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CONTINUOUS-TIME IDENTIFICATION USING LQG-BALANCED MODEL REDUCTION

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Abstract: System identification of continuous-time model based on discrete-time data can be performed using a algorithm combining linear regression and LQG-balanced model reduction. The approach is applicable also to unstable system dynamics and it provides balanced models for optimal linear prediction and control.

Keywords: System Identification, Continuous-time Systems, Model Reduction

INTRODUCTION

A weak point in many approaches to system identification of continuous-time linear systems is how to find an appropriate models for colored noise. For the case of discrete-time data, Johansson (1994) suggested one approach to find a maximum-likelihood (ML) colored-noise model with properties similar to identification of ARMAX models. A drawback with any use of (approximate) ML methods is that they rely on numerical optimization. Another issue is how to apply these methods for multi-input multi-output systems where properties of uniqueness of parametrization become important. The combination of these issues have inspired new efforts to improve pseudolinear regression and subspace-based models using singular value decomposition. Pseudolinear regression is often organized as a two-step method where the first step involves linear regression to find a high-order model and a second step in which the model order is reduced and where the disturbance model is found—e.g., as an iterated Markov estimate. One alternative is to apply balanced model reduction in the second step. As balanced model reduction only can be applied to stable models, there is a limited application range for this method. However, Fuhrmann and Ober (1999) and more recently Salomon et al. (1999) have suggested a modified balanced model that exploited a modified balancing approach. Instead of solving for a pair of Gramians using Lyapunov function, it was suggested to be replaced by Riccati equation. An immediate application in the context of model reduction is that unstable systems may be object for model reduction. The idea goes back at least to Desai and Pal (1982) who suggested LQG-like balanced realization for innovation models and Kalman filters obtained in covariance analysis.

This important observation can also be exploited in the context of system identification. In the context of pseudolinear regression, the benefit is two-fold. Firstly, it permits the application of pseudolinear regression to unstable systems which, in, turn permits derivation of disturbance models. Secondly, by virtue of the LQG properties it permits the formulation of optimal linear model approximation to reduced-order models for application in LQG control and Kalman filtering. Important application is to be found in identification for control, Kalman filter design and spectrum analysis.

PRELIMINARIES

Balanced Model Reduction

Given a a linear time-invariant m-inputs p outputs transfer matrix $G(s)$ with a realization given by

$$\begin{aligned}\frac{dx}{dt} &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t)\end{aligned} \quad (1)$$

where $x \in \mathbb{R}^n$; $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ and A, B, C, D are matrices of the corresponding dimensions. Denote the realization of $G(s)$ given in Eq. (1) by $\mathcal{S}\{A, B, C, D\}$ or

$$\mathcal{S} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \quad (2)$$

The controllability and observability Gramians are defined as

$$P = \int_0^\infty e^{At} B B^T e^{A^T t} dt, \quad (3)$$

$$Q = \int_0^\infty e^{A^T t} C^T C e^{At} dt \quad (4)$$

Note that P, Q are also the solutions to the Lyapunov equations

$$AP + PA^T + BB^T = 0 \quad (5)$$

$$QA + A^T Q + C^T C = 0 \quad (6)$$

In the case where (A, B) is observable and (A, C) observable, there exists a linear transformation T such that $\mathcal{S}\{TAT^{-1}, TB, CT^{-1}, D\}$ is balanced—*i.e*, $TPT^{-1} = (T^T)^{-1} Q T^{-1} = \Sigma$ with $\Sigma = \operatorname{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$. Now partition of the resulting transformed system matrix into

$$\mathcal{S} = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right], \quad \begin{array}{l} A_{11} \in \mathbb{R}^{r\times r}, \\ B_1 \in \mathbb{R}^{r\times m}, \\ C_1 \in \mathbb{R}^{p\times r} \end{array} \quad r < n \quad (7)$$

Then, a reduced-order model $\mathcal{S}_r$ of order $r < n$ can be obtained as one of the following approximants

$$\mathcal{S}_r = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right], \sigma_{r+1} \le \|\mathcal{S} - \mathcal{S}_r\|_\infty \le 2 \sum_{k=r+1}^{n} \sigma_k$$

$$\mathcal{S}_r = \left[\begin{array}{c|c} A_{11} - A_{12}A_{22}^{-1}A_{21} & B_1 - A_{12}A_{22}^{-1}B_2 \\ \hline C_1 - C_2A_{22}^{-1}A_{21} & D - C_2A_{22}^{-1}B_2 \end{array}\right] \quad (8)$$

When A is unstable there is no solution to the Lyapunov equations. Salomon *et al.* (1999) showed that for (A, B) stabilizable, (A, C) detectable there are still relevant solutions obtained by replacing Lyapunov equations with the Riccati equations

$$AP + PA^T + BB^T - PBB^T P = 0 \quad (9)$$

$$QA + A^T Q + C^T C - QC^T CQ = 0 \quad (10)$$

The model reduction scheme obtained using this modification is called LQG-balanced model reduction.

