15th Triennial World Congress, Barcelona, Spain

EXTERNALLY AND INTERNALLY POSITIVE TIME-VARYING LINEAR SYSTEMS

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Abstract. The notions of externally positive and internally positive time-varying linear systems are introduced. Necessary and sufficient conditions for the external positivity and internal positivity of time-varying linear systems are established. Sufficient conditions for the reachability of internally positive time-varying linear systems are presented.

1. INTRODUCTION

Roughly speaking positive systems are systems whose trajectories are entirely in the nonnegative orthant R_+^n whenever the initial state and input are nonnegative. Positive systems arise in modelling of systems in engineering, economics, social sciences, biology, medicine and other areas (Farina and Rinaldi, 2000; d'Alessandro de Santis, 1994; Berman, Neumann and Stern, 1989; Berman and Plemmons, 1994; Kaczorek, 2001; Rumchev, James, 1990; Rumchev, James, 1995). The single-input single-output externally positive and internally positive linear time-invariant systems have been investigated in (Farina and Rinaldi, 2000; Berman, Neumann and Stern, 1989; Berman and Plemmons, 1994). The notions of externally positive and internally positive systems have been extended for singular continuous-time and discrete-time and two-dimensional linear systems in (Kaczorek, 2001). The reachability and controllability of standard and singular internally positive linear systems have been analysed in (Fanti, Maione and Turchsano, 1990; Klamka, 1998; Otha, Madea and Kodama, 1984; Valcher, 1996). The notions of weakly positive discrete-time and continuous-time linear systems have been introduced in (Kaczorek, 2001; 1998). Recently the positive two-dimensional (2D) linear systems have been extensively investigated by Fornasini and Valcher (Fornasini and Valcher, 1997; Valcher, 1996; 1997) and in (Kaczorek 2001).

2. PRELIMINARIES

Let $R^{p\times q}$ be the set of $p\times q$ real matrices and $R^p := R^{p\times 1}$. The set of $p\times q$ real matrices with nonnegative entries will be denoted by $R_+^{p\times q}$ and $R_+^p := R_+^{p\times 1}$.

Consider the linear time-varying system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t),\ x(t_0) = x_0 \tag{1a}$$

$$y(t) = C(t)x(t) + D(t)u(t) \tag{1b}$$

where $\dot{x}(t) = \frac{dx(t)}{dt}$, $x(t) \in R^n$ is the state vector, $u(t)\in R^m$ is the input vector, $y(t)\in R^p$ is the output vector and $A(t), B(t), C(t), D(t)$ are real matrices of appropriate dimensions with continuous-time entries. Solution $x(t)$ of the equation satisfying the initial condition $x(t_0)=x_0$ is given by (Gantmacher, 1959)

$$x(t)=\Phi(t,t_0)x_0+\int_{t_0}^{t}\Phi(t,\tau)B(t)u(\tau)d\tau \qquad (2)$$

where $\Phi(t,t_0)$ is the fundamental matrix defined by

$$\Phi(t,t_0)=I_n+\int_{t_0}^{t}A(\tau)d\tau+\int_{t_0}^{t}A(\tau)\int_{t_0}^{\tau}A(\tau_1)d\tau_1 d\tau+\dots \qquad (3)$$

where I_n is the $n\times n$ identity matrix.

If $A(t_1)A(t_2)=A(t_2)A(t_1)$ for $t_1,t_2\in[t_0,\infty)$ then (3) takes the form (Gantmacher, 1959)

$$\overline{\Phi}(t,t_0)=\exp\left(\int_{t_0}^{t}A(\tau)d\tau\right) \qquad (3a)$$

The fundamental matrix $\Phi(t,t_0)$ satisfies the matrix differential equation

$$\dot{\Phi}(t,t_0)=A(t)\Phi(t,t_0) \qquad (4)$$

and the initial condition $\Phi(t_0,t_0)=I_n$

3. EXTERNALLY POSITIVE SYSTEMS

Definition 1. The system (1) is called externally positive if for all $u(t)\in R_+^m$, $t\geq t_0$ and zero initial conditions ($x_0=0$) the output vector $y(t)\in R_+^p$ for $t\geq t_0$.

