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DYNAMIC ROBOT LOCALIZATION AND MAPPING USING UNCERTAINTY SETS

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Abstract: This paper addresses the simultaneous localization and mapping problem in an environment where indistinguishable landmarks can be detected. A set theoretic approach to the problem is presented. Algorithms for measurement-to-feature matching, estimation of landmark positions, estimation of robot location and heading are derived in terms of uncertainty regions, under the hypothesis that errors affecting all sensor measurements are unknown-but-bounded. The proposed technique is validated in an experimental setup.

Keywords: Mobile robots, localization, mapping, uncertainty, set membership.

1. INTRODUCTION

The problem addressed in this paper is self-localization of an autonomous mobile robot. The considered scenario is that of a vehicle equipped with odometric and exteroceptive sensors, moving in an environment where indistinguishable landmarks are present. The robot has to detect as many landmarks as possible, perform measurements with respect to them, associate each measurement to the corresponding landmark and exploit the correctly matched measurements to estimate its position and orientation.

Landmark-based robot localization has been widely investigated in the last decade. A great variety of solutions is now available for the case in which landmark position is a priori known (i.e., an environment map is given). The proposed localization techniques rely on several different approaches, including Extended Kalman Filtering (EKF) (Leonard and Durrant-Whyte, 1992), Set Membership (SM) estimation (Garulli and Vicino, 2001), mixed statistical/set-theoretic filters (Hanebeck and Schmidt, 1996) and many others.

The problem turns out to be much harder when an environment map is not available and the robot has to estimate both landmark and its own position. This is the so-called *Simultaneous Localization And Map building* (SLAM) problem, which has been recently addressed via EKF (Castellanos and Tardos, 1999), Markov localization (Thrun *et al.*, 1998) and SM techniques (Di Marco *et al.*, 2001*b*).

In this paper, a set-theoretic approach to the localization problem is adopted. Under the assumption that all errors affecting sensor measurements and robot dynamic model are unknown-but-bounded, set-valued estimates of robot position and orientation are computed. Both distance and angle measurements are considered. It is shown that SM uncertainty representation can be exploited to obtain a correct matching between measurements and detected landmarks. Moreover, the proposed approach is naturally extended to tackle the SLAM problem, by including landmark positions in the state vector to be estimated. A SM algorithm for dynamic localization is constructed, exploiting state decomposition and set approximations. The algorithm is validated in an experimental setup, considering both cases of known and unknown environment map.

2. SET-THEORETIC LOCALIZATION

2.1 *Set membership pose estimation*

Let us consider a vehicle navigating in a 2D environment, and let $p(k) = [x(k)\ y(k)\ \theta(k)]' \in \mathcal{Q} \triangleq \mathbb{R}^2 \times [-\pi\ \pi]$ be the pose of the agent at time k (where $x(k), y(k)$ is the robot position, and $\theta(k)$ represents the robot heading w.r.t. the positive x-axis). Under the assumption of slow robot dynamics, if translation and rotation measurements $u(k)$ from odometric sensors are available, the vehicle dynamics can be described by the linear discrete-time model

$$p(k+1) = p(k) + u(k) + Gw(k), \qquad (1)$$

where $w(k) \in \mathbb{R}^3$ models the error affecting measurements $u(k)$ (possibly shaped by a suitable matrix G).
The robot is equipped with exteroceptive sensors, providing measurements on the environment. It is assumed that the environment can be described by landmarks, i.e. indistinguishable features represented by points on a plane. When navigating in a 2D environment, the robot performs two kinds of measurements: distance from a landmark and angle between robot orientation and the direction of a landmark (see Fig. 1). Several sensors (such as laser rangefinders and stereovision systems) provide both kinds of information. These measurements can be modeled as nonlinear functions of the robot pose $p(k)$ and the position $l_i = [x_{l_i}\ y_{l_i}]'$ of the sensed landmark, i.e.

$$\begin{array}{l} D_i(k) = d(p(k), l_i) + v_{d_i}(k) \\ A_i(k) = \alpha(p(k), l_i) + v_{\alpha_i}(k) \end{array} \quad i = 1, \ldots, m, \quad (2)$$

where m is the number of measurements performed at time k, $D_i(k)$ and $A_i(k)$ are the actual readings provided by the sensors at time k; $v_{d_i}(k)$ and $v_{\alpha_i}(k)$ are measurement noises affecting the distance and the heading measurements.
The dynamic localization problem concerns computation of an estimate $\hat{p}(k)$ of the vehicle pose $p(k)$, given an initial pose estimate $\hat{p}(0)$ and model (1)-(2).

