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LQ/LQG CONTROLLER FOR PARALLEL MODELS [1]

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Abstract: The paper presents LQ/LQG optimal controller based on a set of parallel models with given probabilities of individual models (mixture distribution). Both state feedback LQ controller (in case of measurable state) and output feedback LQG controller are described. *Copyright © 2002 IFAC.*

Keywords: Quadratic optimal regulator, state feedback, output feedback, state estimation, mixture distribution.

1. INTRODUCTION

The methodology based on bayesian update of the probability distribution over the set of possible models (Havlena *et al.*, 1996) enables description of a plant by a mixture distribution (Titterington *et al.*, 1985; Böhm and Kárný, 2001), .

LQ/LQG control algorithm based on such mixture distribution by a set of parallel models with given probabilities is developed in this paper. The solution is given for optimal state feedback controller in case of measurable state and optimal output controller if only the outputs of the system are available.

The outline of the paper is as follows: in Section 2 Bayesian approach to state estimation for a set of parallel models is formulated in a general way and implementation based on a bank of Kalman filters is described.

In Section 3, optimal state feedback controller is developed for measurable state of the plant.

In Section 4, optimal output feedback controller is developed and separation principle is discussed for the case of mixture distribution model.

2. BAYESIAN APPROACH TO STATE ESTIMATION

2.1 *Process model and state estimation*

In this section we will review a process model from the bayesian viewpoint.

For the design of manipulated input, the knowledge of the process output based on a finite set of observed input and output data up to time $t-1$

$$\mathcal{D}^{t-1} = \{u(1), y(1), \ldots, u(t-1), y(t-1)\} \quad (1)$$

is required. It can be described by a set of c.p.d.f.

$$p\left(y(t)\,\middle|\,\mathcal{D}^{t-1}, u(t)\right) \quad \text{for} \quad t = 1, \ldots \,. \quad (2)$$

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If there exist a finite-dimensional vector variable $x(t)$ such that

$$p\left(x(t{+}1), y(t)\left|\mathcal{D}^{t-1}, x(t), u(t)\right.\right) = \\ = \; p\left(x(t{+}1), y(t)|x(t), u(t)\right) \qquad (3)$$

i.e. it contains all the relevant information for the prediction of the process output $y(t)$ and state $x(t+1)$, then it is called the *state of the process.*

To obtain the predictive c.p.d.f. (2) given as

$$p\left(y(t)\left|\mathcal{D}^{t-1}, u(t)\right.\right) = \qquad (4) \\ = \int p\left(y(t)|x(t), u(t)\right) \; p\left(x(t)\left|\mathcal{D}^{t-1}\right.\right) \; dx(t),$$

the c.p.d.f.

$$p\left(x(t)\left|\mathcal{D}^{t-1}, u(t)\right.\right) = p\left(x(t)\left|\mathcal{D}^{t-1}\right.\right) \qquad (5)$$

representing our knowledge about the state of the process must be also propagated in time. This c.p.d.f. is called the *state estimate*. The condition (5) introduced by (Peterka, 1981) is called the *natural condition of control*. It will be used repeatedly in the following text.

The c.p.d.f. $p\left(y(t)|x(t), u(t)\right)$ is defined by the output (measurement) equation of the state-space model of the process

$$y(t) = Cx(t) + Du(t) + e(t), \qquad (6)$$

where $e(t)$ is the measurement noise with known distribution with zero mean and covariance matrix $\mathrm{cov}\,\{e(t)\} = \Gamma_e$, independent of the state and input of the process.

The incorporation of the information contained in a new pair of data $\{u(t), y(t)\}$ (the *data-update* step of the algorithm) can be described as

$$p\left(x(t)\left|\mathcal{D}^{t}\right.\right) = \frac{p\left(y(t)|x(t), u(t)\right)}{p\left(y(t)\left|\mathcal{D}^{t-1}, u(t)\right.\right)} \times \qquad (7) \\ \times p\left(x(t)\left|\mathcal{D}^{t-1}\right.\right).$$

