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ORBITAL STABILIZATION OF UNDERACTUATED MECHANICAL SYSTEMS

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Abstract: This paper studies the problem of periodic stabilization of nonlinear underactuated mechanical systems. In opposition to the problem of stabilization of underactuated systems (i.e. acrobots, pendubots, etc.) to a fixed equilibrium, the problem of orbital stabilization of underactuated systems consists in finding control that leads to a stable periodic orbits. The problem is relevant to a class of mechanical systems aimed at operating under periodic motion (orbits), i.e. walking mechanisms (Grizzle *et al.*, 2001), like the one shown in Fig.1.

Keywords: Robot control, control systems, nonlinear analysis, walking, stabilizing controllers.

1. INTRODUCTION

The problem of orbital stabilization arises from applications where the "natural" operation mode is an oscillatory one. The example treated in this paper concerns the stabilization of underactuated mechanisms, more precisely robots with less actuators than degrees of freedom.

Walking mechanisms, like the one shown in Fig.1 are intended to operate under periodic motions, i.e. walking, running, balancing phases. They are complex mechanisms to be controlled because their dynamics may be subject to structural changes during normal operation (loose of degrees of freedom, impacts, sliding, etc.), and the joint trajectory references are not necessarily known in advance. This last point implies that the control design set up is different from typical formulations of output tracking and regulation, where the set point (or the reference trajectory) is a priori given. Instead, the control design here should lead to a closed-loop system that generates its own periodic (stable) motion in the same way as a nonlinear oscillator does. The problem is relevant because there is a need for making the walking (or running) gaits varying on-line.

Here we address the particular case of a n-degrees of freedom walking mechanism, with $m = n - 1$ actuators. We assume that one leg is in contact whereas the other is not. The contact leg is assumed not to slide, and we do not consider any ground impacts. In other words, we look to a system which is equivalent to a n-degree of freedom inverted pendulum. However, the feasible workspace will be assumed to be restricted by the walking surface. Controlled motion under such a consideration leads to a problem that we name *periodic balancing.*

The control design aims at finding a feedback law such that the high-dimensional inverted pendulum is able to reach a periodic motion in a stable way. We also wish to be able to move from one orbit to another, possibly by just changing the speed of the cycle and/or the amplitude of the motion. Although not mandatory, this last point

Fig. 1. The 5-DOF walking robot Rabbit.

is essential when it comes to apply this idea to walking/runnig machines; it will allow to vary the speed and amplitude of the shape and speed of the gaits.

The admissible limit cycles cannot be completely free. To some extent, they should comply with the constraints imposed by the physics of the workspace (i.e. walking motion is constrained by the ground surface). This last restriction disqualifies feedbacks leading to homoclinic orbits (orbits with constant energy levels), that may induce inadmissible motions out of the robot workspace.

A possible control approach leading to a family of periodic orbits consists in finding a particular change of coordinates (generalized angle, and generalized radius deviation) as proposed by (Hauser and Chung, 1994), such that the (local) orbital stability can be determined by looking at the resulting transversal local approximation. The idea was applied to the cart and pendulum system, and requires to find, case-by-case such a transformation.

The control approach proposed here, consists first, in designing a feedback that transforms the full dimensional system motion into a one with lower dimension [1] (zero dynamics), where the limit cycles can be easily studied. Then, a controller is designed such that the low-dimension dynamical system motion reaches a specified target orbit. In this particular setup, the periodic behaviour of the zero dynamics implies also a periodic motion of all the system states.

Our contribution was reformulated recently in the context of an hybrid automaton for the double inverted pendulum (see (Asarin *et al.*, 2001)), where the search for a class of constrained orbits was done by means of the reachability analysis tool **d/dt**, a tool for verification and control synthesis for hybrid systems.

[1] This idea was previously used for walking robots with 3 and 5 DOF, see (Grizzle *et al.*, 2001)

In this paper we introduce a new control approach that imposes periodic behaviour to the zero-dynamics, by means of a dynamic controller. We introduce here the idea of virtual mechanical constraints as a support to our control design.

In the following section we formulate the problem. Section 3 studies the conditions under which the zero-dynamic may exhibe periodic orbits, and present a way to render this orbit stable through a dynamic controller. Finally we present some simulation results.

