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NUMERICAL OBSERVABILITY IN DYNAMICAL SYSTEMS: AN APPROACH BY INTERVAL NUMBERS

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Abstract: This work presents another concept of observability in nonlinear systems, when the states of these types of models are described using a representation by interval numbers. This approach is an extension of classical concepts of nonlinear observability, called ε-observability and is based on a notion of the neighborhood of a state with a given precision of epsilon. Our proposal includes the notion of an indistinguishable trajectory by intervals applied to the IMHSE method, which is the basic concept used to construct the notions of observability that will be presented. *Copyright © 2002 IFAC*

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1. INTRODUCTION

Classical observability properties are described using the notion of undistinguished states in $\mathbb{R}^n$. This type of proposal can be found for example in (Hermann and Krener, 1977). In this reference, a notion of observability of nonlinear systems is given as the possibility of reconstructing the value of the state *x* to the instants t_0, starting from simple knowledge of output evolutions and control law over an interval of time $[t_0, t_0+T[\subset [t_0, t_0+T_{\max}[$.

Frequently, real systems are represented in the following form by nonlinear models:

$$\sum(.):\begin{cases}\dot{x} = f(x,u)\\ y = h(x)\end{cases} \tag{1}$$

Where $x \in \mathbb{R}^n$ is the vector of states, $u \in \mathbb{R}^n$ is the vector of input (or the external applied control), $y \in \mathbb{R}^p$ is the vector of output (measurable parameters of the systems), and *f* and *h* are known nonlinear functions. In other words:

$$\Sigma(.):\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p \tag{2}$$
$$\sum\nolimits_u^y(t,x_0):\{x_0, u(t), t\in[t_0 \quad t_1]\} \to \{y(x_0 \mid t)\}$$

Where $y(x_0 \mid t)$ is the trajectory of the system, starting from an initial condition x_0 to time *t*. When interval arithmetic is used to describe $\Sigma(.)$, (2) is replaced by (3), i.e. $\mathrm{I}_{\sum}(.):\mathbb{IR}^n \times \mathbb{IR}^m \to \mathbb{IR}^p$

$$\mathrm{I}_{\sum}{}_u^y(t,X_0):\{X_0, u(t), t\in[t_0 \quad t_1]\} \to \{Y(X_0 \mid t)\} \tag{3}$$

$\mathrm{I}_{\sum}{}_u^y(t,X_0)$: is the output trajectory of the systems by interval numbers, starting from a initial condition x_0 to time t, ($t \in [t_0, t_0+T_{\max}[$), under the control law *u*.

An interval $\mathrm{I} = [a, b]$ is a closed, bounded and connected set of real numbers. Let a box $B \subset \mathbb{IR}^n$ be a Cartesian product of *n* intervals. The set of all boxes of $\mathbb{R}^n$ is denoted by $\mathbb{IR}^n$ (interval real numbers). The width of an interval vector B is defined as $w(B) = \max_{i=1,\dots,n}\{b_i - a_i\}$. See (Hansen, 1992) for a description of this topic.

Remark 1. The ideas presented are general. They have, however, been proved from a numerical standpoint using interval arithmetic.

A graphical form of (2) and (3), can be found in figure 1.

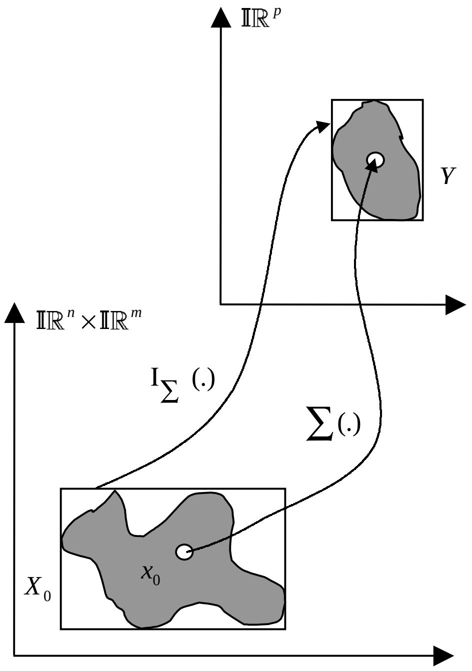

Fig. 1. Application of $\Sigma(.)$ and $I_\Sigma(.)$ over an initial condition x_0 and X_0

This paper is organized as follows: the introduction is followed by a presentation of the ε-observability concepts. Section 3 presents the observability property applied to observable and non-observable systems. An observability index by interval numbers is proposed in section 4. The IMHSE method is then briefly described in section 5. Finally, results from the application of ε-observability to the result of estimation by intervals of state variables are presented and discussed.