Continuous-Time System Identification

Consider a continuous-time time-invariant system $\Sigma_n(A, B, C, D)$ with system equations

$$\begin{aligned}\frac{dx}{dt} &= Ax(t) + Bu(t) + v(t) \\ y(t) &= Cx(t) + Du(t) + e(t)\end{aligned} \quad (11)$$

with input $u \in \mathbb{R}^m$, output $y \in \mathbb{R}^p$, state vector $x \in \mathbb{R}^n$ and zero-mean disturbance stochastic processes $v \in \mathbb{R}^n$, $e \in \mathbb{R}^p$ acting on the state dynamics and the output, respectively. The continuous-time system identification problem is to find estimates of system matrices A, B, C, D from finite sequences $\{u_k\}_{k=0}^N$ and $\{y_k\}_{k=0}^N$ of input-output data.

Discrete-Time Input-Output Data Assume periodic sampling to be made with period h at a time sequence $\{t_k\}_{k=0}^N$, with $t_k = t_0 + kh$ with the corresponding discrete-time input-output data $\{y_k\}_{k=0}^N$ and $\{u_k\}_{k=0}^N$ sampled from the continuous-time dynamic system of Eq. (11). Alternatively, data may be assumed generated by the time-invariant discrete-time state-space system

$$\begin{aligned} x_{k+1} &= A_z x_k + B_z u_k + v_k; \quad A_z = e^{Ah}, \\ y_k &= Cx_k + Du_k + e_k; \quad B_z = \int_0^h e^{As} B ds \end{aligned} \quad (12)$$

with equivalent input-output behavior to that of Eq. (11) at the sampling-time sequence. The underlying discretized state sequence $\{x_k\}_{k=0}^N$ and discrete-time stochastic processes $\{v_k\}_{k=0}^N$, $\{e_k\}_{k=0}^N$ correspond to disturbance processes v and e which can be represented by the components

$$\begin{aligned} v_k &= \int_{t_{k-1}}^{t_k} e^{A(t_k - s)} v(s) ds, \quad k = 1, 2, ..., N \\ e_k &= e(t_k) \end{aligned} \quad (13)$$

with the covariance $\mathcal{Q} \ge 0$,

$$\mathcal{E}\{\begin{pmatrix} v_i \\ e_i \end{pmatrix} \begin{pmatrix} v_j \\ e_j \end{pmatrix}^T\} = \mathcal{Q}\delta_{ij} = \begin{pmatrix} \mathcal{Q}_{11} & \mathcal{Q}_{12} \\ \mathcal{Q}_{12}^T & \mathcal{Q}_{22} \end{pmatrix} \delta_{ij} \quad (14)$$

The continuous-time stochastic processes will have an autocorrelation function according to

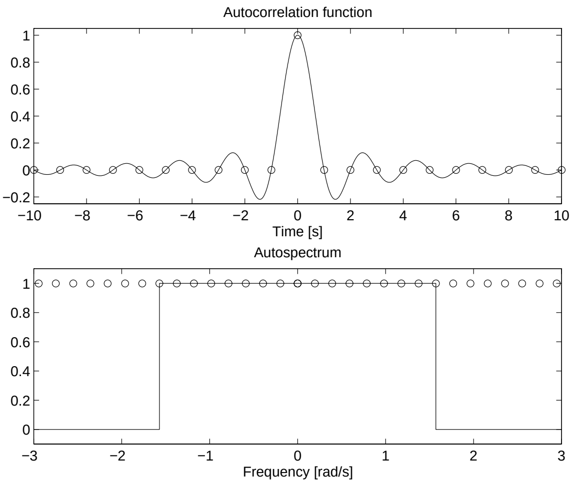

Fig. 1. Autocorrelation functions (*upper diagram*) and autospectra (*diagram below*) of a continuous-time (*solid line* stochastic variable $w(t)$ and a discrete-time (*'o'*) sample sequence $\{w_k\}$. The continuous-time process is bandwidth-limited to the Nyquist frequency $\omega_N = \pi/2$ $[rad/s]$ of a sampling process with sampling frequency 1 Hz. Properties of the sampled sequence $\{w_k\}$ confirm that the sampled sequence is an uncorrelated stochastic process with a uniform autospectrum.

