ESTIMATION OF AN N-L-N HAMMERSTEIN-WIENER MODEL

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Abstract: Estimation of a single-input single-output block-oriented model is studied. The model consists of a linear block embedded between two static nonlinear gains. Hence it is called N-L-N Hammerstein-Wiener model. First the model structure is motivated and the disturbance model is discussed. The paper then concentrates on parameter estimation. A relaxation iteration scheme is proposed by making use of a model structure in which the error is bilinear-in-parameters. This leads to a simple algorithm which minimizes the original loss function. The convergence and consistency of the algorithm are studied. In order to reduce the variance error, the obtained linear model is further reduced using frequency weighted model reduction. Simulation study will be used to illustrate the method.

Keywords: Identification, nonlinear process, block-oriented model, parameter estimation, relaxation algorithm

1. INTRODUCTION

Commonly used block-oriented models are Hammerstein model, Wiener model and their combinations. A Wiener model is a nonlinear model with a linear dynamic block followed by a static nonlinear function and a Hammerstein model has nonlinear block followed by a linear dynamic block. Many researchers have studied the parametric identification of Hammerstein model; see Narendra and Gallman (1966), Hsia (1977), and Stoica and Söderström (1982). Some work has been done on the identification of Wiener models; see, e.g., Billings and Fakhouri (1982), Greblicki (1992) and Verhaegen (1998). Wigren (1993) proposed a recursive algorithm. In the literature, the extension of the Hammerstein and Wiener models is often in the form of an L-N-L model where a nonlinear block is embedded between two linear blocks and it is called a Wiener-Hammerstein model; see Billings (1980), Billings and Fakhouri (1982), and

Boland and Doyle (1982). Often correlation analysis are used and they belong to nonparametric model identification. A recursive estimation of a parametric Wiener-Hammerstein model is proposed in Boutayeb and Darouach (1995).

In this work, another combination of Hammerstein and Wiener models is proposed. The model consists of a linear block embedded between two static nonlinear gains. Hence it is called N-L-N Hammerstein-Wiener model. Very little has been done in the study of this model structure. Falkner (1988) has proposed an iterative scheme to identify nonparametric N-L-N models. Chernyshov (2000) has proposed an output error method for the identification of nonparametric N-L-N models. To our best knowledge, the identification of parametric N-L-N models has not been studied in control literature. In Section 2 we will parametrize the model. Section 3 will treat parameter estimation, convergence and consistency analysis. Simulation studies

are performed in Section 4. Section 5 contains the conclusions.

2. MODEL PARAMETRIZATION

The proposed N-L-N Hammerstein-Wiener model is shown in Figure 1.1. In a process control environment, the N-L-N model can be motivated by considering the input nonlinear block $f_1(u)$ as actuator nonlinearity and the output nonlinear block $f_2(w)$ as process nonlinearity. Mathematically, because the N-L-N model includes both Hammerstein model and Wiener model as its special cases, it will approximate nonlinear systems better than either of the two models.
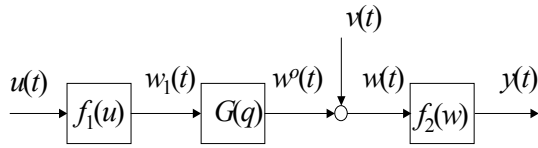

Figure 1.1 N-L-N Hammerstein-Wiener model

The disturbance term $v(t)$ is placed before the output nonlinear block, which is different from the normal assumption that a stationary disturbance acts at the process output. This disturbance model implies *nonlinear* output disturbance: the output disturbance is large when the process gain is large (process is sensitive) and is small when the gain is small (process is insensitive). This assumption is more realistic from a process operation point of view. However, the measurement noise can not be modelled properly in this way. In practice, the influence of (unmeasured) process disturbance is, in general, much greater than that of the measurement noise due to the advances in sensor technologies. The identification test is done in closed-loop and the closed-loop system is assumed to be stable; see Figure 4.2. During the test, a test signal is applied at the setpoint or at the input. It should be clear that an open loop test is a special case of the closed-loop test.
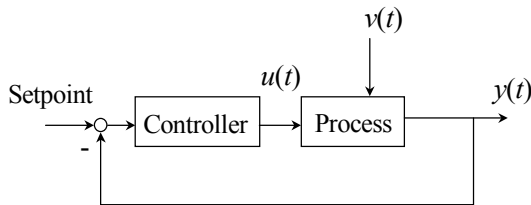