2. NONLINEAR ε-OBSERVABILITY AND ε-INDISTINGUISHABILITY

In this section the system $I_\Sigma(.)$ is described as in the last section. One of the important characteristics in systems using interval numbers is the notion of indistinguishable states, i.e. states described by interval numbers that give an interval indistinguishable output (with a precision that can be arbitrarily small).

A notion based on the distance of the output trajectories, for any precision of cuts of the space of states, seems more suitable to us (considering the possibility that the computer effort to distinguish between two points is substantial).

Proposal I. (ε-indistinguishability). Let ***V*** be any subset of $\mathbb{IR}^n$, as in figure 2. ***V*** is the set of all states (interval vectors) cut with a precision of epsilon (for simplicity and without loss of generality, the cuts will be considered regular over the admissible domain).

1) $x_1, x_2 \subset V$, $V \subset \mathbb{IR}^n$
2) $Max\{w(V)\} = Max\{w(x_V)\} \le \varepsilon$

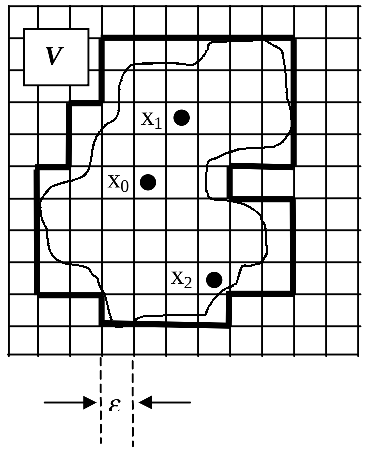

Fig. 2. Indistinguishable states in ***V***.

A pair of points x_1 and x_2 are indistinguishable to a precision of ε, if the output trajectory of the system (starting from these points) is bounded by α, when the external control *u* is applied to the systems. In other words when x_1 and x_2 produce the same indistinguishable output by intervals for every admissible input.

$$I_\varepsilon(x_1, x_2) \Leftrightarrow \forall \varepsilon \ge 0, \exists \alpha \ge 0, (\alpha, \varepsilon) \in \mathbb{R},\quad (x_1, x_2) \in \mathbb{R}^n \,/\, \left\| I_{\Sigma_u}^{y}(t, x_1) - I_{\Sigma_u}^{y}(t, x_2) \right\| < \alpha \tag{4}$$

Proposal 1 is shown in graphical form in figure 3.

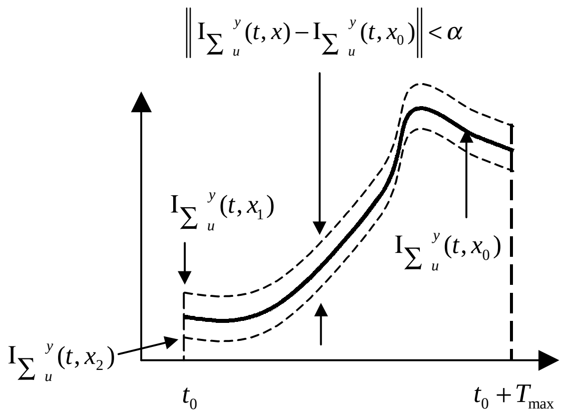

Figure 3.α-neighborhood over the system output.

Proposal II. (ε-Observability). $I_\Sigma(.)$ is observable, if the following proposition is true.