The *time-update* step of the algorithm, i.e. the predictive c.p.d.f. $p\left(x(t{+}1)\left|\mathcal{D}^{t}\right.\right)$ is given as

$$p\left(x(t{+}1)\left|\mathcal{D}^{t}\right.\right) = \qquad (8) \\ \int p\left(x(t{+}1)\left|x(t), \mathcal{D}^{t}\right.\right) \; p\left(x(t)\left|\mathcal{D}^{t}\right.\right) \; dx(t)$$

To complete this step, a state development model defined by the c.p.d.f.

$$p\left(x(t{+}1)\left|x(t), \mathcal{D}^{t}\right.\right) = \qquad (9) \\ = p\left(x(t{+}1)|x(t), u(t), y(t)\right)$$

which can be obtained from (3) is necessary. The c.p.d.f. $p\left(x(t{+}1)|x(t), u(t)\right)$ is usually defined by the state transition equation of the state-space model of the process

$$x(t{+}1) = Ax(t) + Bu(t) + v(t), \qquad (10)$$

where $v(t)$ is the process noise with known distribution with zero mean and covariance matrtix $\mathrm{cov}\,\{v(t)\} = \Gamma_v$, independent of the state and input of the process. The role of the term $y(t)$ in the condition of (9) is discussed in (Havlena, 1993).

2.2 *Multiple state development models*

Suppose a set of h alternative state development models (9) parameterized by the index of active models $m(t) = 1, \ldots, h$

$$p_i\left(x(t{+}1)|x(t), u(t)\right) = \qquad (11) \\ = p\left(x(t{+}1)|x(t), u(t), m(t) = i\right)$$

is given. Then several approaches to simultaneous filtering of the state and detection of the active model can be developed.

The simplest possibility is to propagate the state estimates based on the i-th model in parallel and compute the probability distribution over the set of models. In this setting, no changes in the active model are modeled but the (constant) active model is classified from the set of candidates.

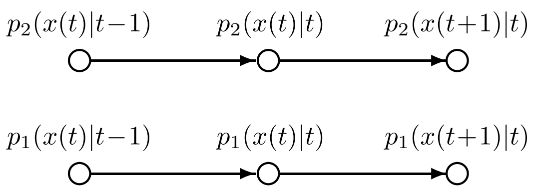

Fig. 1. State filtering and model classification with parallel models

Suppose the initial probability distribution (p.d.) over the set of candidate models is given

$$p\left(m(0) = i\left|\mathcal{D}^{0}\right.\right) = 1/h. \qquad (12)$$

The probabilities $\alpha_i(t) = p\left(m(t) = i\left|\mathcal{D}^{t}\right.\right)$ can be updated by the data as

$$p\left(m(t) = i\left|\mathcal{D}^{t}\right.\right) \propto p\left(y(t)\left|\mathcal{D}^{t-1}, u(t), m(t) = i\right.\right) \\ \times \; p\left(m(t) = i\left|\mathcal{D}^{t-1}\right.\right) \qquad (13)$$

where the predictive c.p.d.f. of the output

$$p\left(y(t)\left|\mathcal{D}^{t-1}, u(t), m(t) = i\right.\right)$$

is given by (4), where the c.p.d.f. of the state based on the i-th model

$$p_i\left(x(t)\,\middle|\,\mathcal{D}^{t-1}\right) = p\left(x(t)\,\middle|\,\mathcal{D}^{t-1}, m(t)-i\right) \quad (14)$$
$$= \mathcal{N}\left(\widehat{x}_i(t|t-1), P_i(t|t-1)\right)$$

is used (see Fig. 1). The time-update step of the algorithm given as

$$p\left(m(t+1)=i\,\middle|\,\mathcal{D}^{t}\right) = p\left(m(t)=i\,\middle|\,\mathcal{D}^{t}\right) \quad (15)$$

corresponds to a hypothesis that the active model is not supposed to vary in time. To enable tracking of the changes in the active model, some form of obsolete information forgetting or model mixing (Blom and Bar-Shalom, 1988) has to be introduced. “Total forgetting” can be implemented using the prior distribution

$$p\left(m(t)=i\,\middle|\,\mathcal{D}^{t-1}\right) = 1/h \quad (16)$$

at each step, resulting in maximum likelihood estimates

$$p\left(m(t)=i\,\middle|\,\mathcal{D}^{t}\right) \propto \quad (17)$$
$$\propto\; p\left(y(t)\,\middle|\,\mathcal{D}^{t-1}, u(t), m(t)=i\right).$$

3. STATE FEEDBACK CONTROLLER

In this section LQ controller for state feedback based on a set of parallel models with known probabilities (mixture distribution) and resulting in ”mixture” form of Riccati equation will be developed.

