider the following change of coordinates:

$$\alpha = V - \bar{V} + 2(\sqrt{gH} - \sqrt{g\bar{H}}) \tag{12}$$

$$\beta = V - \bar{V} - 2(\sqrt{gH} - \sqrt{g\bar{H}}). \tag{13}$$

where $(\bar{H}, \bar{V})$ is an arbitrary steady-state.

With these new coordinates (α, β), system (1)-(2) is rewritten into the following diagonal form:

$$\partial_t \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \begin{pmatrix} c_\alpha(\alpha,\beta) & 0 \\ 0 & c_\beta(\alpha,\beta) \end{pmatrix} \partial_x \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{14}$$

where c_α and c_β are the characteristic velocities now expressed in terms of α, β:

$$c_\alpha(\alpha,\beta) = \frac{3}{4}\alpha + \frac{1}{4}\beta + \bar{V} + \sqrt{g\bar{H}} \tag{15}$$

$$c_\beta(\alpha,\beta) = \frac{1}{4}\alpha + \frac{3}{4}\beta + \bar{V} - \sqrt{g\bar{H}} \tag{16}$$

The solutions $\alpha(x,t)$ and $\beta(x,t)$ of (14) are classically called *Riemann invariants*. Since the change of coordinates (12)-(13) is a bijection H and V can be expressed in terms of Riemann invariants :

$$H = \frac{(\alpha - \beta + 4\sqrt{g\bar{H}})^2}{16g} \tag{17}$$

$$V = \frac{\alpha+\beta}{2} + \bar{V}. \tag{18}$$

3.3 *Stability Theorem*

It is obvious that the equilibrium $\bar{H}$, $\bar{V}$ expressed in the α, β coordinates is:

$$\bar{\alpha} = 0 \qquad \bar{\beta} = 0 \tag{19}$$

The stability of the flow in a neighborhood of this steady state in a single reach can be analyzed with the following theorem. The theorem is stated here in a rather general form because it will be used also later on for the control stability analysis.

We consider the shallow water equations (14), expressed in (α,β) coordinates, defined on the domain $(x,t) \in [0,L] \times [0,\infty)$. The boundary conditions are supposed to be given in the following general form

$$f_0(\alpha(0,t),\beta(0,t)) = 0 \qquad f_L(\alpha(L,t),\beta(L,t)) = 0 \tag{20}$$

with functions $f_0(\alpha,\beta)$ and $f_L(\alpha,\beta)$ being of class C^1. By differentiating these boundary conditions with respect to time and using equations (14), we have the following so-called *boundary compatibility conditions* at the initial instant $t = 0$:

$$\begin{aligned} &c_\alpha(\alpha,\beta)(0,0)\frac{\partial f_0}{\partial \alpha}(\alpha,\beta)(0,0)\frac{\partial \alpha}{\partial x}(0,0) + \\ &c_\beta(\alpha,\beta)(0,0)\frac{\partial f_0}{\partial \beta}(\alpha,\beta)(0,0)\frac{\partial \beta}{\partial x}(0,0) = 0 \\ &c_\alpha(\alpha,\beta)(L,0)\frac{\partial f_L}{\partial \alpha}(\alpha,\beta)(L,0)\frac{\partial \alpha}{\partial x}(L,0) + \\ &c_\beta(\alpha,\beta)(L,0)\frac{\partial f_L}{\partial \beta}(\alpha,\beta)(L,0)\frac{\partial \beta}{\partial x}(L,0) = 0 \end{aligned} \tag{21}$$

Theorem 1. Assume that the initial conditions $\alpha(x,0), \beta(x,0)$ in $C^1([0,L])^2$ satisfy the boundary compatibility conditions (21) and that the following inequality holds:

$$A_1 A_2 < 1 \tag{22}$$

with

$$A_1 = \left| \frac{\frac{\partial f_0}{\partial \beta}(0,0)}{\frac{\partial f_0}{\partial \alpha}(0,0)} \right| \quad \text{and} \quad A_2 = \left| \frac{\frac{\partial f_L}{\partial \alpha}(0,0)}{\frac{\partial f_L}{\partial \beta}(0,0)} \right|. \tag{23}$$

