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RECENT DEVELOPMENTS IN REACHABILITY AND CONTROLLABILITY OF POSITIVE LINEAR SYSTEMS [1]

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Abstract: In this paper, recent results on the structural properties of positive linear systems are collected and analyzed. The study of reachability and controllability properties of positive invariant linear systems is focussed, mainly, in the recent years, either for discrete and continuous systems when results are known. In addition, all known results of positive periodic linear systems are discussed in the paper. Those results are discussed either with algebraic and combinatorial approaches. Canonical forms of reachability for positive invariant and periodic discrete-time linear systems are displayed and analyzed. Finally, the essential reachability and controllability properties are studied for both kind of systems. *©IFAC 2002*

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1. INTRODUCTION

In this paper we deal with positive discrete-time linear control systems in the state-space model, i.e., systems whose states and inputs are nonnegative. Such systems appear in many different real situations such as in economics, biological, environmental and chemical processes, among others. Different studies have been modellized with positive systems, for instance, fugacity models (Bru *et al.*, 1998), crop supply analysis (Kalaitzandonakes and Shonkwiler, 1992) and manpower planning systems (Caccetta *et al.*, 2000). Positive system behavior seems to be intrinsic for many real-life dynamic systems. Positive linear systems are defined on cones and not on linear spaces. Consequently, many well known properties of linear systems cannot be applied to positive systems. The nonnegativity condition yields a different treatment of these control systems based upon the theory of nonnegative matrices.

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In the literature, a lot of work has been done on positive linear systems, either, in papers and books. In the book by (Farina and Rinaldi, 2000), a great number of applications and problems of positive linear systems were presented. it is worth to notice that the books of (Luenberger, 1979) and (Berman *et al.*, 1989) study general dynamic systems and nonnegative matrices, respectively, which both topìcs are basic on the developing of positive linear systems theory. Many authors have studied different problems concerning positive linear systems. New results have appeared and new chapters of the systems theory for positive linear systems such as reachable sets (see (Farina and Benvenuti, 1997) and (Rumchev, 1989)), non-negative and minimal realizations (see (Anderson, 1997), (Farina, 1996), (Kaczorek, 1997) and (Van den Hof, 1997)), feedback control (Rumchev and James, 1995b) and reachability and controllability (see (Murthy, 1986), (Ohta *et al.*, 1984), (Coxson and Shapiro, 1987)). It must be indicated also that reachability and controllability results on singular positive systems, descriptor positive systems and on 2-D positive systems are not included in this survey. Other researchers such as (Bru and Hernández, 1989) and (Bru *et al.*, 1997) deal with the positive periodic case.

The objective of the paper is to collect and analyze the recent results in reachability and controllability theory, focusing in the recent years for positive invariant linear systems and from the first results for positive periodic linear systems. The paper is organized as follows. Reachability and controllability criteria in algebraic form for positive invariant systems are presented in section 2. Criteria in digraph form and related problems are discussed also in that section. Reachability and controllability characterizations are given for positive N-periodic systems throughout section 3. In section 4, canonical forms for positive invariant and periodic cases are shown. In section 5 results on essential reachability and controllability are tackled and characterizations for recognizing these properties. Section 6 concludes the survey.

2. TIME-INVARIANT SYSTEMS

2.1 *Discrete-time systems*

A *positive N-periodic discrete-time linear control system* is given by

$$x(k+1) = A(k)x(k) + B(k)u(k), k \in \mathbb{Z}_+, \quad (1)$$

where the period $N \in \mathbb{N}$, $A(k) = A(k+N) \in \mathbb{R}_+^{n\times n}$, $B(k) = B(k+N) \in \mathbb{R}_+^{n\times m}$, $x(k) \in \mathbb{R}_+^n$ is the nonnegative *state* vector and $u(k) \in \mathbb{R}_+^m$ is the nonnegative *control* or *input* vector. This system is denoted by $(A(\cdot), B(\cdot))_N \geq 0$.