Suppose a set of h state development models are given

$$p_i\left(x(t+1)|x(t),u(t)\right) = \quad (18)$$
$$= \mathcal{N}\left(A_i x(t) + B_i u(t), \Gamma_{v_i}(t)\right)$$

where $\Gamma_{v_i}(t) = \operatorname{cov}\{v_i(t)\}$. Then the state prediction based on the measured state $x(t)$ is

$$p\left(x(t+1)|x(t),u(t)\right) = \quad (19)$$
$$= \sum_{i=1}^{h} \alpha_i p_i\left(x(t+1)|x(t),u(t)\right)$$

Consider a loss function

$$V(x(t), u_t^{N-1}, t) = \mathcal{E}\Big\{x^T(N)Q(N)x(N) \quad (20)$$
$$+ \sum_{\tau=t}^{N-1} x^T(\tau)Q(\tau)x(\tau) + u^T(\tau)R(\tau)u(\tau)\Big\}$$

where

$$u_t^{N-1} = \left\{u(t), \ldots, u(N-1)\right\} \quad (21)$$

and its optimal value

$$V^*(x(t),t) = \min_{u_t^{N-1}} \mathcal{E}\Big\{x^T(N)Q(N)x(N) \quad (22)$$
$$+ \sum_{\tau=t}^{N-1} x^T(\tau)Q(\tau)x(\tau) + u^T(\tau)R(\tau)u(\tau)\Big\}$$

Suppose the state-dependent part of the optimal value of the cost function is given by a quadratic form

$$V^*(x(t),t) = x^T(t)P(t)x(t) + \;\ldots \quad (23)$$

Then one step of the algorithm of dynamic programming, starting with $P(N) = Q(N)$, can be written as

$$V^*(x(t-1), t-1) = \quad (24)$$
$$= \min_{u(t-1)} \mathcal{E}\big\{x^T(t-1)Q(t-1)x(t-1)$$
$$+ u^T(t-1)R(t-1)u(t-1)$$
$$+ V^*(X(t),t)\big|t-1\big\}$$
$$= \min_{u(t-1)} \mathcal{E}\big\{x^T(t-1)Q(t-1)x(t-1)$$
$$+ u^T(t-1)R(t-1)u(t-1)$$
$$+ \mathcal{E}\left\{x^T(t)P(t)x(t)\big|t-1\right\}\big\}$$

where the mean can be obtained as

$$\mathcal{E}\left\{(x^T(t)P(t)x(t)|t-1\right\} = \quad (25)$$
$$= x^T(t-1)\sum_{i=1}^{h} \alpha_i A_i^T P(t) A_i x(t-1)$$
$$+ x^T(t-1)\sum_{i=1}^{h} \alpha_i A_i^T P(t) B_i u(t-1)$$
$$+ u^T(t-1)\sum_{i=1}^{h} \alpha_i B_i^T P(t) A_i x(t-1)$$
$$+ u^T(t-1)\sum_{i=1}^{h} \alpha_i B_i^T P(t) B_i u(t-1)$$
$$+ \sum_{i=1}^{h} \alpha_i \operatorname{trace}\left(P(t)\Gamma_{v_i}(t-1)\right)$$

etc. for $t = N-1, \ldots, 1$. Using completion of squares, the minimization (24) results in optimal control law

$$u^*(t) = -\left(R(t) + \sum_{i=1}^{h} \alpha_i B_i^T P(t+1) B_i\right)^{-1} \quad (26)$$
$$\times \sum_{i=1}^{h} \alpha_i B_i^T P(t+1) A_i x(t)$$

and the Riccati equation

$$P(t) = \sum_{i=1}^{h} \alpha_i A_i^T P(t+1) A_i + Q(t) - \quad (27)$$

$$
-\left(\sum_{i=1}^{h}\alpha_i A_i^T P(t+1)B_i\right)\times
\times\left(R(t)+\sum_{i=1}^{h}\alpha_i B_i^T P(t+1)B_i\right)^{-1}\times
\times\left(\sum_{i=1}^{h}\alpha_i B_i^T P(t+1)A_i\right)
$$

with final condition $P(N) = Q(N)$. The optimal value of the loss function (22) is

$$
J^* = V^*(x(0),0) = x^T(0)P(0)x(0) + \sum_{t=0}^{N-1}\left\{\sum_{i=1}^{h}\alpha_i \mathrm{tr}\ P(t)\Gamma_{v_i}(t)\right\} \tag{28}
$$