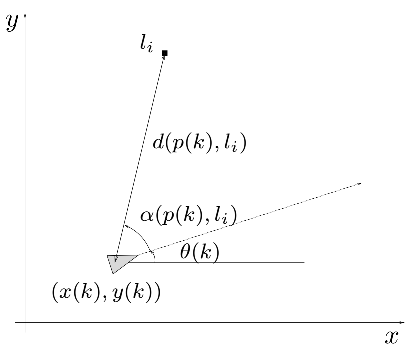


Fig. 1. Distance d and relative orientation α of robot p w.r.t. an identified landmark l_i.

This problem can be tackled in different ways, depending on the hypotheses on the unknown disturbances w, v_{d_i} and v_{α_i}. When statistical assumptions on the errors are considered, the estimate of the pose can be computed via the extended Kalman filter (Leonard and Durrant-Whyte, 1992), or using Markov localization and Bayes rules (Thrun *et al.*, 1998). However, real-world uncertainties may include also systematic errors or nongaussian, non white noise, whose statistical properties are generally very difficult to estimate. In this paper, a different approach is presented, based on the assumption that the disturbances are unknown-but-bounded, i.e.

$$|w_i(k)| \le \epsilon_i^w, \quad (3)$$

$$|v_{d_i}(k)| \le \epsilon^{v_d}, \quad |v_{\alpha_i}(k)| \le \epsilon^{v_\alpha} \quad (4)$$

where ϵ_i^w, ϵ^{v_d} and ϵ^{v} are known positive scalars. We observe that the above bounds need not be be the same for different time instants k and/or for different measurements d and α.
Let us introduce the notion of feasible state vector. Given sensor readings $D_i(k)$, $A_i(k)$, $i = 1, \ldots, m$, the feasible states are those compatible with all the measurements, i.e. the states belonging to the *measurement set*

$$\mathcal{M}(k) = \bigcap_{i=1}^{m} \mathcal{M}_i(k). \quad (5)$$

where

$$\begin{array}{rl} \mathcal{M}_i(k) = \{p : & |D_i(k) - d(p(k), l_i)| \le \epsilon^{v_d} \\ & \text{and } |A_i(k) - \alpha(p(k), l_i)| \le \epsilon^{v_\alpha}\}. \end{array} \quad (6)$$

Notice that, if assumptions (4) are verified, the set $\mathcal{M}$ is not empty. The dynamic localization problem can now be formulated in a set-theoretic framework.
SM localization problem: Let $P(0) \subset \mathcal{Q}$ be a set containing the initial pose $p(0)$. Given the model (1)-(2), find at each time $k = 1, 2, \ldots$, the *feasible pose set* $P(k|k) \subset \mathcal{Q}$ containing all vehicle poses $p(k)$ compatible with the dynamics, the assumptions (3)-(4) and the measurements collected up to time k.

For any distance measurement $D_i(k)$, the corresponding measurement set is a portion of $\mathcal{Q}$ whose shape is a cylindrical circular corona. On the other hand, relative orientation measurements $A_i(k)$ provide sets that are portions of $\mathcal{Q}$ delimited by two helicoids. The intersection of two of these sets, which corresponds to (6), is given by the roto-translation of a sector of corona, around and along the i-th landmark position axis (see Fig. 2).

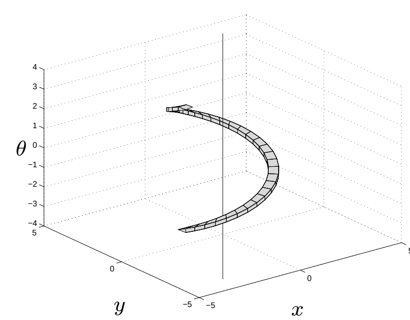


Fig. 2. Measurement set $\mathcal{M}_i$ associated to the pair of measurements (2).