2. PROBLEM FORMULATION

Consider the n-DOF underactuated mechanism,

$$M(q)\ddot{q} + C(\dot{q}, q)\dot{q} + g(q) = B\tau \tag{1}$$

where $q \in R^n$ is the joint position vector, $\tau \in R^m, m < n$ is the input torque, M is the inertia matrix, C the coriolis and centrifugal matrix, and g is the gravity vector. B of rank m, is the input matrix that maps the m torque input vector τ to the joint coordinates space of dimension n.

Assume that q can be partitioned in two sets of coordinates (q_1, q_2), where $q_1 \in R^m$ corresponds to the actuated joints, and $q_2 \in R^{n-m}$ to the non actuated ones. Thus $B = [I_m, 0_{(n-m,m)}]^T$, and (1) can be expressed as

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} N_1(q, \dot{q}) \\ N_2(q, \dot{q}) \end{pmatrix} = \begin{pmatrix} \tau \\ 0 \end{pmatrix} \tag{2}$$

Define the following output

$$y = q_1 - \phi(q_2, \theta(t)) := h(q, t) \tag{3}$$

where $\phi(q_2, \theta)$ is a smooth function, to be defined later, and θ is a parameter vector allowed to vary with the time. Note that $y = 0$ defines a manifold of dimension m, where the system motion is constrained by the relation $q_1 = \phi(q_2, \theta)$. For a 2-DOF system, one can take as an example a linear constraint of the form $q_1 = aq_2 + b$, where $\theta = [a, b]^T$.

Denote $J(q) = \frac{\partial h}{\partial q}$, f the other terms involved in the differentiation of (3), and $N(q, \dot{q}) = C(\dot{q}, q)\dot{q} + g(q)$, the following feedback law

$$\tau = (JM^{-1}B)^{-1}\left[u + JM^{-1}N - f\right] \tag{4}$$

assuming that $JM^{-1}B$ is not singular (at least locally), linearizes the output y, i.e.

$$\ddot{y} = u$$

It is easy to show that the family of non-smooth feedbacks of the form:

$$u = -\lambda\dot{y} - k_s s - k|s|^n \mathrm{sign}(s) \tag{5}$$

$$s = \dot{y} + \lambda y \tag{6}$$

$$\dot{y} = \dot{q}_1 + J\dot{q} + \frac{\partial \phi}{\partial \theta}\dot{\theta} \tag{7}$$

where $0 \leq n < 1$, k, and k_s are positive constants, stabilizes $y \rightarrow 0$ in finite time. Note that in this setup the parameter vector θ may be allowed to vary in time.

Zero dynamics. The zero-dynamics resulting from $y \rightarrow 0$, will be of dimension $(n-m)$. It can be computed by first noticing that $y = \dot{y} = \ddot{y} = 0$ implies $u = 0$, and $\dot{J}\dot{q} = -J\ddot{q}$, which substituted in (4), and then in (1) gives

$$M\ddot{q} + N = B(JM^{-1}B)^{-1}\left[JM^{-1}N + J\ddot{q}\right] \tag{8}$$

that after some manipulations can be rewritten as

$$P(q)\left[M(q)\ddot{q} + N(q,\dot{q})\right] = 0 \tag{9}$$

where $P = I_n - B(JM^{-1}B)^{-1}JM^{-1}$ is the projection operator on the kernel of JM^{-1} in the orthogonal direction of B.

The zero dynamics lies then on a configuration space of dimension $n-m$. Using the constrains

$$\begin{aligned} q_1 &= \phi(q_2,\theta) \\ \dot{q}_1 &= J_2(q_2,\theta)\dot{q}_2, + J_\theta(q_2,\theta)\dot{\theta} \\ \ddot{q}_1 &= J_2(q_2,\theta)\ddot{q}_2 + p_2(q_2,\dot{q}_2,\theta,\dot{\theta},\ddot{\theta}). \end{aligned}$$

with $J_2 = \frac{\partial \phi}{\partial q_2}$, $J_\theta = \frac{\partial \phi}{\partial \theta}$, and $p_2 = \dot{J}_2\dot{q}_2 + J_\theta\ddot{\theta} + \dot{J}_\theta\dot{\theta}$.