In particular, for period $N = 1$ appears the *positive invariant system*, that is,

$$x(k+1) = Ax(k) + Bu(k), \ \ k \in \mathbb{Z}_+, \quad (2)$$

where A and B are constant matrices with nonnegative entries. The system (2) is denoted by $(A,B) \geq 0$. Note that if the initial state vector is nonnegative, that is, $x_0 \geq 0$ and the input vector $u(k)$ is nonnegative for every $k \geq 0$, then the state vector $x(k)$ is also nonnegative in any other instant k.

2.1.1. *Algebraic characterizations of the reachability and controllability properties* In this subsection, we summarize the main results obtained for characterizing the structural properties of positive reachability, controllability and null-controllability for positive N-periodic linear systems and positive invariant systems.

Definition 1. A positive N-periodic system (1) is said to be

(a) *reachable at time s (from* 0*)* if, for any nonnegative state $x_f \in \mathbb{R}_+^n$, there exists a nonnegative input sequence transferring the state of the system from the origin at time s, $x(s) = 0$, to x_f in finite time. It is reachable if it is *reachable at time s*, for all $s \in \mathbb{Z}_+$.

(b) *(completely) controllable at time s* if, for any pair of nonnegative states x_0 and x_f, there exists a nonnegative input sequence transferring the state of the system from x_0 at time s, $x(s) = x_0$, to x_f in finite time. The system is controllable if it is *controllable at time s*, for all $s \in \mathbb{Z}_+$.

Note that if $N = 1$, we have the reachability and controllability concepts for the invariant case (see (Coxson and Shapiro, 1987)). It is worth noting that for positive systems, on the contrary to the general case, reachability from zero does not imply controllability to zero. Further, in this case, complete controllability is obtained only if one adds controllability to zero to reachability from zero (see (Coxson and Shapiro, 1987)).

Given a nonnegative pair $(A,B) \geq 0$ is established the following results (see for example (Caccetta and Rumchev, 2000)).

Theorem 1. The non-negative pair (A,B) is

(i) reachable if and only if the n-step reachability matrix $\mathfrak{R}_n(A,B) = [B, \ AB, A^2B, \ldots, \ A^{n-1}B]$ contains an $n \times n$ monomial submatrix;

(ii) null-controllable if and only if A is a nil-potent matrix;

(iii) controllable (in finite time) if and only if it is reachable and null-controllable.

For remarks regarding the proof of theorem 1 see (Caccetta and Rumchev, 2000). It is worth mentioning that to our knowledge an algebraic proof of the reachability criterion in part (i) of theorem 1 is not known to date. It is still an *open problem*.

When B is an $n \times 1$ matrix (column), denoted as b, (Coxson and Shapiro, 1987) have given an algebraic proof of the reachability criterion (i) in theorem 1 for non-negative matrices A containing a diagonal. They have further conjectured that this result holds for the more general case. (Coxson *et al.*, 1987) have proved the conjecture using a graph-theoretic technique. An algebraic proof of this conjecture is given by (Rumchev, 2000). (Fanti *et al.*, 1989) have also studied controllability and reachability from a graph-theoretic viewpoint and have obtained a similar criterion, namely the pair $(A,b) \geq 0$ is reachable if and only if (i) $\mathfrak{R}_n(A,b)$ is non-singular, and (ii) the inverse $\mathfrak{R}_n^{-1}(A,b) \geq 0$. Since the only class of nonnegative matrices which have non-negative inverses (see (Berman and Plemmons, 1994)) is the class of monomial matrices it is easy to see how (Fanti *et al.*, 1989) result relates to the reachability criterion in theorem 1(i). (Murthy, 1986) has obtained a criterion for the class of nonsingular positive single–input systems.

It is interesting to mention the equivalence between reachability and the totally oscillatory behavior of single-input systems single-output positive systems revealed in (Rumchev and James, 1995a), and the relation between the famous Farcas' Lemma and the reachability properties of positive systems, (see (Rumchev and James, 1989)).

2.1.2. *Digraph characterizations of reachability and controllability properties* In this subsection we collect the characterization results of reachability and controllability in terms of the digraph of the matrix system A. First we recall the basic combinatorial concepts used in the results.