Figure 1.2 Closed-loop system

The equations that describe the N-L-N model in Figure 1.1 are

$$w_1(t) = f_1(u(t))$$
$$w(t) = G(q)w_1(t) + v(t)$$
$$y(t) = f_2(w(t)) = f_2[G(q)f_1(u(t)) + v(t)] \quad (1)$$

It is assumed that: 1) the nonlinear functions $f_1(u)$ and $f_2(w)$ are continuous; 2) the output nonlinear function $f_2(w)$ is monotone and invertible; 3) the output of the linear part is disturbed by a stationary stochastic process $\{v(t)\}$ with zero mean.

The nonlinear functions will be parametrized (approximated) using cubic splines (Lancaster and Šalkauskas, 1986) and the linear block with the disturbance will be parametrized using a Box-Jenkins model. Denote a set of knots $\{u_1, u_2, ..., u_{m_1}\}$ for $u(t)$ which are real numbers and satisfy

$$u_1 = u_{\min} < u_2 < ... < u_{m_1} = u_{\max}$$

A cubic spline function for $f_1(u)$ is given as

$$w_1(u) = f_1(u)$$
$$= \sum_{k=2}^{m_1-1} \beta_k |u - u_k|^3 + \beta_{m_1} + \beta_{m_1+1}u \quad (2)$$

where $[\beta_2, \beta_3, ..., \beta_{m_1+1}]$ are real numbers, namely, the parameters to be estimated. Here m_1 is called the number of knots which can be seen as the "degree" or "order" of the cubic splines. Note that the number of parameters of the cubic splines is also m_1. It is easy to verify that the first and the second derivatives of the function are continuous and hence the function is smooth. Similarly, denote a set of knots $\{w_1, w_2, ..., w_{m_1}\}$ for $w(t)$, a cubic spline function for $f_2(w)$ is given as

$$y(w) = f_2(w) = \sum_{k=2}^{m-1} \alpha_k |w - w_k|^3 + \alpha_m + \alpha_{m+1}w$$
$$(3)$$

Note that the cubic splines are used to approximate the nonlinearities. The true nonlinear functions need not to be cubic splines.

A Box-Jenkins model for the linear part is

$$w(t) = G(q)w_1(t) + H(q)e(t) \quad (4)$$

with

$$G(q) = \frac{b_1 q^{-1} + ... + b_n q^{-l}}{1 + a_1 q^{-1} + ... + a_n q^{-l}} = \frac{B(q)}{A(q)}$$
$$H(q) = \frac{1 + c_1 q^{-1} + ... + c_n q^{-l}}{1 + d_1 q^{-1} + ... + d_n q^{-l}} = \frac{C(q)}{D(q)}$$

where q^{-1} is the unit delay operator, l is the order of the model and $e(t)$ is white noise with zero mean and variance R.

The intermediate signals $w_1(t)$ and $w(t)$ can not be measured, hence two arbitrary gains may be distributed between the linear block and the two nonlinear blocks. This means that the parameters of the model cannot be uniquely determined without introducing further constraints. A solution to this problem is to fix some coefficients of the nonlinear blocks. For example, one can let $\alpha_{m+1} = 2$ and $\beta_{m+1} = 1$.