The solution to the SM localization problem is given by the following set-valued recursion

$$P(0|0) = P(0), \quad (7)$$

$$P(k+1|k) = P(k|k) + u(k) + G\Delta_w \mathcal{B}_\infty, \quad (8)$$

$$P(k+1|k+1) = P(k+1|k) \cap \mathcal{M}(k+1), \quad (9)$$

where Δ_w is a diagonal matrix with elements ϵ_i^w on the diagonal and $\mathcal{B}_\infty$ is the unit ball in the ∞ norm. The main property of recursion (7)-(9) is to provide, for each k, all the pose values that are compatible with all the available information: the true pose is guaranteed to belong to sets $P(k|k)$, and the size of such sets gives a measure of the uncertainty associated to the estimate. Unfortunately, exact computation of such sets is generally a prohibitive task, because the measurement set $\mathcal{M}(k)$ is the intersection of nonlinear and nonconvex sets. Nonetheless, the problem can be tackled by applying set approximation techniques, as it will be shown in Section 3.

2.2 *Measurement matching*

The localization algorithm provided by equations (7)-(9) assumes that the observed landmarks are distinguishable, i.e., the robot is able to correctly associate each measurement to the corresponding measured feature. However, if sensors provide only metric information, all landmarks are indistinguishable, hence matching between measurements and landmarks is of paramount importance. In the set-theoretic approach, one can exploit the feasibility property to evaluate all the admissible matchings, i.e. all the associations measurement-landmark that are compatible with the assumptions on the disturbances.

In the following, subscripts xy and θ denote the projection of a set or a vector on the robot position and orientation subspaces, respectively. At time k, the predicted robot uncertainty set, before measurements are taken, is given by $P(k|k-1)$. On the other hand, measurements D_i and A_i guarantee that, in a robot-centered 2D reference system, landmark l_i lies in the set

$$\mathcal{M}_{l_i}(k) = \Big\{ l \in \mathbb{R}^2 : |D_i(k) - d(0,l)| \le \epsilon^{v_d} \\ \text{and } |A_i(k) - \alpha(0,l) + \hat{\theta}(k)| \le \epsilon^{v_\alpha} + \epsilon^{\hat{\theta}} \Big\} \quad (10)$$

where $[\hat{\theta} - \epsilon^{\hat{\theta}},\ \hat{\theta} + \epsilon^{\hat{\theta}}] = P_\theta(k|k-1)$. Notice that the current robot heading uncertainty $P_\theta(k|k-1)$ has been exploited in (10). The set in which the i-th landmark is guaranteed to lie, on the basis of the robot uncertainty and measurements taken at time k, is $P_{xy}(k|k-1) + \mathcal{M}_{l_i}(k)$. As a consequence, any landmark $l_j, j = 1, \ldots, n$ such that $l_j \in P_{xy}(k|k-1) + \mathcal{M}_{l_i}(k)$, can be associated to the i-th measurement (see Fig. 3). By repeating the process for each pair of measurements (2), it is possible to build a matching graph (see Fig. 4). Even if some measurements are ambiguous, it is often possible to find a unique admissible solution to the matching problem. Indeed, the problem of determining the existence of a perfect matching is widely studied in the operative research field, and efficient algorithms are available to determine the solution of such problems (Korte and Vygen, 2000). Uniqueness of the perfect matching can be tested by removing from the associated graph, one at a time, each solution branch, and verifying that no other solution is available. In principle, even if the solution to the matching problem is not unique, the algorithm described in (7)-(9) is still appropriate to solve the localization problem: as a matter of fact, for any possible perfect matching μ at time k, one has to

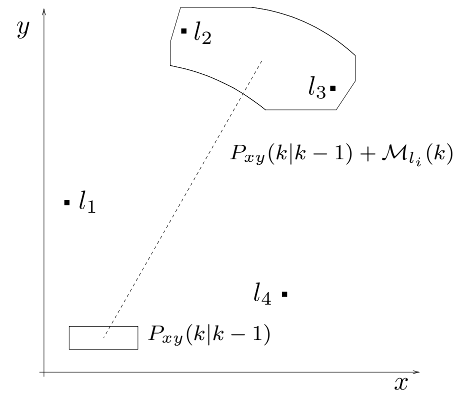


Fig. 3. Uncertainty sets can be employed to perform matching between measurements and features. Measurement i is ambiguous, since it can be associated both to landmark l_2 and to landmark l_3.