Let $D(A)$ be the digraph of an $n \times n$ non-negative matrix A constructed as follows. The set of vertices of $D(A)$ is denoted as $N = \{1, 2, \ldots, n\}$. There is an arc (i, j) in $D(A)$ if and only if $a_{ji} > 0$. The set of all arcs is denoted by U. A *walk* in D(A) is an alternating sequence of vertices and arcs. A walk is called *closed* if the initial and final vertices coincide and *spanning* if it passes through all the vertices of $D(A)$. A walk is said to be a *path* if all of its vertices are distinct, and a *cycle* if it is a closed path. The path length is defined to be equal to the number of arcs it contains. The number of arcs away from a vertex i is called *outdegree of i* and is written $\mathrm{od}(i)$, whilst the number of arcs directed toward a vertex i is called *indegree of i* and is written $\mathrm{id}(i)$ (see (Foulds, 1992)). Notice that zero columns in A correspond to vertices j with $\mathrm{od}(j) = 0$ in $D(A)$; respectively, zero rows correspond to vertices with $\mathrm{id}(i) = 0$. The positive entries in the columns of $B \geq 0$ are identified with the corresponding vertices in $D(A)$.

We distinguish the following *monomial components* of a digraph: simple monomial paths (s.m.p.), blossoms and bunches. A path $\{i_1, i_2, \ldots, i_k, i_{k+1}\}$ is called i_1-*monomial path of length k* if and only if all outdegrees $\mathrm{od}(i_s) = 1$, for $s = 1, \ldots, k$. This notion is the same of deterministic path used in (Valcher, 1996) and (Bru *et al.*, 2000a). Note that the outdegree $\mathrm{od}(i_{k+1})$ of a monomial path is not specified - it can be any. An i_1-monomial path is called *simple* if $\mathrm{od}(i_{k+1}) = 0$ and $\mathrm{id}(i_1) = 0$. The vertex $i_1(s)$ is called *origin*, and the vertex i_{k+1} with zero outdegree - end of the s.m.p. An *isolated vertex* j is a particular kind of s.m.p. of length zero with $\mathrm{id}(j) = \mathrm{od}(j) = 0$. A walk $\{i_1, i_2, \ldots, i_k, i_{k+1}, i_{k+2}\}$ is said to be a *blossom* if all its vertices i_s, $s = 1, 2, \ldots, k, k+1$, are different and $i_{k+2} = i_s$ for some s. Any blossom contains a cycle. Obviously, the *cycle* is a particular kind of blossom with $i_{k+2} = i_1$. The blossom becomes a s. m. p. of length k if the arc (i_{k+1}, i_{k+2}) is removed from it. A *bunch* is a union of a blossom (possibly, a cycle) and monomial paths (possibly, s. m. p.) joined with the tops only to the vertices of the cycle of the blossom. Clearly, $\mathrm{od}(s) = 1$ for $s \in G_j$ but $\mathrm{id}(s) = 0$ if s is an origin and $\mathrm{id}(s) \geq 01$ if s is any other vertex of the bunch.

A canonical decomposition of the digraph $D(A)$ into monomial components has been found recently in (Rumchev, 2000). The idea of decomposing is the following: If all of the outward arcs from vertices with $\mathrm{od}(i) \geq 2$ in $D(A) = (N, U)$ are removed from $D(A)$ then the reduced digraph $D^{(0)}(A_u) = (N, U^{(0)})$ becomes a union of disjoint monomial structures (s.m.p., blossoms or cycles, monomial trees and bunches) since $\mathrm{od}(i) < 2$ for any vertex $i \in D^{(0)}(A_u)$. Then, by using procedures MONTREE and BUNCH (see (Caccetta and Rumchev, 1998)) the digraph $D^{(0)}(A_u)$ can be reduced to a union $D(A_0)$ of disjointed canonical monomial components. The matrix A_0 of the reduced digraph $D(A_0)$ is contained in A. It represents the canonical monomial components of A only.

Lemma 1. If $\mathrm{od}(i) \geq 1$ for $i \in D(A)$. Then a non-monomial column b cannot generate monomial columns in the sequence (3).