It is thus possible to write the dynamics (9) as a function of the unactuated joints q_2 only, by taking $n-m$ independent lines of the system

$$P(\phi(q_2),q_2,\theta)\left[M(\phi(q_2),q_2,\theta)\begin{pmatrix} J_2\ddot{q}_2 + p_2 \\ \ddot{q}_2 \end{pmatrix} + N(\phi(q_2),q_2,J_2,\dot{q}_2,J_\theta,\theta,\dot{\theta})\right] = 0$$

The same expression can be obtained by substituting the above constrains on the unactuated part of the system in (2), i.e.

$$(m_{22} + m_{21}J_2)\ddot{q}_2 + m_{21}p_2 + N_2 = 0 \tag{10}$$

where the $m_{i,j}, \phi, p_2$, and N_2 are functions of $q_2, \dot{q}_2, J_2$, and $\theta, \dot{\theta}, \ddot{\theta}$.

The zero dynamics (10) admits, with $z = [q_2^T, \dot{q}_2^T]^T$, a state-space representation of the form

$$\dot{z} = f(z, \phi(z,\theta), \theta, \dot{\theta}, \ddot{\theta}) \tag{11}$$

The vector θ can be seen here as an additional degree of freedom that will be used to produce an (stable) oscillatory behaviour of the equation (11).

Problem definition. The problem is to find a smooth function ϕ and an adaptation law (dynamic feedback) for $\ddot{\theta}$, such that the zero dynamics (11) exhibits periodic (stable) solutions z^*.

When θ is found to be a constant and thanks to the restriction imposed by $y = 0$, it is clear that the actuated joint coordinates q_1 will exhibit a periodic motion, if z do have one. The zero dynamics will thus acts as a nonlinear autonomous oscillator which drives the rest of the coordinates (through the imposed virtual mechanical constrains $q_1 = \phi(q_2,\theta)$) to a periodic orbit.

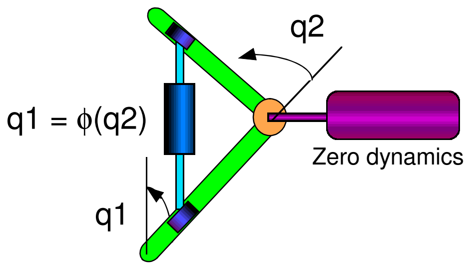

Fig. 2. Principle of the virtually constrained pendubot

This idea is illustrated in Fig.2, which shows a 2-DOF inverted pendulum (pendubot) under the proposed feedback structure. The actuator at joint q_1 is replaced by a virtual mechanical link (constraint), whereas the motion of the unactuated joint q_2 is driven by the zero dynamic generator, represented here as a virtual motor.

In the following sections, we study the conditions under which the functions $\phi(x,\theta)$, ensures the existence of periodic orbits, and studies different ways of defining $\ddot{\theta}$ to render these orbits stable.

3. PERIODIC ORBITS FOR THE ZERO DYNAMICS

Consider systems where the number of actuators is $m = n-1$. This case leads to a zero dynamics of dimension one ($dim(q_2) = 1$), described by a second order nonlinear equation. Its behaviour can thus be studied in the plane. In this case, the equation (10) takes the particular form:

$$\beta_2(q_2,\phi,\theta)\ddot{q}_2 + \beta_0(q_2,\dot{q}_2,\phi,\theta,\dot{\theta},\ddot{\theta}) = 0 \tag{12}$$

Assume that: $\exists\ \phi$, such that $\forall\ \theta$ resulting from the adaptation law $\ddot{\theta}$ (to be defined latter), and $\forall\ q_2 \in \Omega_2 \subseteq Q_2$ (Q_2 is the workspace for the underactuated variable), the solution of (12) are well defined , i.e.

$$\beta_2(q_2,\phi,\theta) > 0 \tag{13}$$

Let us define q_2^d as a desired equilibrium (center), for exemple the steady-state solution of (12). Denote $x = [x_1, x_2]^T$, with $x_1 = q_2 - q_2^d$, and $x_2 = \dot{q}_2$, then (12) writes as

$$\beta_2(x_1,q_2^d,\phi,\theta)\ddot{x}_1 + \beta_0(x,q_2^d,\phi,\theta,\dot{\theta},\ddot{\theta}) = 0. \tag{14}$$

that under assumption (13) has the following state-space representation $x = f(x,\theta,\phi,\dot{\theta},\ddot{\theta})$:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\beta(x,\phi,\theta,\dot{\theta},\ddot{\theta}) \end{cases} \tag{15}$$

with $\beta = \frac{\beta_0}{\beta_2}$.