4. OUTPUT FEEDBACK CONTROLLER

In this section, LQG controller for output feedback based on a set of parallel model (mixture distribution) will be developed and the validity of separation principle for multiple parallel models will be investigated. The set of models is restricted to *compatible models*, i.e. models with different parameters within the same structure.

For the output feedback controller, the set of models available is given by a set of predictive c.p.d.fs

$$
p_i\left(x(t+1)|\mathcal{D}^t\right) = \mathcal{N}\left(\widehat{x}_i(t+1|t); R_{x_i}(t+1|t)\right). \tag{29}
$$

Then the state prediction based on the measured data $\mathcal{D}^t$ is

$$
p\left(x(t+1)|\mathcal{D}^t)\right) = \sum_{i=1}^{h}\alpha_i p_i\left(x(t+1)|\mathcal{D}^t\right) \tag{30}
$$

Then a single step of the algorithm of stochastic dynamic programming can be written as

$$
V^*(t) = \min_{u(t)}\ \mathcal{E}\Big\{x^T(t)Q(t)x(t) + u^T(t)R(t)u(t) + V^*(t+1)\ |\mathcal{D}^{t-1}, u(t)\Big\} \tag{31}
$$

where $V^*(t)$ is the optimal value of loss function (20) based on the limited information $\mathcal{D}^{t-1}$.

Let the backward solution starts for $t = N$

$$
V^*(N) = \mathcal{E}\left\{x^T(N)Q(N)x(N)\ |\ \mathcal{D}^{N-1}\right\} = \sum_{i=1}^{h}\alpha_i\widehat{x}_i^T(N|N-1)P(N)\widehat{x}_i(N|N-1) + \sum_{i=1}^{h}\alpha_i \mathrm{tr}(Q(N)R_{x_i}(N|N-1). \tag{32}
$$

Let us further continue with evaluation of $V^*(t)$ for time $t = N-1,\ldots,1$

$$
V^*(t-1) = \min_{u(t-1)}\mathcal{E}\Big\{x^T(t-1)Q(t-1)x(t-1) + u^T(t-1)R(t-1)u(t-1) + \sum_{i=1}^{h}\alpha_i\widehat{x}_i^T(t|t-1)P(t)\widehat{x}_i(t|t-1) + \sum_{i=1}^{h}\alpha_i\mathrm{tr}\ Q(t)R_{x_i}(t|t-1)\Big|\mathcal{D}^{t-2}, u(t-1)\Big\} \tag{33}
$$

The optimal value of the loss function can be rearranged as

$$
V^*(t-1) = \sum_{i=1}^{h}\alpha_i\mathrm{tr}\ \left(Q(t)R_{x_i}(t|t-1)\right) + \min_{u(t-1)}\Big\{\sum_{i=1}^{h}\alpha_i\widehat{x}_i^T(t|t-1)P(t)\widehat{x}_i(t|t-1) + u^T(t-1)R(t-1)u(t-1) + \mathcal{E}\big\{x^T(t-1)Q(t-1)x(t-1)\big|\mathcal{D}^{t-2}, u(t-1)\big\}\Big\} \tag{34}
$$

The mean in the previous expression equals

$$
\mathcal{E}\left\{x^T(t-1)Q(t-1)x(t-1)|\mathcal{D}^{t-2}, u(t-1)\right\} = \sum_{i=1}^{h}\alpha_i\widehat{x}_i^T(t-1|t-2)Q(t-1)\widehat{x}_i(t-1|t-2) + \sum_{i=1}^{h}\alpha_i\mathrm{tr}\ \left(Q(t-1)R_{x_i}(t-1|t-2)\right) \tag{35}
$$

and the state estimates $\widehat{x}_i(t|t-1)$ are given by Kalman filters for individual models as