Lemma 2. Let $\mathrm{od}(i) \geq 1$ for $i \in D(A)$. Then the digraphs $D(A)$ and $D(A_0)$ have the same monomial structure, that is a monomial column b generates the same sequence of linearly independent monomials

$$b, Ab, A^2b, \ldots, A^kb \tag{3}$$

and $b, A_0b, A_0^2b, \ldots, A_0^kb$.

From the above lemmas, the monomial behavior of the sequence $B, AB, A^2B, \ldots, A^kB$, is completely determined. The next result is a reachability criterion in digraph form (see (Rumchev, 2000)).

Theorem 2. Let $A \geq 0$, and suppose that the associated digraph $D(A)$ has no vertices with $\mathrm{od}(i) = 0$. Let also $I_1 = \{i_1(1), i_1(2), \ldots, i_1(m)\}$ and $J_1 = \{j_1(1), j_1(2), \ldots, j_1(s)\}$ be, respectively, the sets of all origins (of monomial paths and blossoms) and any set of vertices such that $j_1(k) \in C_k$ for $k = 1, 2, \ldots, s$, where C_k are disjointed cycles in the reduced digraph $D(A_o)$. Then the pair $(A, B) \geq 0$ is reachable if and only if the matrix B contains a monomial submatrix $B_0 = DE$ where D is a diagonal matrix and $E = \mathrm{diag}\left[e_{i_1^1} | \cdots | e_{i_1}^{\mu} | e_{j_1^1} | \cdots | e_{j_1^\sigma}\right]$.

For null-controllability property in digraph form is given also in (Rumchev, 2000), that is, the pair $(A, B) \geq 0)$ is null-controllable if and only if there are no cycles in the digraph $D(A)$.

The particular case of theorem 2 for single–input systems has been established by (Coxson *et al.*, 1987). In fact, they give te following result.

Theorem 3. Let $(A, b) \geq 0$. Then, this system is reachable if and only if b is i-monomial and the $D(A)$ is a union of an i-monomial path of length $n-1$ and, possibly, arcs $\{(i_n,i),\ i=1,\ldots,n\}$.

The particular digraph $D(A)$ of the above theorem is called *a palm* in (Bru *et al.*, 2001b), where the authors introduced new monomial components for studying the characterization of reachability of a general pair $(A,B)\geq 0$. In fact, they use the following monomial structures: (i) *monomial trees*: a digraph T is called a *monomial tree* if it is a union of different monomial paths, originating at different vertices connected among them from the last vertices only without forming cycles, and containing at least a single monomial path. In particular, a single monomial path is considered as a monomial tree; (ii) *flowers*: a digraph F is said to be a *flower* if it consists of a monomial path, of lenth $p-1$, $(i_1,i_2,\ldots,i_p)$ linked to a cycle $(i_{p+1},i_{p+2},\ldots,i_{p+k+1})$, such that from the vertex i_p of the monomial path, there are arcs (i_p,T) leading to vertices of the monomial tree T, in addition to the arc (i_p,i_{p+1}); (iii) *palms*: A digraph P is called a *palm* if it consists of a monomial path $(i_1,i_2,\ldots,i_p)$ and, at least one arc of the types $(i_p,i_k),\ k=1,\ldots,p$, (i_p,F) leading to vertices of the flower F, or (i_p,C) leading to vertices of the cycle C.

Consider the nonnegative pair (A,B) and the associated digraph $D(A)$. Recall that the positive entries of the monomial columns of B are identified with the corresponding vertices in $D(A)$ called origins. From these origins, construct the monomial structures, without repeating vertices, as follows: (a) all possible monomial trees. The initials vertices of all monomial paths of the monomial trees form the index set of origins $\mathscr{T}$; (b) all possible flowers. The initials vertices of all monomial paths of the flowers form the index set of origins $\mathscr{F}$; (c) all possible palms. The initials vertices of all monomial paths of the palms form the index set of origins $\mathscr{P}$, and (d) all possible (non) monomial cycles from the nonmonomial columns of B $b_{l_r}=e_{l_r}+w$, where the positive components, if there are, of vector w are identified with vertices in a monomial tree. Indices l_r form the set of origins $\mathscr{C}$.