3. PARAMETER ESTIMATION

Assume that an identification test has been performed, possibly in a closed-loop operation. Denote the input-output data as

$$Z^N = [u(1),\, y(1),\, u(2),\, y(2),\, ...\,,\, u(N),\, y(N)] \tag{5}$$

For given orders l, m and m_1, determine the parameters of the model (1)–(4) using the input-output data from the test by minimizing the loss function

$$V = \frac{1}{N} \sum_{t=1}^{N} \varepsilon^2(t) \tag{6}$$

where

$$\varepsilon(t) = H^{-1}(q)[f_2^{-1}(y(t)) - G(q)f_1(u(t))]$$
$$= \frac{D(q)}{C(q)} [f_2^{-1}(y(t)) - \frac{B(q)}{A(q)} f_1(u(t))] \tag{7}$$

The error defined in (7) is highly nonlinear in model parameters. Direct minimization of the loss function (6) is difficult and may run into numerical problems. It desirable to reduce this complexity by looking for some simpler numerical schemes. It is well know that any linear prediction error model structure can be approximated arbitrarily well by an ARX, or, equation error model with sufficiently high order; see, e.g., Ljung (1987). Based on this fact, approximate the linear part with Box-Jenkins structure in (4) by a high order ARX model

$$w(t) = \frac{B^n(q)}{A^n(q)} w_1(t) + \frac{1}{A^n(q)} e(t)$$
$$= \frac{B^n(q)}{A^n(q)} f_1(u(t)) + \frac{1}{A^n(q)} e(t) \tag{8}$$

where n is the order of the ARX model ($n > l$) and $\{e(t)\}$ is white noise with zero mean and variance R.

Substitute $w(t)$ in (1) using the ARX model, take the inverse of the nonlinear function $f_2(w)$ on both sides and then multiply them by $A^n(q)$. The following equation yields

$$A^n(q)f_2^{-1}(y(t)) = B^n(q)f_1(u(t)) + e(t) \tag{9}$$

where $A^n(q)$ and $B^n(q)$ are polynomials of q^{-1}. Now, parametrize the inverse $f_2^{-1}(y)$ by a cubic spline

$$f_2^{-1}(y) = \sum_{k=2}^{m_2-1} \gamma_k |y - y_k|^3 + \gamma_{m_2} + \gamma_{m_2+1} y \tag{10}$$

where m_2 is the number of knots. For unique representation, we assume that

$$\gamma_{m_2+1} = 1, \;\; \beta_{m_1+1} = 1 \tag{11}$$

Then, the loss function for parameter estimation for model (8) becomes

$$V_{ARX}^N(\theta, Z^N) = \frac{1}{N} \sum_{t=1}^{N} \varepsilon^2(t, \theta) \tag{12}$$

where

$$\varepsilon(t, \theta) = A^n(q)f_2^{-1}(y(t)) - B^n(q)f_1(u(t)) \tag{13}$$

Denote the parameter vector of model (9).as

$$\theta = [a_1,\, ..,\, a_n,\, b_0,\, ..,\, b_n,\, \gamma_2,\, ..,\, \gamma_{m_2+1},\, \beta_2,\, ..,\, \beta_{m_1+1}]^T \tag{14}$$

This vector varies over a set D_M which is a compact subset of R^{n_θ} where n_θ is the number of parameters.

Writing out the nonlinear functions in equation (13) yields

$$\varepsilon(t, \theta) = A^n(q) \left[\sum_{k=2}^{m_2-1} \gamma_k |y(t) - y_k|^3 + \gamma_{m_2} + y(t) \right]$$
$$- B^n(q) \left[\sum_{k=2}^{m_1-1} \beta_k |u(t) - u_k|^3 + \beta_{m_1} + u(t) \right] \tag{15}$$

Note that the error $\varepsilon(t, \theta)$ is bilinear in the parameters of $B^n(q)$, $A^n(q)$, $f_1(u)$ and $f_2^{-1}(y)$. Hence one can use the following relaxation algorithm for parameter estimation.

3.1 *Estimate $f_1(u)$, $f_2^{-1}(y)$ and $[A^n(q), B^n(q)]$*

Initialization.

Set $f_1(u) = u$ and $f_2^{-1}(y) = y$ and estimate $A^n(q)$ and $B^n(q)$ using linear least-squares.