Fig. 1, thereby avoiding the mathematical problems associated with the stochastic processes of Brownian motion.

From the set of first-order linear differential equations of Eq. (11) one finds the Laplace transform

$$\begin{aligned} sX &= AX + BU + V + sx_0; \quad x_0 = x(t_0) \\ Y &= CX + DU + E \end{aligned} \tag{15}$$

and for a corresponding innovations model

$$\begin{aligned} sX &= AX + BU + KW + sx_0; \quad x_0 = x(t_0) \\ Y &= CX + DU + W \end{aligned} \tag{16}$$

Introduction of the complex variable transform

$$\lambda(s) = \frac{1}{1+s\tau} \tag{17}$$

corresponding to a stable, causal operator permits an algebraic transformation of the model

$$\begin{aligned} X &= (I+\tau A)[\lambda X] + (1-\lambda)x_0 \\ &\quad + \tau B[\lambda U] + \tau K[\lambda W] && (18) \\ Y &= CX + DU + W && (19) \end{aligned}$$

In the new operator λ, we define

$$A_\lambda = I + \tau A, \quad B_\lambda = \tau B, \quad K_\lambda = \tau K \tag{20}$$

The transfer-function properties for the innovations model are

$$\begin{aligned} Y(\lambda(s)) &= (C(I-A_\lambda\lambda)^{-1}B_\lambda\lambda + D)U(s) && (21) \\ &\quad + (I_p + C(I-A_\lambda\lambda)^{-1}K_\lambda\lambda)W(s) && (22) \\ &\quad + (1-\lambda)C(I-A_\lambda\lambda)^{-1}x_0 && (23) \end{aligned}$$

Using a matrix fraction description

$$\begin{aligned} A_L(\lambda)^{-1}B_L(\lambda) &= C(I-A_\lambda\lambda)^{-1}B_\lambda\lambda + D && (24) \\ A_L(\lambda)^{-1}C_L(\lambda) &= C(I-A_\lambda\lambda)^{-1}K_\lambda\lambda + I && (25) \\ A_L(\lambda) &= I_p + A_1\lambda + \cdots + A_n\lambda^n \in \mathbb{R}^{p\times p}[\lambda] \\ B_L(\lambda) &= B_0 + B_1\lambda + \cdots + B_n\lambda^n \in \mathbb{R}^{p\times m}[\lambda] \\ C_L(\lambda) &= C_0 + C_1\lambda + \cdots + C_n\lambda^n \in \mathbb{R}^{p\times p}[\lambda] \end{aligned}$$

To the purpose of linear regression for estimation, it is straightforward to formulate this model as counterpart to the autoregressive moving-average model with external input (ARMAX) used in time-series analysis

$$A_L(\lambda)Y(s) = B_L(\lambda)U(s) + C_L(\lambda)W(s) \tag{26}$$

and the linear regresson model

$$\begin{aligned} Y(s) &= -\sum_{k=1}^{n} A_k[\lambda^k]Y(s) + \sum_{k=0}^{n} B_k[\lambda^k]U(s) \\ &\quad + \sum_{k=0}^{n} C_k[\lambda^k]W(s) \end{aligned} \tag{27}$$

LQG BALANCED CONTINUOUS-TIME SYSTEM IDENTIFICATION

Because $W(s)$ is not available to measurement nor as a discrete-time sequence $\{w_k\}$, linear regression cannot be applied. As a substitute, *pseudolinear regression* is often applied as an iterative procedure where the essential step is to find a pseudoregressor sequence to substitute the unknown regressor sequence. It is often suitable to choose the parameter set with the smallest 2-norm. To the purpose of least-squares identification, then, it is suitable to organize model and data according to