$$
\begin{aligned}
\widehat{x}_i(t+1|t) &= A_i\widehat{x}_i(t|t-1) + B_i u(t) + L_i(t)\left(y(t) - C_i\widehat{x}_i(t|t-1)\right)\\
R_{x_i}(t+1|t) &= A_i R_{x_i}(t|t-1)A_i^T + \Gamma_{v_i} - L_i(t)C_i R_{x_i}(t|t-1)A_i^T
\end{aligned} \tag{36}
$$

where Kalman gains equal

$$
L_i(t) = A_i R_{x_i}(t|t-1)C_i^T \times \left(C_i R_{x_i}(t|t-1)C_i^T + \Gamma_{e_i}\right)^{-1} \tag{37}
$$

and the matrices $R_{x_i}(t|t-1)$ are the covariance matrices of state estimation errors.

After the substitution of (35) and (36), the loss function (34) results in

$$
V^*(t-1) = \sum_{i=1}^{h}\alpha_i\mathrm{tr}\ \left(Q(t)R_{x_i}(t|t-1)\right) + \tag{38}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{h}\alpha_i \mathrm{tr}\left(L_i^T(t-1)P(t)L_i(t-1)Q_\varepsilon(t-1|t-2)\right)+\\
&+\min_{u(t-1)}\Big\{u^T(t-1)R(t-1)u(t-1)+\\
&\qquad+\sum_{i=1}^{h}\alpha_i\widehat{x}_i^T(t-1|t-2)Q(t-1)\widehat{x}_i(t-1|t-2)\\
&\qquad+\sum_{i=1}^{h}\alpha_i\left(A_i\widehat{x}_i(t-1|t-2)+B_iu(t-1)\right)^T\\
&\qquad\quad\times P(t)\left(A_i\widehat{x}_i(t-1|t-2)+B_iu(t-1)\right)\Big\}
\end{aligned}
$$

where Q_ε is the prediction error covariance matrix

$$
Q_\varepsilon(t-1|t-2)=CR_{x_i}(t-1|t-2)C^T+\Gamma_{e_i}. \quad (39)
$$

Completing the squares the minimization of (38) results in optimal control law

$$
\begin{aligned}
u^*(t)=&-\Big(R(t)+\sum_{i=1}^{h}\alpha_iB_i^TP(t+1)B_i\Big)^{-1} \quad (40)\\
&\times\sum_{i=1}^{h}\alpha_iB_i^TP(t+1)A_i\widehat{x}_i(t|t-1)
\end{aligned}
$$

Using state feedback gain matrices

$$
\begin{aligned}
K_i(t)=&\Big(R(t)+\sum_{j=1}^{h}\alpha_jB_j^TP(t+1)B_j\Big)^{-1} \quad (41)\\
&\times B_i^TP(t+1)A_i,
\end{aligned}
$$

the optimal output feedback control equals

$$
u^*(t)=-\sum_{i=1}^{h}\alpha_iK_i(t)\ \widehat{x}_i(t|t-1) \qquad (42)
$$

i.e. it is a convex combination of feedback control laws for individual models. This result proves the validity of separation principle, however the individual control laws (41) cannot be evaluated independently.

Riccati equation for $P(t)$ is equivalent with the state feedback case (27) and the optimal value of quadratic criterion equals

$$
\begin{aligned}
J^*=V^*(0)=&\sum_{i=1}^{h}\alpha_i\widehat{x}_i^T(0)P(0)\widehat{x}_i(0)+ \qquad (43)\\
&+\sum_{t=0}^{N}\sum_{i=1}^{h}\alpha_i\Big\{\mathrm{tr}\ Q(t)R_{x_i}(t|t-1)\\
&\quad+\mathrm{tr}\ L_i^T(t)P(t+1)L_i(t)Q_\varepsilon(t-1|t-2)\Big\}
\end{aligned}
$$

where the second term reflects the stochastic input in the system and the last term the uncertainty of the state estimate.

Note that each model i can have a different structure and a different dimension of its state $x_i(t)$.

5. CONCLUSION

In many practical situations a set of multiple process models valid under different process conditions are available. Using a unified bayesian framework, the probability distribution over the set of models can be evaluated.

Then standard LQ/LQG control law can be rigorously designed for plant model given by a mixture distribution.

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