If $\forall x_0 \in \mathbb{R}^n, \forall \delta > \varepsilon, \exists\, \alpha \geq 0, (\varepsilon, \delta, \alpha) \in \mathbb{R}$ such that:

$$\left\| \mathrm{I}\textstyle\sum_u^y(t,x) - \mathrm{I}\textstyle\sum_u^y(t,x_0) \right\| < \alpha \Rightarrow \|x - x_0\| \leq \delta \qquad (5)$$

Remark 2. A system is 0-observable (classic definition in $\mathbb{R}^n$, see (Hermann and Krener, 1977) for a fuller discussion of this point) when:

$$\left\| \mathrm{I}\textstyle\sum_u^y(t,x) - \mathrm{I}\textstyle\sum_u^y(t,x_0) \right\| \equiv 0 \Rightarrow x \equiv x_0$$

Proposal 2 is shown graphically in the following figures.

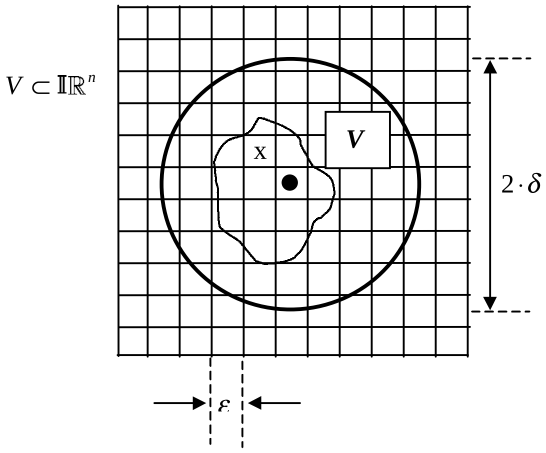

Fig. 4.: δ-neighborhood of *x* over the admissible domain.

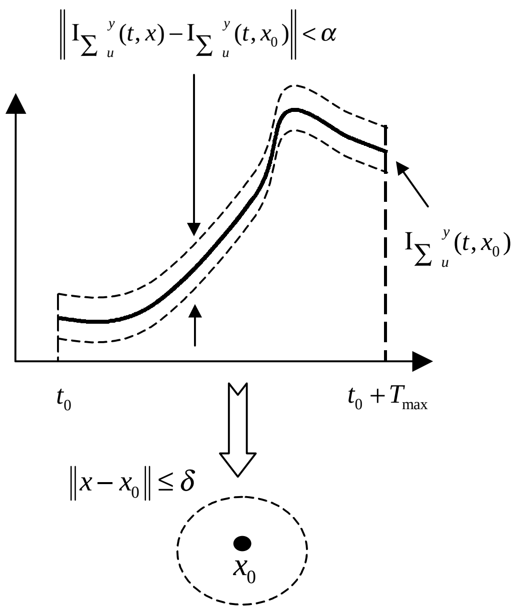

Figure 5. A α-bounded output trajectory implies a δ-neighborhood of *x* over the admissible domain.

Remark 3. Notice that in some cases it is necessary to work for a long time, i.e., a very significant computer effort may be necessary to distinguish the points over the admissible domain. When this is the case, these points are "weakly distinct points." In theory, weakly distinct points exist; from a practical point of view by interval numbers, these types of states can be considered as indistinguishable points for a given precision of epsilon.

3. REDUCTION OF THE ε PRECISION AROUND THE REFERENCE

1.1 Observable systems

The observability property implies that if ε becomes smaller and smaller, the output trajectory contracts towards a smaller α-bounded trajectory that is contained within itself. However, also the neighborhood around x_0 contracts towards a smaller δ-bounded neighborhood contained within itself, such that δ≥ε. In other words:

If $\varepsilon_1 \gg \varepsilon_2 \Rightarrow \alpha_1 \gg \alpha_2 \;\wedge\; \delta_1 \gg \delta_2$. This can be represented graphically as:

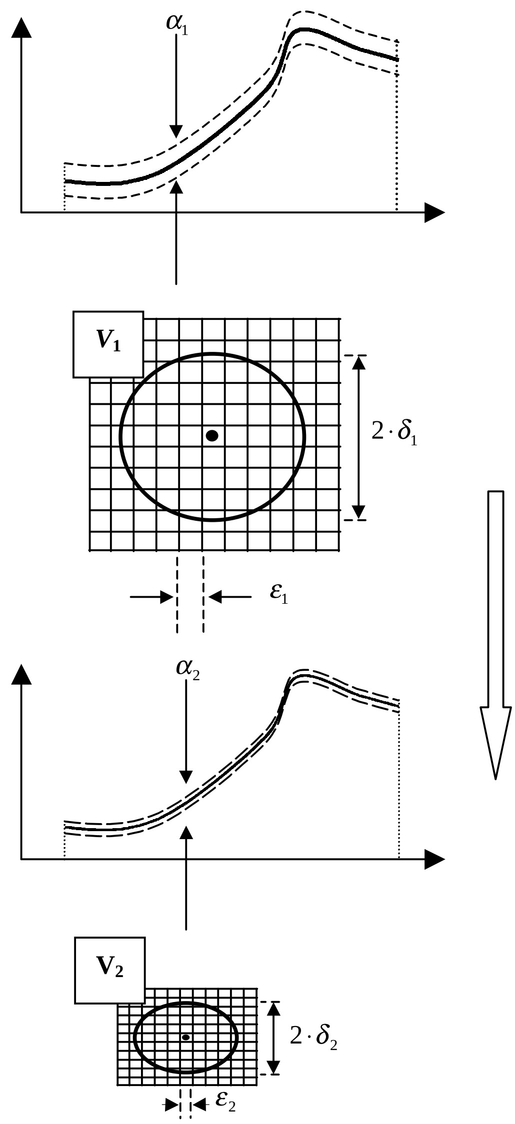

Fig. 6. ε-observability property over an observable system.

1.2 Non-observable systems: Case 1

This is a system with two indistinguishable subsets. Obviously the represented system is a non-observable system. The reason is because there are two different subsets (in the case of real numbers, there are two points) over the admissible domain that give the same bounded output trajectory (as described in section II). Notice that the width of the set V_2 is similar to the width of the set V_1 even if the ε-precision is reduced drastically. In other words, if $\varepsilon_1 \gg \varepsilon_2 \Rightarrow \alpha_1 \gg \alpha_2 \;\wedge\; \delta_1 \approx \delta_2$.

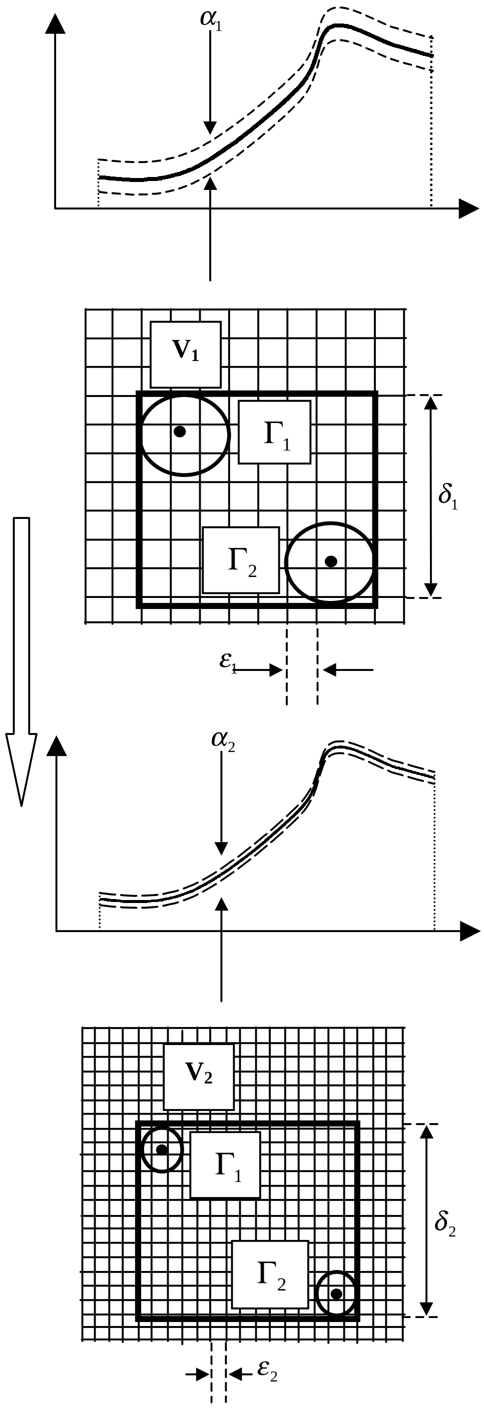

Fig. 7. ε-observability property over a non-observable system with two solutions (subsets).