Then, every vertex of $D(A)$ can belong to exactly one monomial subgraph. Let $D'(A)$ denote the digraph of A formed from the union of all these monomial subgraphs. Note that $D'(A)$ is a spanning subgraph of $D(A)$ and the only arcs of $D(A)$ that are not in $D'(A)$ are those connecting two monomial structures. Let L be the set of such arcs. Then,

$$D(A)=D'(A)\bigcup L$$

Theorem 4. Let $A\geq 0$ and let $D(A)$ be the associated digraph. Let $\mathscr{T}$, $\mathscr{F}$, $\mathscr{P}$, $\mathscr{C}$ the index set of the origins of the monomial subgraph, respectively, monomial trees T, flowers F, palms P, and (non) monomial cycles C of $D(A)$. Then, the pair (A,B) is reachable if and only if $D'(A)$ is a union of these monomial subgraphs, that is:

$$D'(A)=\bigcup_{t=1}^{c_t} T_t \bigcup_{f=1}^{c_f} F_f \bigcup_{p=1}^{c_p} P_p \bigcup_{c=1}^{c_p} C_c$$

where c_t,c_f,c_k and c_p stand for the number of monomial trees, flowers, palms and (non) monomial cycles, respectively.

Theorem 4 gives all possible reachable monomial components for any general system $(A,B)\geq 0$. (Bru *et al.*, 2000a) study that characterization in a different manner. They work with the concept of deterministic path already used in (Valcher, 1996). They establish a reachability criteria from the deterministic paths originating at vertices corresponding to monomial columns of the matrix B and related cycles. Namely, this characterization is in terms of specific subsets of vertices covering the whole set of vertices of $D(A)$. These subsets of $D(A)$ are basic to construct the canonical forms which are described in section 4.

From these characterizations given in terms of the digraph $D(A)$, it could be applied some computational algorithms to decide whether or not a positive pair (A,B) is reachable. For the case of theorem 2, that is when the matrix A has no zero columns that algorithm has been proposed, as it was pointed out in (Rumchev, 2000). However, it remains an *open problem* to construct a computational algorithm for the general case of theorem 4.

2.2 *Continuous-time systems*

There are not many results on continuous-time positive systems, there are only some sufficient conditions for the reachability property detailed below.

Definition 2. The system

$$\dot{x}=Ax+Bu \tag{4}$$

is called *positive* if for $u(t)\geq 0$ the system trajectory $x(t)\geq 0$ is always non-negative for $t\geq 0$ whenever the initial state $x(0)\geq 0$.

Definition 3. The positive invariant system (4) is called *reachable* if for any state $x\geq 0$ there exist a finite t and a non-negative control vector $u(\tau)\geq 0$, t $\tau\in[0,t]$ that transfers the system from the origin $x(0)=0$ to the state $x=x(t)$.

Early results for single-input positive systems can be found in Ohta, Maeda and Kodama (1984). As a matter of fact they do not provide criterion for testing the reachability (controllability) property of the system. Their result can hardly be used to identify such a property.

(Kaczorek, 2001) quite recently has found sufficient conditions for reachability of the system (4).

Theorem 5. The positive invariant system (4) is reachable (from the origin) in time t if the matrix

$$R_t = \int_0^t e^{A\tau} B B^t e^{A^T \tau}\, d\tau.$$

(T denotes the transpose) is a monomial matrix. Moreover, the control vector that steers the system (4) from $x(0) = 0$ to the state $x \geq 0$ is given by the expression

$$u(t) = B^T e^{A^T t} R_t^{-1} x, t \geq 0.$$

Theorem 6. The positive system (4) is reachable in time t if A is a diagonal matrix and $B \geq 0$ is a monomial matrix.

The authors do not know other results on reachability (controllability) of continuous-time positive systems. The proof of necessity is still an *open problem*.