Then, there exist positive constants ϵ, M, μ such that, if the initial condition is small enough:

$$\mid \alpha(x,0) \mid_{C_1[0,L]} + \mid \beta(x,0) \mid_{C_1[0,L]} \ \leq \epsilon \tag{24}$$

there is a unique solution $\alpha(x,t)$, $\beta(x,t)$ of class C^1 on $[0,L] \times [0,\infty)$ which decays to zero with an exponential rate:

$$\mid \alpha(x,t) \mid_{C_1[0,L]} + \mid \beta(x,t) \mid_{C_1[0,L]} \ \leq Me^{-\mu t} \tag{25}$$

■

This theorem is a direct application of Theorem 2 in (Greenberg and Li, 1984).

3.4 *Control design with Riemann invariants*

Let us assume that the inflow rate is constant: $Q(0,t) = \bar{Q}$.

The boundary at $x = 0$ is written as :

$$f_0(\alpha,\beta) = \frac{\alpha+\beta+2\bar{V}}{2}\frac{(\alpha-\beta+4\sqrt{g\bar{H}})^2}{16g} - \bar{Q} \tag{26}$$

The control law at $x = L$ is selected in order to have a linear relationship between α and β :

$$\beta = -\gamma\alpha$$

which implies

$$f_L(\alpha,\beta) = \beta + \gamma\alpha \tag{27}$$

where $\gamma \geq 0$ is a tuning parameter.

It follows that the stability condition A_1A_2 becomes:

$$\left| \frac{\sqrt{g\bar{H}} - \bar{V}}{\sqrt{g\bar{H}} + \bar{V}} \right| |\gamma| < 1 \tag{28}$$

Using the subcritical flow condition (3) in Theorem 1, we have the following sufficient condition for locally exponential convergence of the solution to the equilibrium point :

$$0 \leq \gamma < \left| \frac{\sqrt{g\bar{H}} + \bar{V}}{\sqrt{g\bar{H}} - \bar{V}} \right|. \tag{29}$$

By inverting the change of coordinates using (17)-(18) and (6), we have the expression of u :

$$u = H(L,t) - \left(\frac{(\frac{\bar{Q}}{\bar{H}} + e_v)H(L,t)}{k} \right)^{1/m} \tag{30}$$

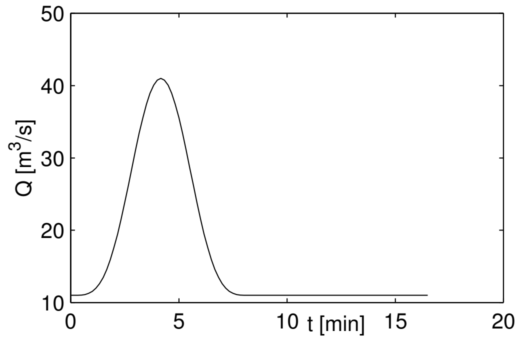

Fig. 3. Inflow at $x = 0$.

where e_v is the error on the water velocity :

$$e_v = V - \bar{V} = 2\frac{1-\gamma}{1+\gamma}(\sqrt{gH} - \sqrt{g\bar{H}}) \qquad (31)$$

From an engineering viewpoint, this control law has several advantages : the control law (30) is local -no communication with other gates is needed- and it depends only on water depth measurements -neither water speed nor flow measurements are needed-.