1: Arrange for data sequences of discrete-time data using the following notation for sampled filtered data

$$\begin{aligned} u_k^{(j)} &= [\lambda^j u]_k = [\lambda^j u](t_k), && (28) \\ y_k^{(j)} &= [\lambda^j y]_k = [\lambda^j y](t_k), \end{aligned}$$

where $\{t_k\}_{k=0}^N$ for $j = 0, 1, \ldots, q$, for some $q > n$.
2: Formulate

$$
\begin{aligned}
y_k &= -A_1 y_k^{(1)} - \cdots - A_n y_k^{(n)} \\
&\quad + B_1 u_k^{(1)} + \cdots + B_n u_k^{(n)}, \quad y_k \in \mathbb{R}^p \\
\theta &= \left(A_1 \ldots A_n \; B_1 \ldots B_n\right)^T, \quad \theta \in \mathbb{R}^{n(m+p)\times p}
\end{aligned}
\tag{29}
$$

which suggests the linear regression model

$$
\mathcal{M}_1: \quad \mathcal{Y}_N = \Phi_N \theta \tag{30}
$$

3: Arrange data sequences into matrices

$$
\phi_k = \left(-[\lambda^1 y]_k^T \; \ldots \; -[\lambda^q y]_k^T \; [\lambda^1 u]_k^T \; \ldots \; [\lambda^q u]_k^T\right)^T
$$

$$
\mathcal{Y}_N = \begin{pmatrix} y_1^T \\ y_2^T \\ \vdots \\ y_N^T \end{pmatrix} \in \mathbb{R}^{N\times p}, \quad \Phi_N = \begin{pmatrix} \phi_1^T \\ \phi_2^T \\ \vdots \\ \phi_N^T \end{pmatrix} \in \mathbb{R}^{N\times n(m+p)}
$$

4: Compute the least-squares estimate $\widehat{\theta}$ and the residual sequence $E_N \in \mathbb{R}^{N\times p}$ with rows $\{\varepsilon_k^T\}_{k=1}^N$

$$
\widehat{\theta}_N = (\Phi_N^T \Phi_N)^{\dagger} \Phi_N^T \mathcal{Y}_N \tag{31}
$$

$$
\begin{aligned}
E_N(\widehat{\theta}) &= \mathcal{Y}_N - \widehat{\mathcal{Y}}_N = \mathcal{Y}_N - \Phi_N \widehat{\theta} \\
&= (I_N - \Phi_N (\Phi_N^T \Phi_N)^{\dagger} \Phi_N^T) \mathcal{Y}_N
\end{aligned}
\tag{32}
$$

5: Formulate a pseudoregression model using $\{\varepsilon_k\}_{k=1}^N$ to replace unknown disturbance $\{w_k\}$

$$
\begin{aligned}
y_k &= -A_1 y_k^{(1)} - \cdots - A_n y_k^{(n)}, \quad y_k, y_k^{(j)} \in \mathbb{R}^p \\
&\quad + B_1 u_k^{(1)} + \cdots + B_n u_k^{(n)} \\
&\quad + C_0 \varepsilon_k + C_1 \varepsilon_k^{(1)} + \cdots + C_n \varepsilon_k^{(n)} \\
\theta &= \left(A_1 \; \ldots \; A_n \; B_1 \; \ldots \; B_n \; C_0 \; \ldots \; C_n\right)^T,
\end{aligned}
\tag{33}
$$

which suggests the linear regression model

$$
\mathcal{M}_2: \quad \mathcal{Y}_N = \Phi_N \theta, \quad \theta \in \mathbb{R}^{n(m+2p+1)\times p} \tag{34}
$$

As a result of the non-uniqueness of parameters, the normal equations of the associated least-squares estimation of θ will exhibit rank deficit in general. It is therefore natural to apply the least-squares solution

$$
\widehat{\theta}_N = (\Phi_N^T \Phi_N)^{\dagger} \Phi_N^T \mathcal{Y}_N \tag{35}
$$

where $(\Phi_N^T \Phi_N)^{\dagger}$ denotes the matrix pseudo-inverse of $\Phi_N^T \Phi_N$. The associated least-squares estimate then obtained has the smallest 2-norm of all possible minimizers of the least-squares criterion.

Step 2—LQG-balanced Model Reduction The regression models $\mathcal{M}_1$, $\mathcal{M}_2$ suggest nonminimal multivariable state-space models which may be objects for model reduction.

