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INPUT-OUTPUT EQUIVALENCE OF NONLINEAR SYSTEMS AND THEIR REALIZATIONS

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Abstract: The notion of realization for single-input single-output nonlinear systems is studied based on a new notion of input/output equivalence. This equivalence relation aims to generalize the equivalence of linear time-invariant systems in the sense of the equality of their transfer functions. Necessary and sufficient conditions are given for the existence of a realization, affine or not. A minimal (*i.e.* accessible and observable) realization may then be derived for those systems which satisfy these conditions, after seeking an equivalent reduced order input/output system.

Keywords: Nonlinear systems, realization, equivalence, algebraic methods.

1. INTRODUCTION

Realization is a longstanding problem in nonlinear control theory as well as its minimality. In the late 1970's the work of Fliess (Fliess, 1980) solved the realization problem for the nonlinear systems with the generating series expressions. Necessary and sufficient conditions were given in terms of the *Lie Rank* of the generating series. More recently, generalized realizations (*i.e.* depending upon derivatives of the input) were derived in (Fliess, 1990; Fliess and Glad, 1993) using a differential algebraic approach. Conditions are given in (Freedman and Willems, 1978; Delaleau and Respondek, 1995) under which there exists a transformation so that the derivatives of the input are eliminated and these results may be viewed as conditions for which a realization exists.

In this paper a classical question is discussed and answered: when does an input/output equation have a state space realization? The approach in this paper is related to (Crouch and Lamnabhi-Lagarrigue, 1992; Crouch *et al.*, 1995; Sontag, 1988). In (Sontag, 1988) attention is restricted to bilinear reachability. A definition of realization is given in (Crouch and Lamnabhi-Lagarrigue, 1992) as well as a necessary condition for an input/output differential equation to have an affine state space realization.

The goal of this paper is to give a definition of realization for an input/output equation, which fully incorporates the linear theory, and to characterize the existence of such a local realization. All statements and computations in this work are valid around regular points, i.e. on a suitable open and dense subset of $I\!R^N$ for some integer N. We explicitly define when a state space system *generates* or *realizes* a given input/output equation. To our best knowledge, the difference between the two notions has never been well described for nonlinear systems.

Consider a nonlinear single-input single-output (SISO) system described by the input-output differential equation:

$$y^{(k)} = \varphi(y, \dot{y}, \ldots, y^{(k-1)}, u, \dot{u}, \ldots, u^{(s)}). \quad (1)$$

with $\partial\varphi/\partial u^{(s)} \neq 0$. The integer k is said to be the order of system (1). Furthermore, assume that $\varphi(y, \dot{y}, \ldots, y^{(k-1)}, u, \ldots, u^{(s)})$ is an analytic function on an open dense subset M of $I\!R^{k+s+1}$.

Example 1. Consider the three input/output differential equations

$$\dot{y} - yu = 0, \quad (2)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{y} - yu) = \ddot{y} - \dot{y}u - y\dot{u} = 0 \quad (3)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{y} - yu) - u(\dot{y} - yu) = \ddot{y} - yu^2 - y\dot{u} = 0 \quad (4)$$

Note that any solution of (2) is a solution of (3) and (4). With respect to the associated state space representation and its minimality, the three systems may hardly be considered to be equivalent. Consider the accessible and observable state space system

$$\begin{aligned} \dot{x} &= xu \\ y &= x \end{aligned} \quad (5)$$

The elimination of the state x in (5) yields (2) and suggests that (5) is a minimal realization for (2). Let

$$\begin{aligned} \dot{x}_1 &= x_1 u + x_2 \\ \dot{x}_2 &= 0 \\ y &= x_1 \end{aligned} \quad (6)$$

which is not accessible. The elimination of the state in (6) suggests again that (6) is a (non minimal) realization for (3). In a similar vein,

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1 u \\ \dot{x}_2 &= -x_2 u \\ y &= x_1 \end{aligned} \quad (7)$$

is accessible and observable and yields (4).

Deciding which state space system is a realization of which input/output equation deserves special attention and motivates this paper. The theory which is developed will establish that (5) is a minimal realization of (3), but is not a realization of (4). Note that any solution of (2) is also a solution of both (3) and (4).