3.5 *Simulation result*

We consider a reach of length $L = 5000\text{m}$ and 40m width, the initial state and steady state are:

$$(Q, H)_{t=0} = (\bar{Q}, \bar{H}) = (10\text{m}^3/\text{s}, 4\text{m}).$$

The tuning parameter γ has been set to 0.1, so the product A_1A_2 equals to 0.098. The inflow is perturbated by a flood wave as depicted in Figure 3. The simulation has been made using a semi-implicit Preissman scheme with a time step of 30 s and a spatial step of 100 m. The proposed control law (30) is compared to an open loop strategy where u is simply constant, $u = \bar{u}$. The deviation of the canal state with respect to the equilibrium is measured by the entropy of the fluid, R :

$$R = \int_0^L H\frac{(V-\bar{V})^2}{2} + g\frac{(H-\bar{H})^2}{2}\,\mathrm{d}x \qquad (32)$$

In Figure 4, we see $H(.,t)$ for different simulation times. One can see that the wave is almost totally dampened by the proposed control law. In Figure 5, we see that control law (30) asymptotically stabilizes the channel even for a large deviation (see Figure 3) of the inflow state.

4. PRACTICAL CASE: SAMBRE RIVER

The purpose of this last section is to investigate the applicability of this control to the Sambre river in Belgium

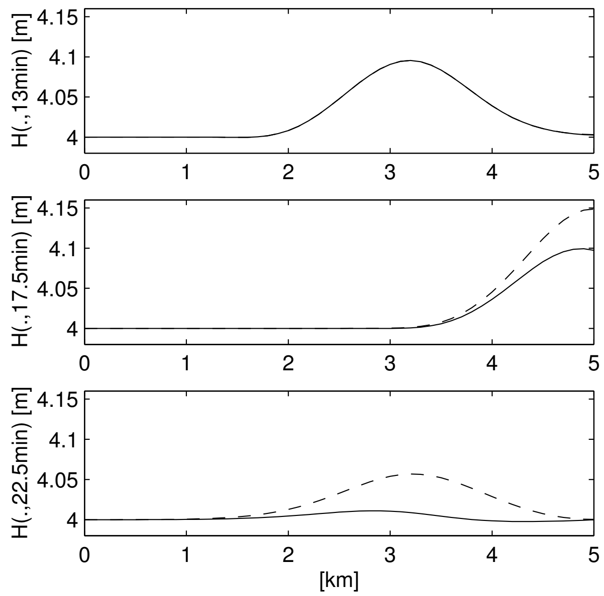

Fig. 4. $H(.,t)$ at different simulation time. Plain curve for our control, dotted curve for open loop.

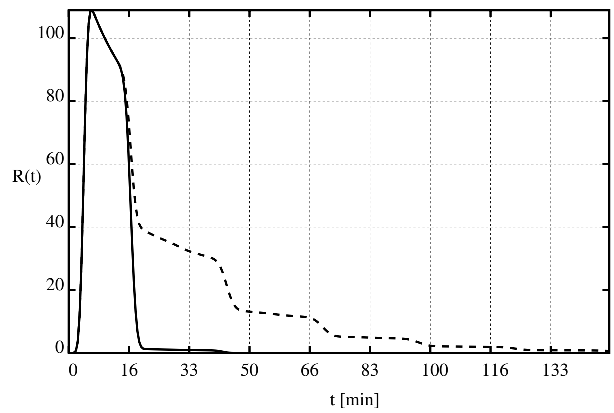

Fig. 5. Entropy R. Plain curve, Riemann based control. Dotted curve, open loop control.

4.1 *Description and model*

The Sambre is a class IV (up to 1350T boat) water-way that starts in France and is a tributary of the *Meuse* river in Belgium as depicted in Figure 6. The studied part of the Sambre - from Marcinelle to Salzinnes - is composed of 7 reaches separated by spillways with an average width of $40m$. The river is modelled by a cascade of prismatic reaches as depicted in the Figure 6.

4.2 *Simulation results*

In order to get a more realistic model, the reaches are described by Saint Venant equations with friction terms of the Manning-Strickler type ($K = 50$) and a slight slope for some of them.