evaluate the corresponding measurement set $\mathcal{M}^\mu(k)$, defined as in (5). To account for all possible distinct choices in the matching step, one has to replace equation (9) with

$$P(k+1|k+1) = P(k+1|k) \cap [\cup_\mu \mathcal{M}^\mu(k+1)] \,,$$

where set $\cup_\mu \mathcal{M}^\mu$ replaces the single measurement set available when landmark identification is a priori known. Notice that, when performing the matching, robot uncertainty needs to be considered in order to correctly determine all the admissible perfect matchings. Once that these matchings are available, the evaluation of each measurement set $\mathcal{M}^\mu$ does not depend on the robot position uncertainty. Hence, some of the sets $\mathcal{M}^\mu$ may turn out to be empty, thus allowing to deem that matching μ is not correct.

2.3 *Simultaneous localization and mapping*

In real-world applications, landmarks position may be unknown, e.g. when the robot is used for exploration and mapping of unknown environments. This calls for the solution of the Simultaneous Localization And Map building (SLAM) problem.

When landmark locations are not known, their position can be included among the quantities that must be estimated at each time step. As a consequence,

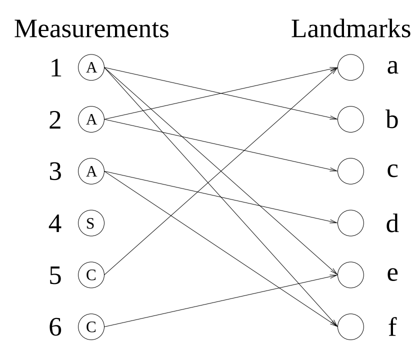


Fig. 4. Example of matching graph: three measurements are ambiguous (A), one is spurious (S), and two are compatible with only one landmark (C).

in addition to the robot motion model (1), one has to introduce also a model for the landmarks. The resulting state estimation problem involves a system whose state dimension can be very large, as it depends on the number of remarkable features present in the environment, which may vary in time. When n landmarks are considered, the state vector is given by $\xi(k) = [p'(k)\ l_1'(k) \dots l_n'(k)]' \in \mathbb{R}^{(3+2n)}$. If static landmarks are used, the state update equation is

$$\xi(k+1) = \xi(k) + E_3 u(k) + E_3 G w(k), \quad (11)$$

where $E_3 = [I_3\ 0 \dots 0]' \in \mathbb{R}^{(3+2n)\times 3}$.
Under the SM assumption on the disturbances, the measurement set is still given by (5) where

$$\mathcal{M}_i(k) = \{\xi : |D_i(k) - d(p(k), l_i)| \le \epsilon^{v_d} \\ \text{and } |A_i(k) - \alpha(p(k), l_i)| \le \epsilon^{v_\alpha}\}.$$

Then, the SLAM problem can be formulated as follows.

SM SLAM Problem: Let $\Xi(0) \subset \mathbb{R}^{(3+2n)}$ be a set containing the initial position of the vehicle and the landmarks $\xi(0)$. Given the model (11),(2), find at each time $k = 1, 2, \dots$ the set $\Xi(k|k)$ of state vectors $\xi(k)$ which are compatible with the robot dynamics, the assumptions (3)-(4) on the disturbances, and the measurements collected up to time k.

The solution to the above problem is still provided by the algorithm outlined by equations (7)-(9), where the robot feasible pose set $P(k)$ is replaced by the extended feasible state set $\Xi(k)$.
Concerning the initialization of the algorithm, a possible choice is to set $\Xi(0|0) = \mathbb{R}^{(3+2n)}$. Since all measurements are relative, one is allowed to choose an arbitrary reference system. Hence, without loss of generality, it is possible to set the origin of the reference system in the initial position of the robot, choosing as x-axis the robot initial heading.