1.3 Non-observable systems: Case 2

This case represents a non-observable system with infinite solutions, i.e., a system with a subset in which many different indistinguishable states are contained. As in section 1.2, the width of the set V_2 is similar to the width of the set V_1 even if the ε-precision is reduced drastically. Here also:
if $\varepsilon_1 \gg \varepsilon_2 \Rightarrow \alpha_1 \gg \alpha_2 \;\wedge\; \delta_1 \approx \delta_2$.

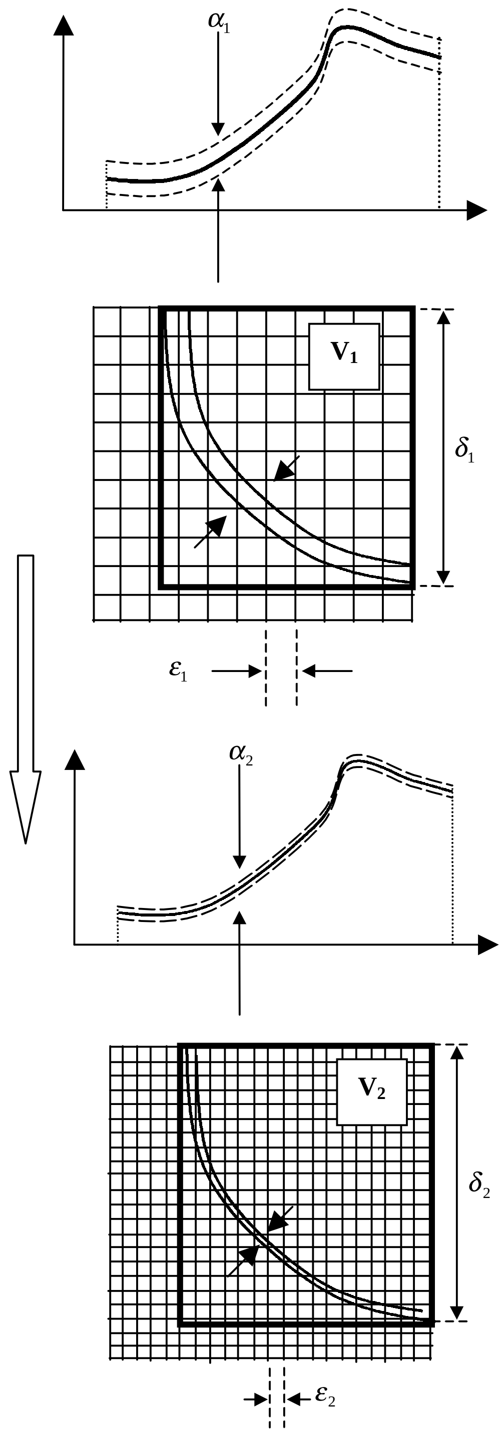

Fig. 8. ε-observability property over a non-observable system with a set that encloses many different states as solutions.

4. OBSERVABILITY INDEX

In an observable system, people also want to know if there is sufficient amplitude of the link between states to permit estimation of these states by interval numbers. Our aim is to propose an index number called *Numerical observability index* denoted by I_{NO}. The main characteristic of this index is to know the width of the neighborhood of states in the subset ***V***. In other words, this work is concerned with an index that is the ratio computed for the level δ (of each states variable) obtained for a given margin of α on the output trajectory over an admissible domain. In other words:

If $\forall x_0 \in \mathbb{IR}^n,\ x_i \in V(.), \forall \delta_i > \varepsilon,\ \exists\ \alpha_i \ge 0,\ (\varepsilon, \delta_i, \alpha_i) \in \mathbb{R}$ (i:1→n) such that:

$$\| y(x_i / t) - J_{\Omega}(x_o / t) \| < \alpha \Rightarrow \|x_0 - x_i\| < \delta_i \qquad (6)$$