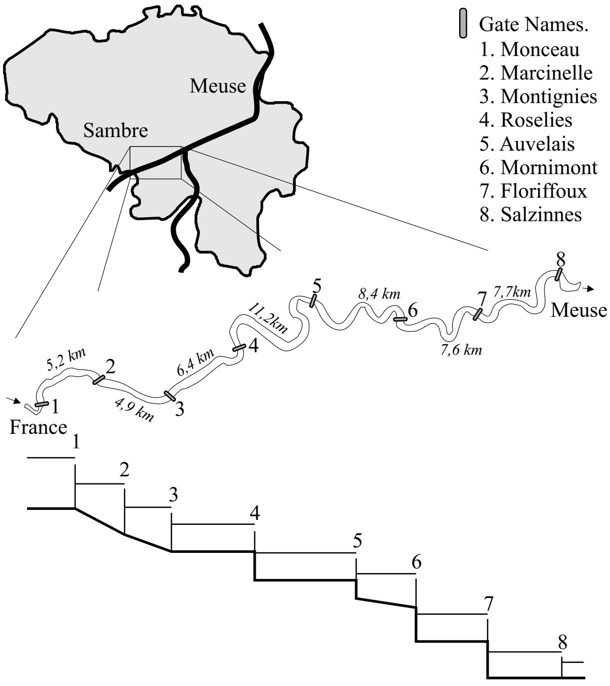

Fig. 6. Schematic view of Sambre river from *Monceau* to *Salzinnes* gate.

A flood wave is injected at the first gate ($Monceau$). The simulation results for open loop and closed loop strategy are presented at Figures 7, 8, 9.

Figure 7 shows the hydrogram of each gate - the water flow passing through each gate -. One can see that the flood wave is constantly dampened for the open loop case, whether in the closed loop case, the flood wave crosses the gates with little modification. As this will be emphasized in the next paragraph, the open loop case "pays the price" of such behavior by producing greater deviation of the water depth than the closed loop case. We can also appreciate the acceleration of convergence for the closed loop case.

Figure 8 shows the limnigram of each gate - the deviation of the water depth at each gate -. The open loop, by dampening the water flow, generates high deviation of the water depth (up to 40cm). The closed loop, which does not hold the flow, yields to smaller deviation of the water depth (up to 12cm). Again, one can see that the equilibrium is reached much faster in the closed loop case.

The acceleration of the convergence can also be assessed in Fig. 9 where the entropy is represented. Indeed the response time in closed loop (9 hours) is much smaller that in open loop (20 hours).

5. CONCLUSION

Our main contribution in this paper has been to propose a control strategy based on Riemann invariants for level and velocity control in open-channels described by the Saint Venant equations.