3. SET MEMBERSHIP ALGORITHM FOR GUARANTEED LOCALIZATION

The algorithm solving the set membership pose estimation problem requires the computation of highly complex sets P or Ξ. Set approximation must be pursued, in order to devise algorithms that, by performing a tradeoff between complexity and accuracy, prove themselves suitable for real-time applications. In particular, approximating sets belonging to a class $\mathcal{R}$ of regions with simple, fixed structure, will be considered.
The structure and size of the approximating regions $\mathcal{R}$ are chosen so that recursive updating according to (7)-(9) is performed through efficient algorithms. Moreover, at each time instant k, the approximating regions $\mathcal{R}(k+1|k)$, $\mathcal{R}(k|k)$ must contain the corresponding exact sets $P(k+1|k)$, $P(k+1|k+1)$ ($\Xi(k+1|k)$, $\Xi(k+1|k+1)$, respectively) so that the true state vector $p(k)$ ($\xi(k)$) is guaranteed to belong to the approximating set.
To satisfy the above requirements, approximations are introduced at different stages of the state estimation procedure: (i) decomposition of the state vector into subsets of state variables; (ii) guaranteed approximations of the true feasible subsets through classes of simple regions.

3.1 *Set membership estimation of state subvectors*

The state vector can be decomposed into different subsets of variables: robot position $x(k), y(k)$, robot heading $\theta(k)$ and (in the SLAM case) landmark positions $l_i(k)$, $i = 1, \dots, n$.
Let Ξ_{xy} denote the feasible robot position set, Ξ_θ the feasible robot heading set, and Ξ_{l_i} denote the feasible i-th landmark position set. In addition, let $\Xi_{xy}(0)$, $\Xi_\theta(0)$ and $\Xi_{l_i}(0)$ be the corresponding initial sets. Initialization (7) splits into

$$\Xi_{xy}(0|0) = \Xi_{xy}(0), \quad (12)$$
$$\Xi_\theta(0|0) = \Xi_\theta(0), \quad (13)$$
$$\Xi_{l_i}(0|0) = \Xi_{l_i}(0). \quad (14)$$

The time update equation (8) boils down to

$$\Xi_{xy}(k+1|k) = \Xi_{xy}(k|k) + u_{xy}(k) + \\ + G_{xy}\Delta_{w_{xy}}\mathcal{B}_\infty, \quad (15)$$
$$\Xi_\theta(k+1|k) = \Xi_\theta(k|k) + u_\theta(k) + G_\theta \Delta_{w_\theta}\mathcal{B}_\infty, \quad (16)$$
$$\Xi_{l_i}(k+1|k) = \Xi_{l_i}(k|k). \quad (17)$$

The advantage of state decomposition is obtained in the measurement update step (9), since it allows for sequential update of simple 2D and 1D sets. The approximated measurement update process, described in Fig. 5, can be employed. Efficient algorithms to separately perform position and orientation refinements are available (details are reported in (Di Marco *et al.*, 2001*a*)). First, all the orientation measurements are processed in order to reduce the uncertainty on robot orientation (thus allowing for smaller uncertainty sets (10)). Then, distance and orientation measurements are processed to get a set of robot positions

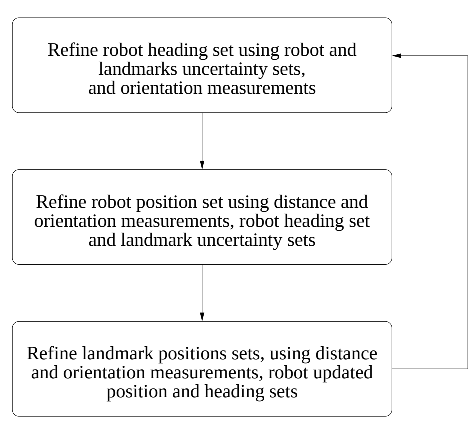


Fig. 5. Modified measurement update step, after state decomposition.

that are compatible with all the information about landmarks. During this step landmarks uncertainty is not reduced. In the third step (performed in the SLAM case) the same measurements are reprocessed to (possibly) tighten the uncertainty set of each landmark. Since the evaluation of each set is based on the previously determined approximations, a further refinement of the estimated sets can be achieved by repeating the three step measurement update. The guiding idea of this approach is similar to the Baum-Welch algorithm (Thrun *et al.*, 1998), where map and robot location are alternatively updated by maximization in the likelihood space. Nevertheless, in this case, no probabilistic meaning is attributed to the measurements: this allows one to reprocess the same measurements several times, to iteratively reduce the size of the approximated uncertainty sets.