Iteration. Denote $\hat{A}_{(i)}^n(q)$, $\hat{B}_{(i)}^n(q)$, $\hat{f}_{1(i)}(u)$ and $\hat{f}_{2(i)}^{-1}(y)$ as the estimates from iteration i, then

1) Calculate the parameters of $\hat{f}_{1(i+1)}[(u(t)]$ for fixed $\hat{f}_{2(i)}^{-1}[(y(t)]$, $\hat{A}_{(i)}^n(q)$ and $\hat{B}_{(i)}^n(q)$ by minimizing

$$\sum_{t=1}^{N} \{\hat{A}_{(i)}^n(q)\hat{f}_{2(i)}^{-1}[y(t)] - \hat{B}_{(i)}^n(q)f_{1(i+1)}[u(t)]\}^2 \tag{16}$$

2) Calculate the parameters of $\hat{f}_{2(i+1)}^{-1}[(y(t)]$ for fixed $\hat{f}_{1(i+1)}[(u(t)]$, $\hat{A}_{(i)}^n(q)$ and $\hat{B}_{(i)}^n(q)$ by minimizing

$$\sum_{t=1}^{N} \{\hat{A}_{(i)}^n(q)f_{2(i+1)}^{-1}[y(t)] - \hat{B}_{(i)}^n(q)\hat{f}_{1(i+1)}[u(t)]\}^2 \tag{17}$$

3) Calculate $\hat{A}_{(i+1)}^n(q)$ and $\hat{B}_{(i+1)}^n(q)$ for fixed $\hat{f}_{1(i+1)}(u)$ and $\hat{f}_{2(i+1)}^{-1}(y)$ by minimizing

$$\sum_{t=1}^{N} \{A_{(i+1)}^n(q)\hat{f}_{2(i+1)}^{-1}[y(t)] - B_{(i+1)}^n(q)\hat{f}_{1(i+1)}[u(t)]\}^2 \tag{18}$$

Go back to 1). Stop when convergence occurs.

All the three steps are linear least-squares problems which are numerically simple and reliable.

Now, we will show that this relaxation scheme is a theoretically sound optimization method. To this end, we need to define persistent excitation condition on the test input. Denote

$$\bar{u}(t) = [|u(t) - u_2|^3, |u(t) - u_3|^3, \cdots |u(t) - u_{m_1-1}|^3, u(t)]$$

We say that the input signal $u(t)$ is *strongly persistently exciting* with orders (n, m_1) over the knots $\{u_1, u_2, ..., u_{m_1}\}$, if matrix

$$\Phi_u = \begin{bmatrix} \bar{u}(1) & \bar{u}(2) & \cdots & \bar{u}(n) \\ \bar{u}(2) & \bar{u}(3) & \cdots & \bar{u}(2n+1) \\ \vdots & \vdots & & \vdots \\ \bar{u}(N-n) & & & \bar{u}(N) \end{bmatrix} \quad (19)$$

is nonsingular for all $N >> \max(n, m_1)$. This definition is similar to the strongly persistent excitation condition for polynomial nonlinearity introduced in Stoica and Söderström (1982).

Theorem 1. *Assume that the input is strongly persistently exciting with orders greater than $(2n, m_1)$ over the knots $\{u_1, u_2, ..., u_{m_1}\}$ and the input is not determined purely by linear output feedback (see a5 bellow for the definition). Then the relaxation algorithm (16)–(18) minimizes the criterion in (12) locally if it converges.*

Proof. See the proof of Theorem 2.1 of Golub and Pereyra (1973)

Golub and Pereyra (1973) recommended to use the separable least-squares if possible.

In the following we will established the convergence of criterion $V_{ARX}^N(\theta, Z^N)$ and consistency of parameter estimate $\hat{\theta}$. First, some assumptions on the process, the model and on the test conditions are given.

a1 The data set Z^N is generated by the process

$$A^{on}(q)f_2^{o-1}(y(t)) = B^{on}(q)f_1^o(u(t)) + e_o(t) \quad (20)$$

The noise $\{e_o(t)\}$ is zero mean white noise with bounded moments of order $4 + \delta$ for some $\delta > 0$. There is at least one delay in the process, or in the controller. The polynomial $A^{on}(q)$ is monic (its first coefficient is unity.). The nonlinear functions $f_2^{o-1}(y(t))$ and $f_1^o(u(t))$ are continuous.
a2 The input-output data are bounded.