Let $g(t)\in R^{p\times m}$ be the matrix impulse response of the system (1). It is well-known that the output vector $y(t)$ of the system (1) with zero initial conditions for an input vector $u(t)$ is given by the formula

$$y(t)=\int_{t_0}^{t}g(t,\tau)u(\tau)d\tau,\ t\geq t_0 \qquad (5)$$

where

$g(t,\tau)=C(t)\Phi(t,\tau)B(t)+D(t)\delta(t-\tau)$ for $t\geq\tau$ and $\delta(t)$ is the Dirac impulse. (6)

Theorem 1. The system (1) is externally positive if and only if

$$g(t)\in R_+^{p\times m} \text{ for } t\geq t_0 \qquad (7)$$

Proof. The necessity follows immediately from definition 1 and the definition of impulse response. To show the sufficiency let us assume that (7) holds. Then from (5) for $u(t)\in R_+^m$, $t\geq t_0$ we have $y(t)\in R_+^p$ for $t\geq t_0$ ■

4. INTERNALLY POSITIVE SYSTEMS

Definition 2. The system (1) is called internally positive if for every $x_0\in R_+^n$ and all $u(t)\in R_+^m$ the state vector $x(t)\in R_+^n$ and $y(t)\in R_+^p$ for $t\geq t_0$.

From comparison of the definitions 1 and 2 it follows that every internally positive system (1) is always externally positive.

Lemma. The fundamental matrix

$$\Phi(t,t_0)\in R_+^{n\times n} \text{ for } t\geq t_0 \qquad (8)$$

if and only if the off-diagonal entries $a_{ij}, i\neq j, i,j=1,\dots,n$ of the matrix $A(t)$ satisfy the condition

$$\int_{t_0}^{t}a_{ij}(\tau)d\tau\geq 0 \text{ for } i\neq j, i,j=1,\dots,n \qquad (9)$$

Proof. First we shall show that (9) implies (8).

Let $x_i(t)$ ($z_i(t)$) be the i-th component of the vector $x(t)$ ($z(t)$) and

$$x_i(t)=z_i(t)\exp\left(\int_{t_0}^{t}a_{ii}(\tau)d\tau\right),\ i=1,\dots,n \qquad (10)$$

Substitution of (10) into the equation (1a) for $u(t)=0$, $t\geq t_0$ yields (Ratajczak, 1967)

$$\dot{z}(t)=\overline{A}(t)z(t) \qquad (11)$$

where $\overline{A}(t)=[\overline{a}_{ij}(t)]\in R^{n\times n}$

$$\bar{a}_{ij}(t) = \begin{cases} a_{ij}(t)\exp\left(\int_{t_0}^{t}[a_{jj}(\tau) - a_{ii}(\tau)]d\tau\right) & \text{for } i \neq j \\ 0 & \text{for } i = j \end{cases}$$

(12)

From (10) it follows that

$$z_i(t_0) = x_i(t_0) \geq 0 \text{ for } i = 1,\dots,n \text{ if } x_0 \in R_+^n \qquad (13)$$

Using (2) for $u(t) = 0$, $t \geq t_0$ and (3) for (11) we obtain

$$z(t) = \overline{\Phi}(t, t_0) z_0 \qquad (14)$$

where

$$\overline{\Phi}(t,t_0) = I_n + \int_{t_0}^{t} \bar{A}(\tau)d\tau + \int_{t_0}^{t} \bar{A}(\tau)\int_{t_0}^{\tau} \bar{A}(\tau_1)d\tau_1 d\tau + \dots \qquad (15)$$

From (12) it follows that if (9) holds then $\bar{A}(t) \in R_+^{n\times n}$ and by (15) this implies $\overline{\Phi}(t,t_0) \in R_+^{n\times n}$ and $z(t) \in R_+^n$, $t \geq t_0$ for any $z_0 \in R_+^n$. Hence by (10) and (13) $x(t) \in R_+^n$, $t \geq t_0$ for any $x_0 \in R_+^n$. Therefore, (9) implies (8). Necessity follows immediately from (3a) and the fact that $\overline{\Phi}(t,t_0) \in R_+^{n\times n}$ only if $\int_{t_0}^{t} \bar{A}(\tau)d\tau$ is a Metzler matrix for any $t \geq t_0$ (Kaczorek, 1998). ■

Remark 1. If the matrix $A(t)$ is independent of t, i.e. $A(t) = A = [a_{ij}]$ and $a_{ij} \geq 0$, for $i \neq j$ then A is the Metzler matrix (Farina nd Rinaldi, 2000; Kaczorek, 2001) and $\Phi(t,t_0) = \exp(A(t - t_0))$.