The proposed approach allows one to simplify the measurement update process, but it also introduces an approximation because information about correlation between robot and landmark position is lost.

We point out that the approximated measurement update described in this section is more conservative than the exact set membership algorithm (7)-(9). On the other hand, notice that the set $\Xi_{xy}(k|k) \otimes \Xi_\theta(k|k) \otimes \Xi_{l_1}(k|k) \otimes \ldots \otimes \Xi_{l_n}(k|k)$, where $\otimes$ denotes the Cartesian product, is guaranteed to include the true feasible set $\Xi(k|k)$.

3.2 *Set approximation*

Since we are interested in fast (on-line) algorithms, which are able to provide regions guaranteed to contain the robot pose (and possibly landmark positions), outer approximation of the feasible sets are looked for. The approximating sets belong to classes of simple structure sets. The choice of the specific element in the approximating class is performed by selecting minimum area sets in the chosen class containing $P(k+1|k)$, $P(k+1|k+1)$ (the same holds for sets Ξ). Let us consider a class of regions $\mathcal{R}$ of fixed structure, and let us denote by $\overline{\mathcal{R}}\{\mathcal{Z}\}$ the minimum area set in the class $\mathcal{R}$ containing the set $\mathcal{Z}$. For the general set membership localization procedure described by (7)-(9), the desired outer approximation is given by the following recursion

$$\mathcal{R}(0|0) = \overline{\mathcal{R}}\{P(0)\} \tag{18}$$

$$\mathcal{R}(k+1|k) = \overline{\mathcal{R}}\{\mathcal{R}(k|k) + u(k) + G\Delta_w \mathcal{B}_\infty\} \tag{19}$$

$$\mathcal{R}(k+1|k+1) = \overline{\mathcal{R}}\{\mathcal{R}(k+1|k) \cap \mathcal{M}(k+1)\}. \tag{20}$$

Notice that, with the introduction of the approximating class, all the sets operations in (18)-(20) are now performed on sets of known and simple structure. The proposed approximation algorithm can be easily extended to the SLAM case, including the decomposition in subvectors proposed in Sect. 3.1. In this case, the desired approximation is obtained computing the outer set $\overline{\mathcal{R}}$ for each set provided by equations (12)-(17).

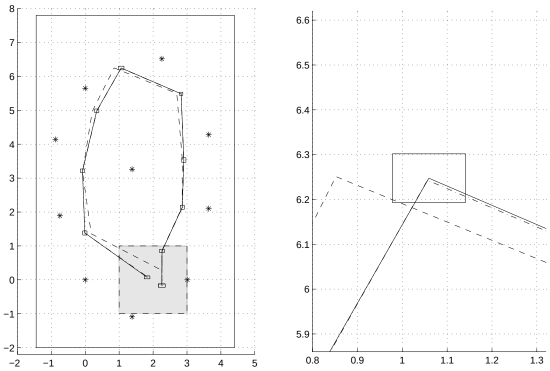
The choice of the class $\mathcal{R}$ of approximating regions is performed taking into account the trade-off between accuracy and computational complexity. Common choices found in literature are ellipsoids (Hanebeck and Schmidt, 1996), boxes and parallelotopes (Di Marco *et al.*, 2001*b*). Fast algorithms, performing set sum and outer approximation required by (19), are available (Garulli and Vicino, 2001). Moreover, algorithms providing suitable approximations of measurement sets have been developed for distance and angle measurements (Di Marco *et al.*, 2001*b*). These results allow one to devise computationally efficient algorithms providing guaranteed set-valued solutions to the localization problem.

3.3 *Computational complexity*

With respect to the number of landmarks, the proposed SM localization procedures perform a fixed number of operations for each measurement pair, i.e. for each of the m landmarks detected and correctly associated to the corresponding measurements, at time k. Since $m \leq n$ (the total number of landmarks), the complexity at each time step is at most $O(n)$. Notice that, thanks to state decomposition, the basic tasks of the measurement update step are approximations of 2D regions via simple sets like boxes or parallelotopes (a typical example being the computation of the minimum area box containing the intersection between a box and a sector of corona), which can be done very efficiently evaluating suitable functions on a finite number of points.