3. PERIODIC SYSTEMS

3.1 *Discrete-time systems*

A *positive N-periodic discrete-time linear control system*

$$x(k+1) = A(k)x(k) + B(k)u(k), k \in \mathbb{Z}_+, \quad (5)$$

is equivalent to N *positive invariant systems* defined by

$$\begin{aligned} x_s(k+1) &= A_s x_s(k) + B_s u_s(k), \\ s &= 0, 1, \ldots, N-1 \end{aligned} \quad (6)$$

where

$$\begin{aligned} &A_s = \phi_A(s+N, s) \\ &B_s = [B(s+N-1), \phi_A(s+N, s+N-1)B(s+N+2), \\ &\ldots, \phi_A(s+N, s+1)B(s)], s = 0, 1, \ldots, N-1. \end{aligned}$$

By $\phi_A(k, k_0)$, we denote the *transition matrix* of the system (5),

$$\begin{aligned} &\phi_A(k, k_0) = A(k-1)A(k-2)\cdots A(k_0),\ k > k_0, \\ &\phi_A(k_0, k_0) = I. \end{aligned}$$

This equivalence was proved independently in (Bittanti and Bolzern, 1985) and (Hernández and Urbano, 1987), where

$$\begin{aligned} &x_s = x(kN+s) \\ &u_s(k) = \mathrm{col}\left[u(kN+s+N-1), \ldots, u(kN+s)\right], \end{aligned}$$

are the relationships between the state and input vectors of the two systems.

The state $x(k)$ of the system (5) at the time k when is applied the control sequence $u(k_0)$, $u(k_0 + 1), \ldots, u(k-1)$, from the initial state x_0 at time k_0 is given by

$$\begin{aligned} x(k) = &\phi_A(k, k_0)x_0 + \\ &+ \sum_{j=k_0}^{k-1} \phi_A(k, j+1)B(j)u(j),\ k \geq k_0 \end{aligned}$$

With the positive restrictions, $x_0 \geq 0$, $x_f \geq 0$, $u(j) \geq 0$, $j = s-k_f, s-k_f+1, \ldots, s-1$, $A(k) \geq 0$ and $B(k) \geq 0$ for all $k = 0, 1, \ldots, N-1$, the structural properties of the systems (5) and (6) are related, by means of the above equivalence, as follows.

Proposition 1. (see (Bru and Hernández, 1989)) The positive periodic system (5) is completely positive controllable at s if and only if the positive invariant system (6) corresponding to index s is completely positive controllable, for every $s = 0, 1, \ldots, N-1$.

Moreover, for every $s = 0, 1, \ldots, N-1$, all system (5) completely controllable at s is completely reachable at s. And, for all system (5) completely reachable at s, the k-step reachability matrix of the associated invariant system (6) given by

$$C_k^{(s)} = \left[B_s, A_s B_s, \ldots, A_s^{k-1} B_s\right]_{n \times kNm}$$

has rank n for some k. This assertions are equivalent in the case without positive restrictions (see (Urbano, 1987)).

Reachability and complete controllability properties were studied in terms of the cones of reachability.

Definition 4. For each $s = 0, 1, \ldots, N-1$:

(i) The k-reachable cone at s of the positive periodic system (5), $R_k(A(\cdot), B(\cdot), s)$, is defined as the set of reachable states in k-steps from zero at time s.
(ii) The k-reachable cone of the positive invariant system (6), $R_k(A_s, B_s)$, corresponding to index s is defined as the set of reachable states in k-steps from zero at time zero.

The above cones are related as follows.

Lemma 3. (see (Bru and Hernández, 1989)) For each $s = 0, 1, \ldots, N-1$:

$$R_k(A(\cdot), B(\cdot), s) = R_p(A_s, B_s),\ k = pN.$$

Remark 1. For each $s = 0, 1, \ldots, N-1$:

(i) For the positive periodic system (5), the set of states which are reachable at s in finite time with nonnegative inputs is given by

$$R_\infty(A(\cdot), B(\cdot), s) = \bigcup_{k \in \mathbb{N}} R_k(A(\cdot), B(\cdot), s)$$

(ii) For the positive invariant system (6), corresponding to index s, the set of states which are reachable in finite time with nonnegative inputs is given by

$$R_\infty(A_s, B_s) = \bigcup_{k\in\mathbb{N}} R_k(A_s, B_s)$$

Therefore, completely reachable at time s of the system (5) is equivalent to $R_\infty(A(\cdot), B(\cdot), s) = \mathbb{R}^n_+$. Hence, following result holds.