Theorem 2. The system (1) is internally positive if and only if

i) the off-diagonal entries of $A(t)$ satisfy the condition (9)

ii) $B(t) \in R_+^{n\times m}$,

$C(t) \in R_+^{p\times n}$, $D(t) \in R_+^{p\times m}$ for $t \geq 0$.

Proof. Necessity. Let $u(t) = 0$ for $t \geq t_0$ and $x_0 = e_j$. The trajectory does not leave the orthant R_+^n only if $\dot{x}(t_0) = A(t_0)e_j \geq 0$ and this implies (9). For the same reasons for $x_0 = 0$ we have $\dot{x}(t_0) = Bu(t_0) \geq 0$ and this implies $B(t) \in R^{p\times m}$, $t \geq t_0$ since $u(t_0) \in R_+^m$ may be arbitrary. From (1b) for $u(t_0) = 0$ we have $y(t_0) = C(t_0)x_0 \in R_+^p$ and $C(t) \in R_+^{p\times n}$, $t \geq 0$ since $x_0 \in R_+^n$ may be arbitrary. Similarly, from (1b) for $x_0 = 0$ we obtain $y(t_0) = D(t_0)u(t_0) \in R_+^p$ and $D(t) \in R_+^{p\times m}$ for $t \geq 0$ since $u(t_0) \in R_+^m$ may be arbitrary.

Sufficiency. If the condition (9) is satisfied then by Lemma (8) holds and from (2) we obtain $x(t) \in R_+^n$ for any $x_0 \in R_+^n$ and $u(t) \in R_+^m$, $t \geq t_0$, since $B(t) \in R_+^{n\times m}$. If $C(t) \in R_+^{p\times n}$ and $D(t) \in R_+^{p\times m}$ for $t \geq 0$ then from (1b) we obtain $y(t) \in R_+^p$ since $x(t) \in R_+^n$ and $u(t) \in R_+^m$ for $t \geq t_0$. ■

5. REACHABILITY

Definition 3. The state $x_f(t) \in R_+^n$ of the system (1) is called reachable in time $t_f - t_0$ if there exist an input vector $u(t) \in R_+^m$ for $[t_0, t_f]$ which steers the state of the system from $x_0 = 0$ to x_f.

Definition 4. If every state $x_f(t) \in R_+^n$ of the system (1) is reachable in time $t_f - t_0$ then the system is called reachable in time $t_f - t_0$.

Definition 5. If for every state $x_f(t) \in R_+^n$ there exist $t_f > t_0$ such that the state is reachable in time $t_f - t_0$ then the system (1) is called reachable.

A matrix $A \in R_+^{n\times n}$ is called monomial (or the generalised permutation matrix) if in each row and in each column only one entry is positive and the remaining entries are zero.

<u>Theorem 3.</u> The internally positive system (1) is reachable in time $t_f - t_0$ if the matrix

$$R(t_f, t_0) := \int_{t_0}^{t_f} \Phi(t_f, \tau) B(\tau) B^{\mathrm{T}}(\tau) \Phi^{\mathrm{T}}(t_f, \tau) d\tau$$

(T denotes the transpose) (16)

is a monomial matrix.

The input vector which steers the state vector of (1) from $x_0 = 0$ to x_f is given by

$$u(t) = B^{\mathrm{T}}(t)\Phi^{\mathrm{T}}(t_f,t)R^{-1}(t_f,t)x_f \qquad (17)$$

for $t \in [t_0, t_f]$

Proof. If $R(t_f,t_0)$ is a monomial matrix then the inverse matrix $R^{-1}(t_f,t_0) \in R_+^{n\times n}$ and $u(t) \in R_+^m$ for $[t_0, t_f]$. We shall show that (17) steers the state of (1) from $x_0 = 0$ to x_f. Substituting (17) into (2) for $t = t_f$ and $x_0 = 0$ we obtain

$$x(t_f) = \int_{t_0}^{t_f} \Phi(t_f,\tau)B(\tau)B^{\mathrm{T}}(\tau)\Phi^{\mathrm{T}}(t_f,\tau)R^{-1}(t_f,t_0)x_f\,d\tau =$$

$$= \left[\int_{t_0}^{t_f} \Phi(t_f,\tau)B(\tau)B^{\mathrm{T}}(\tau)\Phi^{\mathrm{T}}(t_f,\tau)d\tau\right]R^{-1}(t_f,t_0)x_f = x_f$$