Section 2 is devoted to observable realizations *generating* a given input/output system. Definition 2 states the so-called notion of *generating system*. Necessary and sufficient conditions are given for the existence of affine or non affine realizations. Section 3 is devoted to the key point of input/output equivalence; an equivalence relation is defined between input-output differential equations. It enables to formalize the order reduction in the nonlinear setting. This order reduction generalizes the process of reduction of transfer functions in the case of linear time invariant systems. The irreducibility of an input/output system is defined as well, and it is used to tackle minimality of the realization. A general notion of realization is displayed in Definition 20.

2. OBSERVABLE REALIZATIONS

Before introducing an input/output equivalence relation and defining general realizations, let us first define the notion of *generating system* for a given input/output equation. The latter will appear to be a special case of realization. Then, conditions for the existence of such an observable generating system will be derived.

Consider

$$y^{(k)} = \varphi(y, \dot{y}, \ldots, y^{(k-1)}, u, \ldots, u^{(s)}) \quad (8)$$

with $s < k$ and a state space system:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \\ y &= h(x) \end{aligned} \quad (9)$$

where x belongs to an open subset of $I\!R^n$. The system (9) yields a set of algebraic equalities:

$$\begin{aligned} y &= h(x) \\ \dot{y} &= h_1(x, u) \\ &\vdots \\ y^{(n-1)} &= h_{n-1}(x, u, \ldots, u^{(n-2)}) \end{aligned} \quad (10)$$

If (9) is an observable control system, then the observable codistribution, which spanned by the differentials of all functions in observation space, is n-dimensional ((Isidori, 1995; Zheng and Cao, 1993)). I.e. over the field of meromorphic functions of x, u and the time derivatives of u

$$\operatorname{rank} \frac{\partial(h, h_1, \cdots, h_{n-1})}{\partial(x_1, x_2, \cdots, x_n)} = n$$

Let $\bar{x}_0 = (x_0, u_0, ..., u_0^{(n-1)})$ be a point in $I\!R^{2n}$ and a neighborhood $V(\bar{x}_0)$ of $\bar{x}_0$ such that the rank of $\frac{\partial(h, h_1, \cdots, h_{n-1})}{\partial(x_1, x_2, \cdots, x_n)}$ is constant (when evaluated over $I\!R$) over $V(\bar{x}_0)$. It turns out that (10) can be solved , in the state variables $x_i; i = 1, \ldots, n$ as

$$x = \Phi(y, \dot{y}, \ldots, y^{(n-1)}, u, \dot{u}, \ldots, u^{(n-2)}) \quad (11)$$

for any $(x, u, ..., u^{(n-1)} \in V(\bar{x}_0)$. Consider

$$
\begin{aligned}
y^{(k)} &= L_f^k h + L_g L_f^{k-1} hu + \ldots + (L_g hu)^{(k-1)} \\
&= h_k(x, u, \ldots, u^{(k-2)}).
\end{aligned} \tag{12}
$$

Definition 2. The observable system (9) is said to be a generating system locally around $(y_0, ..., y_0^{(n-1)}, u_0, ..., u_0^{(n-2)})$ for the input/output system (8) if there exists a neighborhood $V(y_0, ..., y_0^{(n-1)}, u_0, ..., u_0^{(n-2)})$ such that substituting $x_i; i = 1, \ldots, n$ in (11) into (12) yields (8).

A similar definition can be derived for a non-affine state space system

$$
\begin{aligned}
\dot{x} &= f(x, u), \\
y &= h(x)
\end{aligned} \tag{13}
$$

generating a given input/output equation.

A consequence from Definition 2 is that if (9) is a generating system for (8), then $n \leq k$.

Example 3. The system

$$
\begin{cases}
\dot{x}_1 = x_2 + u \\
\dot{x}_2 = 0 \\
y = x_1
\end{cases} \tag{14}
$$

is a generating system for the input/output equation $\ddot{y} = \dot{u}$ but it does *not* generate the input/output equation $\dot{y} = u$. The system

$$
\begin{cases}
\dot{x} = u \\
y = x
\end{cases}
$$

is a generating system for both $\ddot{y} = \dot{u}$ and $\dot{y} = u$.