Target orbit (exosystem). Introduce the generalized target orbit $\Omega_d(x)$ defying a closed path in the phase plane. Let also $\Omega_d(x)$ define an invariant and (at least locally) attractive set of the solution of the generalized exosystem (orbit generator),

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\beta_d(x) \end{cases} \tag{16}$$

By thereby, we assume that the function $\beta_d(x)$ vanishes at the equilibrium $x = 0$ ($\beta_d(0) = 0$) only, which is included in a convex set where $\Omega_d(x)$ is assumed to be. In other words, there exist a closed set $\mathcal{M} \in R^2$, s.t. $\mathcal{M}$ contains no equilibrium points and is positive invariant. The bounded semi-positive orbit $\Omega_d(x)$ is thus contained in $\mathcal{M}$.

Example 1 (parabolic orbits). Consider the "centered [2]" family of parabolic target orbits,

$$\Omega_d = \left\{ x : V_d = \frac{1}{2}(\alpha_d x_1^2 + x_2^2) \right\}$$

as a function of the parameters set $\{V_d, \alpha_d, q_2^d\}$ defining: the desired orbit level V_d, the desired orbit shape α_d, and the desired orbit center q_2^d (see footnote).

These orbits attract the solutions of the exosystem (16), with $\beta_d(x)$ defined as

$$\beta_d(x) = \alpha_d x_1 + k_V x_2 \tilde{V}(x)$$

where $\tilde{V}(x)$ is given by

$$\tilde{V}(x) = V(x) - V_d = \frac{1}{2}(\alpha_d x_1^2 + x_2^2) - V_d$$

that is,

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\alpha_d x_1 - k_V x_2 \left(\frac{1}{2}(\alpha_d x_1^2 + x_2^2) - V_d \right)$$

To see that, define $v = \tilde{V}^2/2$, and note that $\dot{v} = -2k_V x_2^2 v \leq 0$. The only two cases where $\dot{v}$ cancels are: 1) when the solution have reach the target orbit ($v = 0$), and 2) when initial condition are taken at the equilibrium $x = 0$. The former case shows the positive invariance of Ω_d, the later is a consequence that the orbit should be centered at the equilibrium point $(q_2^d, 0)$.

Example 2 (nonlinear oscillator). The previous examples require the explicit knowledge of the target orbit. It is however possible to re-formulate a similar problem by only defining the exosystem (orbit generator) without explicitly using the expression of the invariant set (orbital attractor) $\tilde{V}$. As an example consider the exosystem defined by the nonlinear oscillator

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - \epsilon h'(x_1) x_2$$

where $\epsilon > 0$ and $h(x)$ fulfills the following well known properties:

i) $h(0) = 0$,
ii) $h'(0) < 0$,
iii) $\lim_{x_1 \to \infty} h(x_1) = \infty$,
iv) $\lim_{x_1 \to -\infty} h(x_1) = -\infty$.

As a consequence of this, there exists (at least) one strip $S = [-s_{min}, s_{max}]$, of length L_s such that $h(x_1)x_1 < 0$ inside of S, and $h(x_1)x_1 \geq 0$ outside of S. The length L_s will allow to modify the amplitude of the oscillation. As a particular example we have the Van der Pol oscillator which obtained by letting $h(x_1) = -x_1 + \frac{1}{3}x_1^3$.

This leads to define the exosystem (16) with $\beta_d(x) = x_1 + \epsilon(-1 + x_1^2)x_2$, i.e.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + \epsilon(1 - x_1^2)x_2$$

In both examples, we have thus that for all $x(0) \neq (0,0)^T$, the system trajectories of (15) converge to the target orbit Ω_d (example 1), or to the solutions of the nonlinear oscillator (example 2), provided than we can find a function ϕ, and a feedback law for $\ddot{\theta}$ such that assumption (13), and the equality,

$$\beta(x, \phi, \theta, \dot{\theta}, \ddot{\theta}) = \beta_d(x)$$

holds simultaneously. This is formally stated next.