These assumptions are needed to prove the convergence of the method. The following two assumptions are necessary to prove the consistency.

a3 The true process (20) has the same parametrization as the model (9), (2), (10) and (11) with the same orders. Polynomials $A^{on}(q)$ and $B^{on}(q)$ are coprime.
a4 The input $u(t)$ is strongly persistently exciting with orders $(2n, m_1)$. If the input-output data is generated in closed-loop, the input is not determined purely by linear output feedback.

Theorem 2. *Assume that conditions a1 and a2 are true, then the following convergence results hold:*

$$\sup_{\theta \in D_M} |V_{ARX}(\theta, Z^N) - E\varepsilon^2(t, \theta)| \to 0 \text{ w.p.1 as } N \to \infty \quad (21)$$

$$\hat{\theta}^N \to \theta^* \quad \text{w. p. 1 as } N \to \infty \quad (22)$$

where

$$\hat{\theta}^N = \arg \min_{\theta \in D_M} V_{ARX}^N(\theta, Z^N)$$

$$\theta^* = \arg \min_{\theta \in D_M} E\varepsilon^2(t, \theta)$$

Moreover, assume that conditions a3 and a4 also hold and denote the true parameter vector as θ^o. Then the estimate $\hat{\theta}^N$ is consistent, namely,

$$\hat{\theta}^N \to \theta^o \quad \text{with probability 1 as } N \to \infty \quad (23)$$

The proof can be found in Zhu (2002). The convergence result holds under very general and weak conditions. The true nonlinear functions need not be cubic splines and the model orders need not to be correct. The limiting model will always minimize the loss function of the prediction error of $w(t)$. Unstable process can be treated without problems, because the predictor is stable when an ARX model is used.

The determination of the nonlinear function $f_2(w)$ from the estimate of its inverse $\hat{f}_2^{-1}(y)$ is an approximation problem. This can be done using a linear least-squares estimate.

3.2 *Model Reduction for Linear ARX Model*

The obtained N-L-N model with ARX model is unbiased, provided that the process can be modelled by an N-L-N model, but the variance error of the ARX model is high due to its high order. Model reduction can be used to reduce the variance error.

Assume that the input to the linear block $w_1(t)$ and the input to the output nonlinearity $w(t)$ are known exactly, then it can be shown (see Ljung, 1985) that the estimated frequency response of the high order model is unbiased and its error follows a Gaussian distribution with variance given as

$$var[\hat{G}^n(e^{i\omega})] \approx \frac{n}{N} \frac{\Phi_v(\omega)R}{\Phi_{w_1}(\omega)R - |\Phi_{w_1 e}(\omega)|^2} \quad (24)$$

where $\hat{G}^n(e^{i\omega})$ is the frequency response of the estimated high order ARX model, n is the order of the ARX model, N is the number of data points, $\Phi_v(\omega)$ is the power spectrum of the disturbance, R is the variance of white noise $\{e(t)\}$ that generated the disturbance, $\Phi_{w_1}(\omega)$ is the power spectrum of input to the linear block $w_1(t)$ and $\Phi_{w_1 e}(\omega)$ is the cross-spectrum between the white noise $\{e(t)\}$ and $w_1(t)$. Based on

(24), it can be shown that the asymptotic negative log-likelihood function for the linear model is given by (Wahlberg, 1989):

$$\int_{-\pi}^{\pi} \left| \hat{G}^m(e^{i\omega}) - G^l(e^{i\omega}) \right|^2 \frac{\Phi_{w_1}(\omega)\lambda^2 - |\Phi_{w_1 e}(\omega)|^2}{\Phi_v(\omega)\lambda^2} d\omega$$

(25)

where $G^l(e^{i\omega})$ is the frequency response of the reduced model to be calculated.