Example 4. The system

$$
\begin{cases}
\dot{x}_1 = x_2 + u \\
\dot{x}_2 = x_2^2 \\
y = x_1
\end{cases}
$$

is a generating system for $\ddot{y} = (\dot{y} - u)^2 + \dot{u}$.

The following material was first established for discrete-time nonlinear systems in (Kotta *et al.*, 1997).

Consider the SISO nonlinear system

$$
y^{(k)} = \varphi(y, \ldots, y^{(k-1)}, u, \ldots, u^{(s)}) \tag{15}
$$

where $\partial\varphi / \partial u^{(s)} \neq 0$ with $s \geq 0$.

Associate a so-called *extended system* Σ_e to (15), defined as

$$
\begin{aligned}
\Sigma_e : \frac{d}{dt} \begin{bmatrix} y \\ \vdots \\ y^{(k-2)} \\ y^{(k-1)} \\ u \\ \vdots \\ u^{(s-1)} \\ u^{(s)} \end{bmatrix} &= \begin{bmatrix} \dot{y} \\ \vdots \\ y^{(k-1)} \\ \varphi \\ \dot{u} \\ \vdots \\ u^{(s)} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u^{(s+1)} \\
&= f_e + g_e u^{(s+1)}
\end{aligned} \tag{16}
$$

For completeness, let us adapt to system (16) the setting introduced in (di Benedetto *et al.*, 1989; Conte *et al.*, 1999; Fliess, 1990). Define the differential field $\mathcal{K}$ of meromorphic functions in a finite number of variables y, u and their time derivatives, associated to system (15). Let $\mathcal{E}$ be the formal vector space $\mathcal{E} = \text{span}_{\mathcal{K}}\{\text{d}\varphi \mid \varphi \in \mathcal{K}\}$, $\mathcal{K}_{(x,u)}$ denote the field of meromorphic functions of x, u and the time derivatives of u. Then we have the following preliminary result whose proof follows from Definition 2.

Lemma 5. Assume that the system (9) is a generating system for (15). The field $\mathcal{K}_{(x,u)}$ associated with system (9) and the field $\mathcal{K}$ associated with equation (15) can be identified. That is, for any $\phi \in \mathcal{K}_{(x,u)}$ there exists an unique element $\psi(y, \dot{y}, \cdots, u, \cdots) \in \mathcal{K}$ such that, after the substitutions (10) and (11), the two functions ϕ and ψ are equal.

Conversely, for every element $\psi(y, \dot{y}, \cdots, u, \cdots) \in \mathcal{K}$ there exists unique element $\phi \in \mathcal{K}_{(x,u)}$ such that, after the substitutions (10) and (11), the two elements ϕ and ψ are equal.

Define the following subspaces of $\mathcal{E}$

$$
\mathcal{H}_1 = \text{span}_{\mathcal{K}}\{\text{d}y, \text{d}\dot{y}, \ldots, \text{d}y^{(k-1)}, \text{d}u, \ldots, \text{d}u^{(s)}\}
$$

and more generally

$$
\mathcal{H}_{i+1} = \text{span}_{\mathcal{K}}\{\omega \in \mathcal{H}_i \mid \dot{\omega} \in \mathcal{H}_i\} \tag{17}
$$

The following theorem gives an intrinsic necessary and sufficient condition for the existence of an observable realization. Alternative (algorithmic) conditions may be found in (Delaleau and Respondek, 1995): starting with a so-called generalized state realization (Fliess, 1990), necessary and sufficient conditions for the reduction of the order of time derivation of u may be checked step after step.

Theorem 6. There exists an observable state space system

$$
\begin{cases}
\dot{x} = f(x, u) \\
y = h(x)
\end{cases} \tag{18}
$$

which is a generating system for (15) – locally around any point $(y_0, ..., u_0^{(s)})$ in some suitable open dense subset of $I\!R^{k+s+1}$ – if and only if

- $k > s$
- and, $\mathcal{H}_i$ is integrable for each $i = 1, \ldots, s+2$.