Main result. Theorem. Let $\theta_1 = \theta$, $\theta_2 = \dot{\theta}$, $\Theta = (\theta_1, \theta_2)^T$. Consider the following *extended* zero dynamics (with $\phi = \phi(x_1, \Theta)$):

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\beta(x, \phi, \Theta, k(x, \phi, \Theta)) \\ \dot{\theta}_1 = \theta_2 \\ \dot{\theta}_2 = k(x, \phi, \Theta) \end{cases} \tag{17}$$

where $k(x, \phi, \Theta)$ defines the adaptation law for $\ddot{\theta}$. Assume that a $k(x, \phi, \Theta)$ can be found such that the following holds:

(1) A target orbit (or a target exosystem) is defined by the equation set (16), throughout the definition of a particular $\beta_d(x)$,
(2) There exists a smooth function $\phi(x_1, \Theta)$, and a set Ω_Θ of suitable initial conditions for $\Theta(0) = \Theta_0$, such that:
  A1) $\beta_2(x_1(t), \phi(x_1(t), \Theta(t)), \Theta(t)) > 0$,
  A2) $\beta(x(t), \phi(x_1(t), \Theta(t)), \Theta(t)) = \beta_d(x)$.

  for all $\Theta(t), x_1(t), t \geq 0$ resulting from the solution of (17), with $x(0) \neq 0$, and $\Theta(0) \in \Omega_\Theta$.
(3) The resulting sub-system, with $x^*(t) = x^*(t + T)$, and $|x^*(t)| < \infty$,

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[2] Due to the change of coordinates $x_1 = q_2 - q_2^d$, the orbits will be centered about $x_1 = 0$. In practice, the user specs are also function of the center point q_2^d

$$\begin{cases} \dot{\theta}_1 &= \theta_2 \\ \dot{\theta}_2 &= k(x^*, \phi, \Theta) \end{cases} \tag{18}$$

yields bounded solutions.

Then, for all $x(0) \neq 0$ the solution $x(t)$ converge to the target orbit.

4. STUDY CASE: THE PENDUBOT

Consider the pendubot equations in the form (2), with

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix} \tag{19}$$

and

$$N = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} + \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \tag{20}$$

where

$$\begin{array}{l} m_{11} = m_1 l_1^2 + m_2(l_2^2 + L_1^2 + 2L_1 l_2 c_2) \\ m_{12} = m_2(l_2^2 + L_1 l_2 c_2) \\ m_{22} = m_2 l_2^2 \\ C_{11} = -m_2 L_1 l_2 s_2 \dot{q}_2 \\ C_{12} = -m_2 L_1 l_2 s_2(\dot{q}_1 + \dot{q}_2) \\ C_{21} = m_2 L_1 l_2 s_2 \dot{q}_1 \\ C_{22} = 0 \\ g_1 = g((m_1 l_1 + m_2 L_1) s_1 + m_2 l_2 s_{12}) \\ g_2 = g m_2 l_2 s_{12} \end{array} \tag{21}$$

with $s_i := sin(q_i)$, $c_i := cos(q_i)$, $s_{ij} := sin(q_i + q_j)$. The parameters values are: $L_1 = 0.52m$; $l_1 = 0.3m$; $l_2 = 0.29m$; $m_1 = 6kg$; $m_2 = 4kg$, and $g = 9.81$. We have here $B = (1, 0)^T$

Target orbits. For this study case, we consider the two possible structures of equation (16) proposed in the previous section, i.e.

- Parabolic orbits,

$$\beta_d(x) = \alpha_d x_1 + k_V x_2 \tilde{V}(x)$$

- Nonlinear oscillator,

$$\beta_d(x) = x_1 + \epsilon(-1 + x_1^2)x_2$$

Zero-dynamics. Consider the particular case of $\phi(q_2, \theta) = aq_2 + b(t)$, $\theta = (a, b(t))^T$ with a constant, and $b(t)$ to be adjusted as described previously. Then, the output y is:

$$y(t) = q_1(t) - aq_2(t) - b(t) \tag{22}$$

Let q_2^d be the desired equilibrium about which the limit cycle will be defined, then the resulting zero dynamics expressed in the *shifted* coordinates is given by (14), with