4. SIMULATION STUDIES

The process is given as

$$y(t) = f_2 \left[\frac{0.5q^{-1} + 0.25q^{-2}}{1 - 1.5q^{-1} + 0.7q^{-2}} \cdot f_1[u(t)] + v(t) \right]$$

with

$$f_1[u(t)] = \frac{u(t)}{\sqrt{0.1 + 0.9u^2(t)}}$$

$$f_2(w(t)) = w(t) + 0.2w^3(t)$$

and

$$v(t) = \frac{\alpha}{1 - 0.9q^{-1}} e(t)$$

The range of $u(t)$ is $[-0.7, 2.5]$. This process has severe nonlinearities which are not cubic splines.

Simulation 1. First noise free simulation is used to check the correctness and the convergence of the algorithm. Open loop test is used and 1500 samples are simulated. GMN signal with average switch time $T_{sw} = 10$ is applied at the input. The range of $u(t)$ is $[-0.7, 2.5]$. The true model order of the linear part is used; the degrees of the two nonlinear functions are both 12. The algorithm converges in about 15 iterations. The identified model is almost perfect. The fit error is 0.07% in variance at $w(t)$ and 0.04% at $y(t)$. Because the plots of the process and of the model coincide with each other, they are not shown. Linear Box-Jenkins and output-error models are also identified using the same input-output data. The fit error of a linear Box-Jenkins model is 26.8% at $y(t)$ and that of a linear output error model is 25.5%.

Simulation 2. The same as in Simulation 1, but disturbance $v(t)$ is added at $w(t)$. Disturbance level is 3% (in variance) at $w(t)$ and is about 10% at $y(t)$. The true and identified nonlinear functions are plotted in Figure 4.1 and the true step response of linear part and its estimation are shown in Figure 4.2. One can see that the identified model is very good considering the disturbance level. The algorithm converges in about 15 iterations.
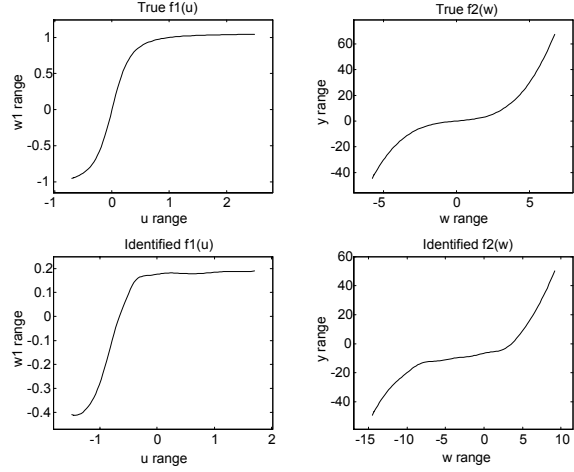

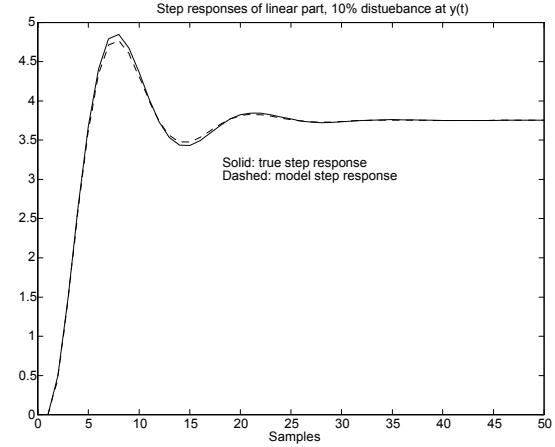Figure 4.1 True and identified nonlinear functions.

Figure 4.2 True and estimated step responses of the linear block. Model gain is corrected.

5. CONCLUSIONS

The identification of a SISO N-L-N Hammerstein-Wiener model is studied. The disturbance is introduced that is nonlinear at the output. In parameter estimation, the bilinear-in-parameters property of the high order model is used to derive the relaxation algorithm which is numerically simpler and more reliable than general nonlinear search algorithms. The convergence and consistency are proved. The effectiveness of the proposed method have been shown in the simulation examples. It is easy to extend the algorithm to multi-input single-output (MISO) processes, provided that there is no joint nonlinearity among the inputs.

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