Proof

Sufficiency. Let $\{\mathrm{d}\xi_1, \ldots, \mathrm{d}\xi_k\}$ be a basis of $\mathcal{H}_{s+2}$. From the construction of the subspaces $\mathcal{H}_i$ we have

$$\begin{aligned}\mathcal{H}_{s+1} &= \mathcal{H}_{s+2} \oplus \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u\}\\ \mathcal{H}_s &= \mathcal{H}_{s+2} \oplus \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u, \mathrm{d}\dot{u}\}\\ &\vdots\\ \mathcal{H}_1 &= \mathcal{H}_{s+2} \oplus \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u, \ldots, \mathrm{d}u^{(s)}\}\end{aligned} \tag{19}$$

Introduce the following coordinate transformation for the system (16):

$$\begin{aligned}x_1 &= \xi_1(y, \dot{y}, \ldots, u^{(s)})\\ &\vdots\\ x_k &= \xi_k(y, \dot{y}, \ldots, u^{(s)})\\ x_{k+1} &= u\\ &\vdots\\ x_{k+s+1} &= u^{(s)}\end{aligned} \tag{20}$$

By (17) and that $\mathcal{H}_{s+2} \subset \mathcal{H}_{s+1}$, one has $\mathrm{d}\dot{x}_i = \mathrm{d}\dot{\xi}_i \in \mathcal{H}_{s+2} \oplus \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u\}$, $i = 1, \cdots, k$. Thus, $\mathrm{d}\dot{x}_i = \sum_{j=1}^{k} \alpha_{ij}\mathrm{d}x_j + \beta \mathrm{d}u$, for each $i = 1, \cdots, k$. Let $x = (x_1, \ldots, x_k)$. Thus, at least locally $\dot{x} = f(x, u)$. The assumption $k > s$ indicates that the output y depends only on x.

Necessity. Assume that the observable state space system (18) is a generating system for the input/output system (15). Since the state space system is proper, necessarily $k > s$. From Lemma 5 the subspaces $\mathcal{H}_i$ can be identified with the $\tilde{\mathcal{H}}_i$:

$$\begin{aligned}\tilde{\mathcal{H}}_1 &= \mathrm{span}_{\mathcal{K}(x,u)}\{\mathrm{d}x, \mathrm{d}u, \ldots, \mathrm{d}u^{(s)}\}\\ &\vdots\\ \tilde{\mathcal{H}}_{s+1} &= \mathrm{span}_{\mathcal{K}(x,u)}\{\mathrm{d}x, \mathrm{d}u\}\\ \tilde{\mathcal{H}}_{s+2} &= \mathrm{span}_{\mathcal{K}(x,u)}\{\mathrm{d}x\}\end{aligned}$$

From (20) the spaces $\mathcal{H}_i$ are integrable as expected.

Example 7. Let $\ddot{y} = \dot{u}^2$, and compute

$$\begin{aligned}\mathcal{H}_1 &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}\dot{y}, \mathrm{d}u, \mathrm{d}\dot{u}\}\\ \mathcal{H}_2 &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}\dot{y}, \mathrm{d}u\}\\ \mathcal{H}_3 &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}\dot{y} - 2\dot{u}\mathrm{d}u)\}\end{aligned}$$

Since $\mathcal{H}_3$ is not integrable, there does not exist any state space system generating $\ddot{y} = \dot{u}^2$.

Example 8. Let $\ddot{y} = u^2$. The conditions of Theorem 6 are fulfilled and the state variables $x_1 = y$, and $x_2 = \dot{y}$ yield $\dot{x}_1 = x_2$, $\dot{x}_2 = u^2$, $y = x_1$ which is a generating system for $\ddot{y} = u^2$.

The conditions given by Theorem 6 can be extended to derive a characterization of the existence of an affine generating system as follows.