Therefore if (16) is a monomial matrix then the positive system (1) is reachable in time $t_f - t_0$. ■

Theorem 4. The internally positive system (1) is reachable in time $t_f - t_0$ if

$$A(t) = \mathrm{diag}[a_1(t), a_2(t), \ldots, a_n(t)] \qquad (18)$$

($a_i(t)$ $i = 1,\ldots,n$ is continuous-time function) and $B(t) \in R_+^{n\times n}$ is a monomial continuous-time matrix.

Proof. It is well-known (Gantmacher, 1959) that if $A(t)$ has the form (18) then $A(t_1)A(t_2) = A(t_2)A(t_1)$ for $t_1, t_2 \in [t_0, \infty)$ and

$$\Phi(t,t_0) = \exp\left(\int_{t_0}^{t} A(\tau)d\tau\right)$$

is also diagonal nonnegative matrix for $t \ge t_0$. Hence the matrix $\Phi(t,t_0)B(t) \in R_+^{n\times n}$ is a monomial matrix and the matrix

$$R(t_f,t_0) = \int_{t_0}^{t_f} \Phi(t_f,\tau)B(\tau)B^{\mathrm{T}}(\tau)\Phi^{\mathrm{T}}(t_f,\tau)d\tau =$$

$$= \int_{t_0}^{t_f} \Phi(t_f,\tau)B(\tau)[\Phi(t_f,\tau)B(\tau)]^{\mathrm{T}}\,d\tau$$

is also a monomial matrix. Then by Theorem 3 the system (1) is reachable in time $t_f - t_0$. ■

Remark 2. If the diagonal matrix (18) and $B(t)$ are independent of t, then from theorems 3 and 4 we obtain the corresponding theorems 3.10 and 3.11 in (Kaczorek, 2001)

Similar results can be obtained for the controllability of time-varying linear systems.

6. EXAMPLE

Consider the system (1) with $t_0 = 0$ and

$$A(t) = \begin{bmatrix} 2 & 0 \\ 0 & t \end{bmatrix},\; B(t) = \begin{bmatrix} 0 & \mathrm{e}^t \\ \sqrt{t} & 0 \end{bmatrix} \qquad (19)$$

By theorem 4 the system is reachable in time $t_f - t_0$. Therefore, there exist input $u(t)$ which steers the state of the system from $x_0 = 0$ to $x_f = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ in time $t_f = 1$. Using (3a), (16) and (17) we obtain

$$\Phi(1,\tau) = \exp\left(\int_{\tau}^{1} A(\tau)d\tau\right) =$$

$$= \begin{bmatrix} \exp(2(1-\tau)) & 0 \\ 0 & \exp\left(\frac{1}{2}(1-\tau^2)\right) \end{bmatrix}$$

$$R(t_f,t_0) = R(1,0) = \int_0^1 \Phi(1,\tau)B(\tau)B^{\mathrm{T}}(\tau)\Phi^{\mathrm{T}}(1,\tau)d\tau =$$

$$= \begin{bmatrix} \frac{\mathrm{e}^4}{2}(1-\mathrm{e}^{-2}) & 0 \\ 0 & \frac{1}{2}(\mathrm{e}-1) \end{bmatrix}$$

$$u(t) = B^{\mathrm{T}}(t)\Phi^{\mathrm{T}}(1,t)R^{-1}(1,0)x_f =$$

$$= \begin{bmatrix} 0 & \frac{2t}{\mathrm{e}-1}\exp\left(\frac{1}{2}(1-t^2)\right) \\ \frac{4\cdot\exp(-t)}{\mathrm{e}^2-1} & 0 \end{bmatrix}$$

7. CONCLUDING REMARKS

The notions of externally positive and internally positive time-varying linear systems have been introduced. Necessary and sufficient conditions for the external and internal positivity of time-varying linear systems have been established. The concept of reachability has been extended for internally positive time-varying linear systems and sufficient conditions for the reachability of internally positive time-varying linear systems have been established. With slight modifications the consideration can be extended for discrete-time varying linear systems. An extension of these considerations for 2D linear systems with variable coefficients is also possible. An open problem is an extension of these considerations for singular time-varying linear systems.

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