A nonminimal state-space model may be suggested as

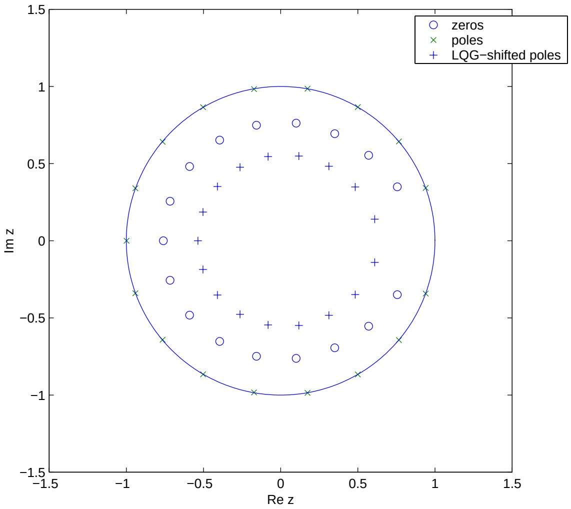

Fig. 2. Pole-zero map of empirical transfer function estimate before model reduction exhibits many pole-zero cancellations inside and outside the unit circle. Pole pattern after LQG-balanced pole shifting shows a shift into stable region. Original poles are indicated by '×' and the LQG-shifted poles as '+'.

$$
A_x = \left[\begin{array}{ccc|ccc}
-I_p - A_1\tau & -A_2\tau & \cdots - A_n\tau & B_1\tau & \cdots & B_n\tau \\
I_p & -I_p & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & & \vdots \\
0 & I_p & -I_p & 0 & \cdots & 0 \\
\hline
0 & \cdots & 0 & -I_m & 0 \cdots & 0 \\
0 & & & I_m & -I_m \cdots & 0 \\
\vdots & \ddots & \vdots & & \ddots \; \ddots & 0 \\
0 & \cdots & 0 & 0 & I_m & -I_m
\end{array}\right] \frac{1}{\tau}
$$

$$
C_x = \left(-A_1 \; \ldots \; -A_n \; B_1 \; \ldots \; B_n\right) \tag{36}
$$

$$
B_x = \left(0 \; \cdots \; 0 \; | I_m \; 0 \; \cdots \; 0\right)^T, \tag{37}
$$

and the intemediate high-order result

$$
\begin{pmatrix} \dot{x} \\ y \end{pmatrix} = \begin{pmatrix} A_x & B_x \\ C_x & 0 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \tag{38}
$$

Application of LQG-balanced model reduction to the state-space description of Eq. (38) which is observable but not controllable.

LQG-balancing for Frequency-domain Methods

Frequency response fitting based on least-squares identification in the complex frequency domain is a natural idea which also benefits from LQG balancing. Let the polynomial ratio

$$
\begin{aligned}
\widehat{G}(i\omega) &= A^{-1}(i\omega) B(i\omega) \\
A(s) &= s^n I_p + A_1 s^{n-1} + \cdots + A_{n-1} s + A_n \\
B(s) &= B_1 s^{n-1} + \cdots + B_{n-1} s + B_n
\end{aligned}
\tag{39}
$$

denote a transfer function estimate to be fitted to the experimental data $G(i\omega_k)$ and known at the frequency points ω_k, $k = 1, 2, \ldots, N$. A natural goal of optimization is to minimize the error criterion

$$\min_{A,B} \sum_k \|A(s)Y(z) - B(s)U(z)\|^2_{s=i\omega_k,\ z=e^{i\omega_k}} \quad (40)$$

where $Y(z)$, $U(z)$ denote z-transformed input-output data. The linear regression problem with parameter vector θ takes on the format

$$\mathcal{Y}_N = \Phi_N \theta = \begin{pmatrix}\Phi_Y & \Phi_U\end{pmatrix} \theta \quad (41)$$

$$\theta = \begin{pmatrix}A_1 & \cdots & A_n & B_1 & \cdots & B_n\end{pmatrix}^T \quad (42)$$

with

$$\mathcal{Y}_N = \begin{pmatrix} (i\omega_1)^n Y^T(z^{i\omega_1}) \\ (i\omega_2)^n Y^T(z^{i\omega_2}) \\ \vdots \\ (i\omega_N)^n Y^T(z^{i\omega_N}) \end{pmatrix} \quad (43)$$

and the regressor matrices

$$\Phi_Y = \begin{pmatrix} -(i\omega_1)^{n-1} Y^T(e^{i\omega_1}) & \cdots & i\omega_1 Y^T(z^{i\omega_1}) & Y^T(e^{i\omega_1}) \\ -(i\omega_2)^{n-1} Y^T(e^{i\omega_2}) & \cdots & i\omega_2 Y^T(z^{i\omega_2}) & Y^T(e^{i\omega_2}) \\ \vdots & & \vdots & \\ -(i\omega_N)^{n-1} Y^T(e^{i\omega_N}) & \cdots & i\omega_N Y^T(z^{i\omega_N}) & Y^T(e^{i\omega_N}) \end{pmatrix}$$