Corollary 9. There exists an affine state space system

$$\begin{cases}\dot{x} = f(x) + g(x)u\\ y = h(x)\end{cases}$$

which is a generating system for (15) – locally around any point $(y_0, ..., u_0^{(s)})$ in some suitable open dense subset of $I\!R^{k+s+1}$ – if and only if

(i) $\mathcal{H}_i$ is integrable for each $i = 1, \ldots, s+2$
(ii) $\dot{\xi} = F(\xi) + G(\xi)u$ for some F and G, where $\{\mathrm{d}\xi\}$ is a basis of $\mathcal{H}_{s+2}$.

[Note that condition (ii) does not depend on the choice of the basis $\{\mathrm{d}\xi\}$ of $\mathcal{H}_{s+2}$.]

Proof: This follows from the proof of Theorem 6 and from the fact that the input affine structure remains unchanged under state transformation.

3. INPUT-OUTPUT EQUIVALENCE AND MINIMAL REALIZATIONS

The heart of the analysis of (15) will be extensively based on the accessibility property of Σ_e. The standard strong accessibility distribution associated with (16) is denoted as:

$$\mathcal{L} := \overline{\mathrm{span}_{\mathcal{K}}\{ad^i_{f_e} g_e,\, i \geq 0\}} \tag{21}$$

In this section we will formalize the notion of *reduction* to obtain the notion of input/output equivalence, and a definition of realization.

Definition 10. (Irreducible input-output system). The system (15) is said to be an *irreducible input-output system* if the associated system (16) satisfies

$$\mathcal{L}^{\perp} = 0$$

Example 11. (Example 1 cont'd). $\ddot{y} = yu^2 + y\dot{u}$ is irreducible since

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y\\ \dot{y}\\ u\\ \dot{u}\end{pmatrix} = \begin{pmatrix}\dot{y}\\ yu^2 + y\dot{u}\\ \dot{u}\\ 0\end{pmatrix} + \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\ddot{u}$$

satisfies the strong accessibility rank condition. It is worth to note that the set of solutions $(u(t), y(t))$ of $\dot{y} = yu$ is a subset of the set of solutions of $\ddot{y} = yu^2 + y\dot{u}$, but $\ddot{y} - yu^2 - y\dot{u} = 0$ is irreducible.

Example 12. $\ddot{y} = \dot{u} + (\dot{y} - u)^2$ is not irreducible since

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y \\ \dot{y} \\ u \\ \dot{u} \end{pmatrix} = \begin{pmatrix} \dot{y} \\ \dot{u} + (\dot{y}-u)^2 \\ \dot{u} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \ddot{u}$$

does not satisfy the strong accessibility rank condition. However, $\mathrm{d}(\dot{y}-u) \perp \mathcal{L}$ and we will claim that $\dot{y} = u$ is an irreducible input-output system of $\ddot{y} = \dot{u} + (\dot{y}-u)^2$.

We are interested in minimal, *i.e.* observable and accessible realizations and will assume from now on that the input/output system (15) admits an observable generating system. Introduce the following definitions of reduced differential form and of reduced input-output system.

Definition 13. (Reduced differential form). An exact
form $\mathrm{d}\phi'$ is said to be a *reduced differential form* of system (15) if

(a) $\mathrm{d}\phi' \not\equiv 0$
(b) and $\mathrm{d}\phi' \in \mathcal{L}^{\perp}$ where $\mathcal{L}$ is the accessibility distribution of (16) defined in (21).

Definition 14. (Reduced input-output system). Let $\mathrm{d}\phi'$ be a reduced differential form, which produces the differential equation

$$\phi'(y, \cdots, y^{(k'-1)}, y^{(k')}, u, \cdots, u^{(s')}) = 0 \quad (22)$$

such that $\partial\phi'/\partial y^{(k')} \neq 0$, $\partial\phi'/\partial u^{(s')} \neq 0$, $\partial^2\phi'/\partial {y^{(k')}}^2 \equiv 0$ with $k' > 0$, $s' \geq 0$. Equation (22) has an unique solution

$$y^{(k')} = \varphi'(y, \cdots, y^{(k'-1)}, u, \cdots, u^{(s')}). \quad (23)$$

Then (23) is called a *reduced input-output system* of system (15).