$$\beta_2(x_1, a, q_2^d) = m_2 l_2((1 + a)l_2 + aL_1 \cos(x_1 + q_2^d))$$
$$\beta_0(x, a, b, \dot{b}, \ddot{b}, q_2^d) = \sigma_2(\cdot)x_2^2 + \sigma_1(\cdot)x_2 + \sigma_0(\cdot)$$

where,

$$\begin{aligned} \sigma_2(\cdot) &= m_2 l_2 L_1 a^2 \sin(x_1 + q_2^d) \\ \sigma_1(\cdot) &= 2m_2 l_2 L_1 a\dot{b} \sin(x_1 + q_2^d) \\ \sigma_0(\cdot) &= m_2 l_2 \Big[(l_2 + L_1 \cos(x_1 + q_2^d))\ddot{b} + L_1 \dot{b}^2 \sin(x_1 + q_2^d) \\ &\qquad + g \sin((a + 1)(x_1 + q_2^d) + b)\Big] \end{aligned}$$

A sufficient condition satisfying hypothesis $(A1)$ of the theorem for the existence of feasible solutions is $\beta_2 = m_{22} + am_{12} > 0, \ \ \forall q_2$, which can be simplified to a simple bound on a

$$|a| < \frac{l_2}{L_1 + l_2} := a_{max} \tag{23}$$

however, less conservative bounds can be found (i.e. negative values for a).

Adaptation law. The state-space representation of the zero dynamics is given by (15) with $\beta = \beta_0/\beta_2$, with β_0 and β_2 as defined previously. Interestingly enough, it can be shown by direct calculations that the expression resulting from the equality $(A2)$ of the main theorem, is affine in $\ddot{b}$, i.e.

$$\gamma(x_1, a)\ddot{b} + \xi(x, a, b, \dot{b}) = \beta_d(x) \tag{24}$$

with,

$$\begin{aligned} \gamma &= \frac{m_2 l_2(l_2 + L_1 \cos(x_1 + q_2^d))}{\beta_2(x_1, a)} \\ \xi &= \frac{\sigma_2 x_2^2 + \sigma_1 x_2 + \zeta}{\beta_2(x_1, a)} \\ \zeta &= m_2 l_2 \Big[L_1 \dot{b}^2 \sin(x_1 + q_2^d) \\ &\qquad + g \sin((a + 1)(x_1 + q_2^d) + b)\Big] \end{aligned}$$

therefore the adaptation law $k(x, \phi, \Theta)$, with $b = \theta_1$, $\dot{b} = \theta_2$, and $\ddot{b} = k(x, \phi, \Theta)$, ($a$ is not consider anymore as an adaptation parameter) has the form:

$$\ddot{b} = \frac{1}{\gamma(x_1, a)} \Big(-\xi(x, a, b, \dot{b}) + \beta_d(x)\Big) \tag{25}$$

Note that, solutions of this equation are well defined as long as $\gamma > 0$, that is if a is selected according to (23), and $L_1 > l_2$. This last condition can be interpreted as a physical constraint needed to generate the required oscillations.

It remains to check under which conditions the assumption of the main theorem holds. In other words, under which conditions, the subsystem

$$\begin{aligned} \dot{\theta}_1 &= \theta_2 \\ \dot{\theta}_2 &= \frac{1}{\gamma(x_1^*, a)} \Big(-\xi(x^*, a, b, \dot{b}) + \beta_d(x^*)\Big) \end{aligned}$$

has, for the specified target orbit solutions $x^*(t)$, a bounded solution. This issue, will not be studied here in detail, but only presented simulation show that it is indeed the case.

Simulations results. Figures 3, 4, 5, and 6 show several examples of motion, where the trajectories converge to the orbit Ω_d, for the parabolic and the nonlinear oscillator target orbits. For the parabolic orbits, we take any desired parameters set $\{V_d, \alpha_d, q_2^d\}$, i.e. $V_d = 0.01[Rad^2/s^2]$, $\alpha_d = 0.3286[1/s^2]$, $q_2^d = \pi/2$[Rad], $k_V = 50$, initial conditions: $q_2^d(0) = \pi/3$[Rad], $a(0) = -1.0072$, $b(0) = 0.0083$[Rad], and $\dot{b}(0) = 0$[Rad/s]. For the nonlinear oscillator, the following desired parameters set was taken : $q_2^d = 1.1551$[Rad], initial conditions: $q_2^d(0) = 2.1551$[Rad], $a(0) = -1.0072$, $b(0) = 0.0083$[Rad], and $\dot{b}(0) = 0$[Rad/s].