In order to clarify (6), a 2D subset will be used as an example: this is shown in the next figure:

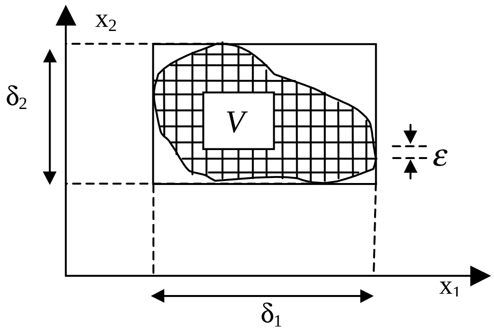

Fig. 9. Subset $V \subset \mathbb{IR}^2$, where ***V*** is the set of indistinguishable points by interval numbers of $I_\Sigma(.)$.

In this context, the following proposals can be given:

Proposal III. (Numerical observability index 1). Is defined by each state variable considering (6), as an index that is the ratio computed for the level δ_i obtained for a given margin of α on the output trajectory.

$$\pi_i = \frac{\alpha}{\max(\delta_i)} = \frac{\alpha}{w(x)} \qquad (7)$$

Proposal IV. (Amplitude factor). Is defined by each state variable considering (6), as the ratio computed between the amplitude and the width of the admissible domain for this state variable.

The normalized amplitude factor is given by:

$$\psi = \frac{\delta_i}{w(\Omega_i)} \qquad (8)$$

Proposal V. (Occupation factor). Is defined as a volumetric index for the subset of indistinguishable states, that is:

$$\Gamma = \frac{\prod_{i=1}^{n} \delta_i}{\delta_{\max}^n} \qquad (9)$$

Proposal VI. (Numerical observability index 2). Is defined as a compromised index between the amplitude factor and the occupation factor.

$$I_{NO} = \Gamma \cdot \prod_{i}^{n} (1 - \psi) \qquad (10)$$

In other words:

$$I_{NO} = \frac{\prod_{i=1}^{n} \delta_i}{\delta_{\max}^n} \cdot \prod_{i=1}^{n} \left(1 - \frac{\delta_i}{w(\Omega_i)} \right) \qquad (11)$$

The following properties (from (11)) can be demonstrated easily using interval arithmetic:

a) I_{NO} is bounded by [0 1], i.e., $0 \le I_{NO} \le 1$.

b) If $I_{NO} = 0$ then the control law (*u(t)* in (1)) is a singular input.

c) If the system is observable, the I_{NO} converge to the maximal possible value, i.e., $I_{NO} = 1$.

5. STATE ESTIMATION BY INTERVALS

IMHSE converts the problem of state estimation from a dynamic system into a static problem of nonlinear optimization. The main purpose of the technique is to find the value of a vector of states at the start time *sh*. The goal is to minimize the difference between the simulated output of the system, and the measured output, which corresponds to the experimental measurements. This is then converted into the minimization of a nonlinear function as (12) over the time horizon. IMHSE can be formulated as a nonlinear programming problem with the following structure:

$$\text{Minimize (globally)} \quad J(x_{sh}) = \frac{1}{2} \sum_{j=sh}^{sh+lh-1} v_j^T \cdot W \cdot v_j \qquad (12)$$

Subject to:

$$I\Sigma(.): \begin{cases} [\dot{x}] = f([x],[u]) \\ [y] = h([x]) \end{cases} \qquad (13)$$
$$x \in \mathbb{IR}^n$$

Where (12) corresponds to the objective function, (13) is the interval nonlinear model described by (1) over an admissible domain in $\mathbb{IR}^n$. In (12) *v* corresponds to an equation of remainders and is defined as $v_j = y_j - \hat{y}_j$ and *lh* is the horizon length.

Remark 4: A complete description of the IMHSE globally convergent method, the involved global optimization technique and the interval algorithms used are explored and can be found in (Valdés-González and Flaus, 2001).