Definition 15. (Irreducible differential form). If (23) is an irreducible input-output system in the sense of Definition 10, then $\mathrm{d}(y^{(k')} - \varphi')$ is said to be an *irreducible differential form* of (15).

Example 16. (Example 3 cont'd). $\mathrm{d}(\dot{y}-u) \perp \mathcal{L}$ and $\dot{y} = u$ is an irreducible system. Thus, $\phi' = \dot{y} - u = 0$ is an irreducible input-output system of $\ddot{y} = \dot{u} + (\dot{y}-u)^2$.

An irreducible input-output system cannot be associated to every input-output system. Consider

$$\ddot{y} = \frac{\dot{y}\dot{u}}{u} \quad (24)$$

$d\phi' = \mathrm{d}(\dot{y}/u)$ is a reduced differential form of (24). Thus, system (24) is not irreducible. Let $\phi' = \dot{y}/u = 0$, which is not an *irreducible input-output system* in the sense of the above Definition. Therefore, system (24) does not admit any irreducible input-output system.

We now restrict our attention to the family of input/output systems which admit an irreducible input-output system. It is possible to introduce an equivalence relation. First we need to prove the unicity of the irreducible input-output system associated to a given system, if any.

Lemma 17. If the system

$$y^{(k)} = \phi(y, \ldots, y^{(k-1)}, u, \ldots, u^{(s)}) = 0 \quad (25)$$

admits an irreducible input-output system, then the order of the irreducible input-output system equals $k - \mathrm{dim}\mathcal{L}^{\perp}$.

Proof Let $\dim \mathcal{L}^{\perp} = r$, it follows from (Zheng *et al.*, 1995) that $\mathcal{H}_{\infty} = \mathcal{L}^{\perp}$ and then the order of irreducible input-output system associated to (25) equals $\kappa = k - \dim \mathcal{H}_{\infty} = k - \dim \mathcal{L}^{\perp} = k - r$.

By means of the results of (**?**; Zheng *et al.*, 1993) we can show that

Lemma 18. If (25) admits an irreducible input-output system, then its irreducible input-output system is uniquely defined in the form

$$y^{(\kappa)} = \varphi(y, \ldots, y^{(\kappa-1)}, u, \ldots, u^{(\sigma)}). \quad (26)$$

It is now possible to introduce the following equivalence relation within the class of systems which admit an irreducible differential form.

Definition 19. (Input-output equivalence). Two systems Σ_1 and Σ_2, which are supposed to admit an irreducible differential form, are said to be input/output equivalent if they have the same irreducible input-output system representation of the form (26).

Now we give a general definition of *realization.*

Definition 20. (Realization). A state space system (9) is said to be a realization of the input/output system (15) if (9) is a generating system for an input/output equation – locally around any point $(y_0, ..., y_0^{(\kappa)}, u_0, ..., u_0^{(\sigma)})$ in some suitable open dense subset of $I\!R^{\kappa+\sigma+1}$ for some suitable κ and σ – which is input/output equivalent to (15).

The system (15) is said to be realizable if there exists a realization in the sense of Definition 20. Note that the case of generating system as in Section 2 is a special case of this general notion of realization.

The notion of minimality here is standard for linear systems and means that a state space system is both observable and accessible. For nonlinear systems (Isidori, 1995) it is commonly accepted as well, although other approaches to minimality exist in the recent literature, where only observability is requested (Diop and Fliess, 1991). From the previous results we can now state the following.

Theorem 21. Given an input/output system (15), assume that the conditions in Theorem 6 are fulfilled. Then there exists an observable and accessible, *i.e.* minimal, realization of order k for (15), if and only if (15) is an irreducible input/output system.

Proof: Given (15), the generating system (18) obtained from Theorem 6 is observable. The extended system (16) can be written in the coordinates (20). It then reads as the composite system of system $\dot{x} = f(x,u)$ and the controllable string of integrators $\dot{u}^{(i)} = u^{(i+1)}$, $i = 0, ..., s$. Thus, (16) is accessible if and only if (**??**) is accessible. The result of Theorem 21 follows since (16) is accessible if and only if (15) is irreducible, by Definition 10.