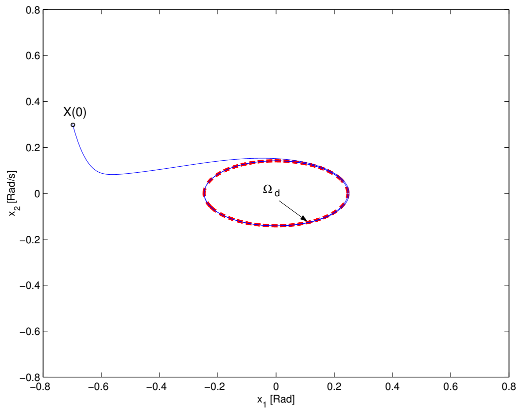

Fig. 3. Convergence to a parabolic target orbit Ω_{di}.

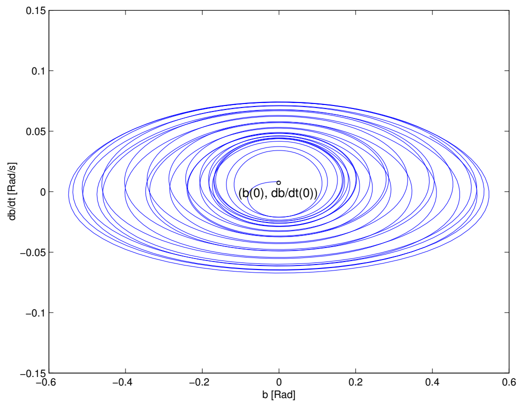

Fig. 4. Evolution of $b(t)$ in the case of parabolic orbit.

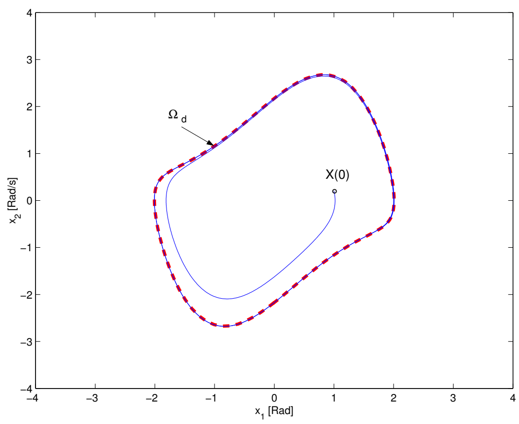

Fig. 5. Convergence to a nonlinear oscillator target orbit Ω_{di}.

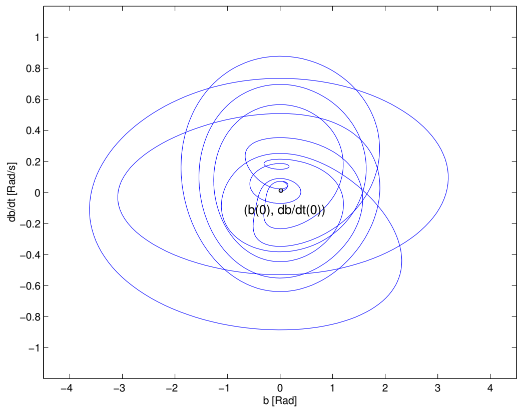

Fig. 6. Evolution of $b(t)$ in the case of nonlinear oscillator target orbit.

5. CONCLUSIONS

We have dealt with the problem of orbital stabilization of underactuted mechanical system. The case when the number of degrees of freedom is equal to the number of actuators minus one, has been treated in a rather general framework. Conditions for the local convergence have been established via an *implicit* adaptive hybrid algorithm. Possibility of finding an *explicit* solution has to be studied case by case. We have presented the pendubot example for which an explicit algorithm can be found. The case of the acrobot can also be treated, but the analytic expressions are more extensive. This case has not been presented here.

6. REFERENCES

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