5.1 Observability by intervals in IMHSE

It can be seen that the system (1) is observable over the prescribed horizon, if and only if the solution to the nonlinear optimization problem (12) at the beginning of the defined time horizon exists and is, in addition unique, according to the classical definition of observability. In other words, different states give different output trajectories. However, with IMHSE if more than one state is found as a result of the global optimization problem at the beginning of the horizon (there are many different indistinguishable states that generate the same bounded trajectory of output as described in section 2), the system will not be observable. Theoretically, it is possible to show that for the IMHSE method, global observability exists over an admissible domain (Ω), if the injection property for $\mathrm{I}_{\Sigma_u}^{y}(t,x)$ is true for each step, i.e., when we shift the horizon in one step, see (Valdés-González and Flaus, 2001).

5.2 Application to a generic fermentation process over a finite horizon

The bioprocess considered works in batch mode. The dynamic nonlinear model describes this kind of bioprocess as follows:

$$\begin{aligned} \dot{X} &= \mu(S)\cdot X \\ \dot{S} &= -\frac{\mu(S)}{\gamma}\cdot X \end{aligned} \qquad (14)$$

Where *X* represents the concentration of biomass in the reactor (g/l) and *S* represents the concentration of the substrate (g/l). The term *μ(s)* is called the specific growth rate of the biomass, is expressed in (1/h), and is modeled by the following hybrid mixed formulation:

IF $S < 2.5$

$$\mu(S) = 0.064\cdot 10^{-3}\cdot S$$

ELSE (15)

$$\mu(S) = 0.16$$

ENDIF

The output model in the case of aerobic processes is combined with the respiration of micro-organisms, and the gaseous balance calculated allows the rate of respiration to be evaluated, which is typically of the form (where a is generally greater than b):

$$y = \left(a\cdot\mu(S)+b\right)\cdot X \qquad (16)$$

The goal is to estimate the concentrations of biomass and substrate from the output measurements of the process (simulation). In this study, the process is considered under the following initial condition for the state and the parameters: x(0)=0.15 (g/l), s(0)=4.9 (g/l), a=0.01, b=1 and γ=0.8. The solution space Ω in the global optimization problem for X and S for this test is $X = [0 \quad 4.5]$, $S = [0 \quad 5.5]$ and $\Omega = X\times S$.

Remark 5: In this work, the method used assumes the non-linear model is known exactly, i.e., it does not consider model uncertainty. However, arithmetic by intervals enables us to define (if required) the parameters as interval numbers, which will give another type of representation of the uncertainties of the real model.

Application of IMHSE over the systems described by (14)-(16) shows that the model involved is a non-observable system. This is due to the fact that there are many different indistinguishable states, and is ratified graphically (when looking at the entire admissible domain) for the optimization tolerance prescribed (ε=0.01), see figure 10. In this case $\psi_X = 1.78\cdot 10^{-3}$, $\psi_S = 1$, $\Gamma = 1.45\cdot 10^{-3}$ and the index I_{NO} is $I_{NO} = 0$

General conditions and other interval estimation results for this type of methodology and model can be found in (Valdés-González and Flaus, 2001).

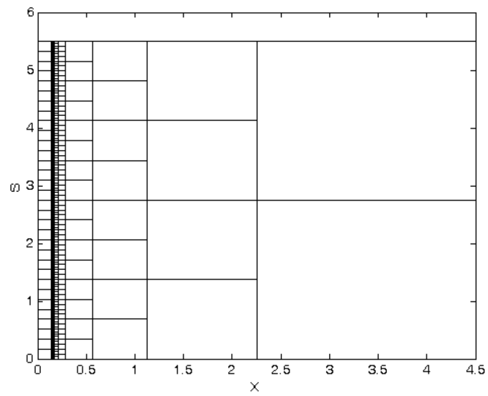

Fig. 10. Indistinguishable states determined when the beginning of the horizon is t=0 hrs and Bε=0.01.

6. CONCLUSIONS

Notions of ε-observability by interval numbers combined with the IMHSE method in nonlinear systems have been presented. This approach is assumes/takes the form of a subset that contains the global solution for a determinate system by interval numbers. An index combining a level δ_i obtained (over a subset of neighborhood) for a given margin of α on the output trajectory is also presented.

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