Example 22. Consider (2) and (3) in Example 1. Let $\phi = \ddot{y} - \dot{y}u - y\dot{u}$. Compute $f_e = \begin{pmatrix} \dot{y} \\ \dot{y}u + y\dot{u} \\ \dot{u} \\ 0 \end{pmatrix}$ and $g_e = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. The distribution $\mathcal{L}$ is spanned by $\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ y \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} y \\ uy \\ 0 \\ 0 \end{pmatrix} \right\}$. Thus, $\mathrm{d}\phi_r = \mathrm{d}(\dot{y} - yu) \in \mathcal{L}^{\perp}$. An irreducible differential form of $\phi = 0$ is $\mathrm{d}\phi_r = \mathrm{d}(\dot{y} - yu)$. A minimal realization is obtained for $\dot{y} = yu$.

4. CONCLUSION

General necessary and sufficient conditions have been obtained for the existence of affine and non-affine (observable) realizations of a nonlinear system which is a generating system for a given input/output differential equation. A notion of input/output equivalence has been given which

- yields a new definition of realization that is consistent with a standard notion of minimality, including minimal linear state space representations,
- generalizes equality of transfer functions, up to factor reduction.

5. REFERENCES

di Benedetto, M.D., J.W. Grizzle and C.H. Moog (1989), Rank invariants of nonlinear systems. *SIAM J. Contr.*, **27**, 658–672.

Conte, G., C.H. Moog and A.M. Perdon (1999). *Nonlinear Control Systems: An Algebraic Setting*, Lect. N. Contr. Inf. Sci., vol. 242, Springer-Verlag, London.

Crouch, P.E. and F. Lamnabhi-Lagarrigue (1992), Realizations of input-output differential equations. In *Recent Advances in Mathematical Theory of Systems, Control, Networks and Signal Processing II Proceeding MTNS-91*, Mita Press, 259–264.

Crouch, P.E., F. Lamnabhi-Lagarrigue and D. Pinchon (1995), Some realization algorithms for nonlinear input-output systems. *Int. J. Contr.*, **62**, 941–960.

Delaleau, E., and W. Respondek (1995), Lowering the orders of derivatives of controls in generalized state space systems. *J. Math. Systems Estim. Contr.*, **5**, 1–27.

Diop, S. and M. Fliess (1991), Nonlinear observability, identifiability, and persistent trajectories. *Proc. 30th CDC*, Brighton, UK, 714–719.

Fliess, M. (1980), Realizations of nonlinear systems and abstract transitive Lie algebras. *Bull. of the American Mathematical Society*, **2**, 444–446.

Fliess, M. (1990), Generalized controller forms for linear and nonlinear dynamics. *IEEE Trans. Aut. Contr.*, **35**, 994–1001.

Fliess, M. and S.T. Glad (1993), An algebraic approach to linear and nonlinear control. In *Essays on Control: Perspectives in the Theory and its Applications*,Birkhäuser, Boston.

Freedman, M.I. and J.C. Willems (1978), Smooth representations of systems with differentiated inputs. *IEEE Trans. Aut. Contr.*, **23**, 16–21.

Isidori, A. (1995). *Nonlinear control systems* (Third Edition), Springer-Verlag, Berlin.

Kotta, Ü., P. Liu and A.S.I Zinober (1997), State-space realization of input-output nonlinear difference equations. *Proc. ECC'97 Conf.*.

Sontag, E.D. (1988), Bilinear realizability is equivalent to existence of a singular affine differential i/o equation. *Systems and Control Letters*, **11**, 181–187.

Zheng, Y.F. and L. Cao (1993), Reduced inverse for controlled systems. *MCSS*, **6**, 363–379.

Zheng, Y.F., P. Liu, A. Zinober and C.H. Moog (1995), What is the dimension of the minimal realization of a nonlinear system?. *Proc. IEEE CDC*, New Orleans, 4239–4244.

Zheng, Y.F., J.C. Willems and C. Zhang (1993), A polynomial approach to nonlinear system controllability. *IEEE Trans. Aut. Contr.*, in press.