4. EXPERIMENTAL RESULTS

In order to test the proposed algorithm in a real-world setting, several experiments with the mobile robot Nomad XR4000, equipped with a SICK LMS 200 laser rangefinder, have been carried out. In a first set of experiments, the robot was programmed to follow a nominal path in a room with 10 artificial landmarks (whose positions were available to the robot) using only information provided by the encoders. Periodically, laser scans of the environment were taken in order to estimate the vehicle pose. Boxes have been used as approximating sets. To allow for landmark matching, a bounded region for the initial pose was used, assuming $\mathcal{R}(0)$ equal to a 3D box of sides $1m$, $1m$ and $10°$, respectively. The odometry error bound was set to $\epsilon^w = [0.25m\ 0.25m\ 5°]'$, while $\epsilon^{v_d} = 0.05\ m$ was used as distance measurement error bound (from the rangefinder data sheet). Bounds on the angular measurement errors were obtained directly as a byproduct of landmark extraction phase. Fig. 6 shows the result of a typical experiment. As expected, odometry information (dashed line) rapidly drifts away from the real trajectory (solid line). The proposed algorithm provides an efficient method for



a b

Fig. 6. Localization experimental results with 10 landmarks (*): (a) nominal trajectory (dashed line), true trajectory (solid line), estimated trajectory (dash-dotted line); (b) magnified view of the top part of the path, with uncertainty position region (solid box).

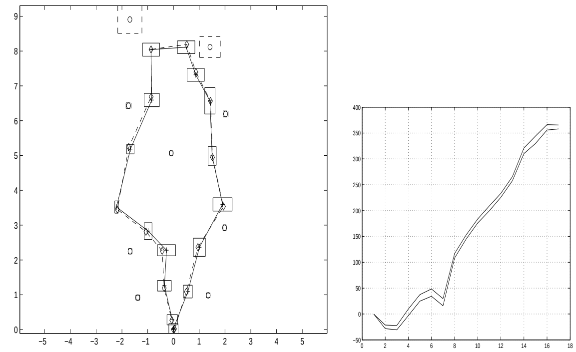


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Fig. 7. SLAM experimental results with 9 landmarks (○): (a) true trajectory (dashed line), estimated trajectory (solid line); landmark uncertainty regions (dashed boxes); (b) Robot heading uncertainty bounds.

an accurate estimation of the vehicle pose. In practice, the estimated trajectory (dash-dotted line) is not distinguishable from the real one (the average position and heading errors are less than 1 cm and $0.2°$). Moreover, the associated uncertainty regions are very small, with average values of $150\ cm^2$ and $3.1°$ for the position and heading, respectively. Notice that the nominal trajectory is often outside the estimated feasible set (see Fig. 6b). Due to the sensor used to extract environment information, only a halfplane of the whole surrounding environment was visible at a given time. This means that at each measurement step only a subset of the landmarks was detected. Nevertheless, the proposed matching strategy allowed to correctly associate every landmark to the corresponding measurements, so that all the detected landmarks could be used in the localization procedure.

Also the SLAM algorithm was tested in a similar scenario, where landmark positions were unknown. The result of a typical experiment is reported in Fig. 7; in this case, the average robot uncertainty regions are 0.18 m^2 (position) and 9° (heading). Landmark uncertainty strongly depends on the pose uncertainty affecting the vehicle at each time instant the landmark is detected.

Concerning the computational burden, a non-optimized Matlab code, performing a typical localization step in presence of 7 landmarks, took about $0.1\,s$ on a $1.3\,GHz$ Athlon processor, thus confirming the suitability of this technique for real-time applications.

5. CONCLUSIONS

The set-theoretic approach presented in the paper is able to cope with both localization based on a given map and the much harder SLAM problem. The use of indistinguishable landmarks is allowed, because the robot can exploit the SM uncertainty representation to associate each collected measurement to the corresponding landmark. The use of set approximation techniques exploiting the specific geometry of the involved sets, lead to limited complexity algorithms which can be employed in real-time experiments, like those presented in Section 4.

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