$$\Phi_U = \begin{pmatrix} (i\omega_1)^{n-1} U^T(e^{i\omega_1}) & \cdots & i\omega_1 U^T(z^{i\omega_1}) & U^T(e^{i\omega_1}) \\ (i\omega_2)^{n-1} U^T(e^{i\omega_2}) & \cdots & i\omega_2 U^T(z^{i\omega_2}) & U^T(e^{i\omega_2}) \\ \vdots & & \vdots & \vdots \\ (i\omega_N)^{n-1} U^T(e^{i\omega_N}) & \cdots & i\omega_N U^T(z^{i\omega_N}) & U^T(e^{i\omega_N}) \end{pmatrix}$$

The least-squares solution minimizing is then

$$\widehat{\theta} = (\Phi^*\Phi)^{-1}\Phi^*\mathcal{Y} \quad (44)$$

where Φ^* denotes the transpose and complex conjugate of Φ.

LQG-balanced model reduction can be applied to the intermediate result

$$\frac{dx}{dt} = \begin{pmatrix} -\widehat{A}_1 & -\widehat{A}_2 & \cdots & -\widehat{A}_n \\ I_p & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots 0 & I_p & 0 \end{pmatrix} x + \begin{pmatrix} I_m \\ 0 \\ \vdots \\ 0 \end{pmatrix} u,$$
$$y = \begin{pmatrix}\widehat{B}_1 & \widehat{B}_2 & \ldots & \widehat{B}_n\end{pmatrix} x \quad (45)$$

DISCUSSION AND CONCLUSIONS

Major application areas are to be found for optimal control and optimal prediction using reduced-order models. Assume that LQG-balanced model reduction exploits in the Riccati equations

$$0 = AP + PA^T + BB^T - PBB^TP, \quad (46)$$
$$0 = QA + A^TQ + C^TC - QC^TCQ \quad (47)$$

Then, for a model $\mathcal{S}\{A_r, B_r, C_r, D_r\}$ of reduced order, we have the reduced-order Riccati equations

$$\begin{aligned} 0 &> (A_r - B_rB_r^TP^{-1})P_r + P_r(A_r - B_rB_r^TP_r^{-1})^T \\ &= -B_rB_r^T - P_rB_rB_r^TP_r - R_P \\ 0 &> Q_r(A_r - Q_r^{-1}C_r^TC) + (A_r - Q_r^{-1}C_r^TC)_r^TQ_r \\ &= -C_r^TC_r - Q_rC_r^TC_rQ_r - R_Q \end{aligned} \quad (48)$$

for some matrices R_P, R_Q which represent the difference between the higher-order model and thereduced-order model. An immediate interpretation is that R_P, R_Q represent the approximation cost associated with the model approximation. For example, the observer

$$\begin{aligned} \frac{d\widehat{x}_r}{dt} &= (A_r - K_rC_r)\widehat{x}_r + (B_r - K_rD_r)u + K_ry, \\ \widehat{y} &= C_r\widehat{x}_r + D_ru, \quad K_r = Q_r^{-1}C_r^T \end{aligned} \quad (49)$$

has an asymptotic covariance function from Q_r and its convergence rate described by

$$\begin{aligned} V(\widetilde{x}_r) &= \widetilde{x}_r^TQ_r\widetilde{x}_r, \quad \widetilde{x}_r = \widehat{x}_r - x_r \qquad (50) \\ \frac{dV(\widetilde{x}_r)}{dt} &= \widetilde{x}_r^T(-C_r^TC_r - Q_rC_r^TC_rQ_r - R_Q)\widetilde{x}_r \\ &< 0, \quad \|\widetilde{x}_r\| \neq 0 \end{aligned}$$

Thus, the cost of optimal approximation can be quantified by R_Q in the context of optimal prediction. Similarly, the degradation in optimal control can be quantified by means of R_P.

Another interesting application which is opened up by the LQG-balanced model reduction is state-space model identification based on empirical transfer function estimates—*e.g.*, input-output spectrum ratios or cross-spectrum ratios. Previously, such approaches were hampered by the presence of unstable pole-zero cancellation in the rational functions obtained. A remaining problem, though, is how to treat cases with eigenvalues of A on the imaginary axis.

An interesting question for further investigation is how to exploit relationships to subspace-based identification and to the Krylov-Arnoldi methods—see Gugercin and Antoulas (2000), Antoulas et al. (2001).

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