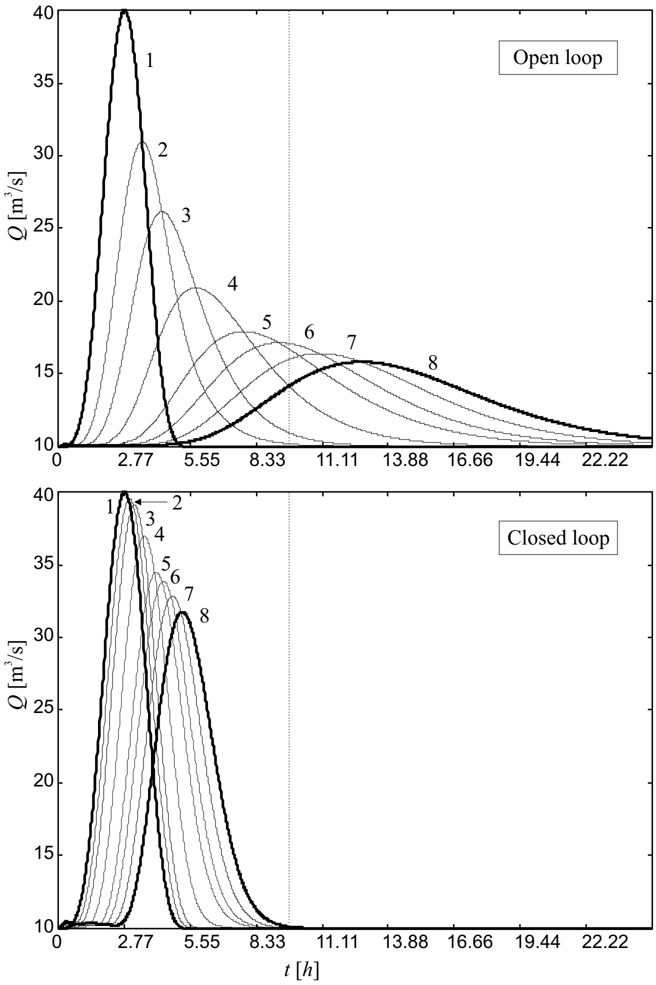

Fig. 7. Hydrogram for gate 1 to 8.

Using the characteristic of the spillways, we have derived control laws for a single reach. The extension to multi-reach canals has been illustrated by a simulation on a practical case : the Sambre river in Belgium.

Related results on the feedback stabilization of open-channels in the context of the theory of quasilinear hyperbolic systems can also be found in (Greenberg and Li, 1984; Li and Yu, 1985; Leugering and Schmidt, 2002). Other relevant references concerning the control of open-channels systems are e.g. (Litrico and Georges, 1999; Xu and Sallet, 1999; Malaterre, 1994; Coron *et al.*, 1999; de Halleux *et al.*, 2001).

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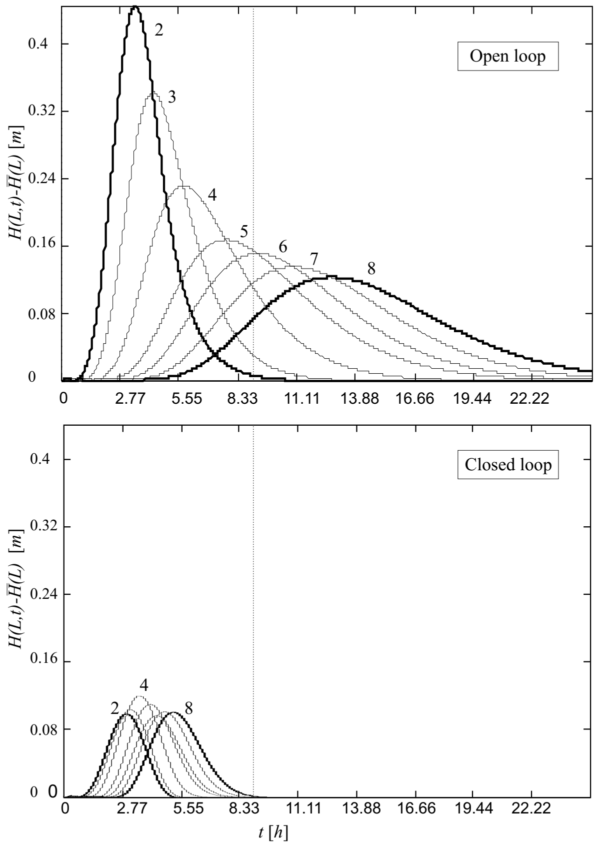

Fig. 8. Limnigram for gate 2 to 8.

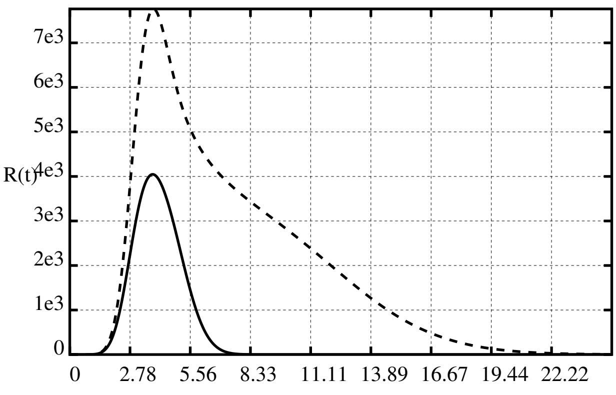

Fig. 9. Entropy R for Sambre river. Plain curve, Riemann based control. Dotted curve, open loop control.

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