Proposition 2. (see (Bru and Hernández, 1989)) The positive periodic system (5) is completely positive controllable at s if and only if $\phi_A(s+N, s)$ is nilpotent and $R_\infty(A(\cdot), B(\cdot), s) = \mathbb{R}^n_+$.

Till now, we have seen characterizations of the structural properties of a positive periodic system (5) using N positive invariant systems (6). In the following, we indicate new characterizations of the structural properties of such system (5) obtained by an associated positive invariant system, the *positive invariant cyclically augmented system*.

Park and Verriest (see (Park *et al.*, 1989)) introduced the *positive invariant cyclically augmented system* associated with a positive N-periodic system (5), which is given by

$$z(k+1) = A_e z(k) + B_e u_e(k), \qquad (7)$$

where $A_e \in \mathbb{R}^{nN\times nN}_+$ is weakly cyclic of index N (see (Varga, 1962)), that is,

$$A_e = \begin{bmatrix} O & A(0) \\ A & 0 \end{bmatrix},$$

with $A = \text{diag}[A(1), \ldots, A(N-1)]$ and $B_e = \text{diag}[B(0), B(1), \ldots, B(N-1)] \in \mathbb{R}^{nN\times mN}_+$. Moreover, the state vector and the input vector of system (7) are associated with the stacked vectors of the inputs and the states of (5), $\hat{x}(k) = \text{col}\,[x(k), x(k+1), \ldots, x(k+N-1)]$ and $\hat{u}(k) = \text{col}\,[u(k), u(k+1), \ldots, u(k+N-1)]$, by the following relations

$$z(k) = M_n^{k-1}\hat{x}(k), \;\; u_e(k) = M_m^k \hat{u}(k),$$

where

$$M_j = \begin{bmatrix} O & I_j \\ I_{(N-1)j} & 0 \end{bmatrix},$$

and I_q is the identity matrix of order q. We denote the invariant system given in (7) by (A_e, B_e).

As the invariant system (7) is constructed from the associated periodic system (5) then their respective cones can be related as follows.

Proposition 3. (see (Bru *et al.*, 1997)) Let $x \in \mathbb{R}^{nN}_+$, where $x = \text{col}[x^1 x^2 \cdots x^N]$ and $x^j \in \mathbb{R}^n_+$. Then, $R_k(A_e, B_e)$ if and only if $x^j \in R_k(A(\cdot), B(\cdot), j)$, $j = 1, \ldots, N$.

Hence,

Corollary 1. (see (Bru *et al.*, 1997)) For each $k \in \mathbb{Z}$, $R_k(A_e, B_e) = \mathbb{R}^n_+$ if and only if $R_k(A(\cdot), B(\cdot), j) = \mathbb{R}^n_+$, $\forall s \in \mathbb{Z}$.

Therefore, the positive periodic system (5) is (completely controllable) reachable if and only if the positive invariant system (7) is (completely controllable) reachable (see (Bru *et al.*, 1997) and (Romero, 2001)).

3.2 *Continuous-time periodic systems*

To our knowlegde, there are not any result on continuous-time periodic linear systems. Then, many *open problems* remain to work in this case.

4. CANONICAL FORMS

In (Valcher, 1996) was obtained a first general canonical form of reachability property of a positive invariant system (2) with state matrix devoid of zero columns.

Theorem 7. Let A be an $n \times n$ positive matrix devoid of zero columns. The positive system $(A, B) \geq 0$ is reachable if and only if there exist permutation matrix, P and Q, of suitable dimensions, such that

$$[P^T AP |\, P^T BQ] =$$

$$\left[\begin{array}{ccccccccccccc|cccc}
* & + & \ldots & 0 & * & 0 & \ldots & 0 & & * & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ldots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
* & 0 & \ldots & + & * & 0 & \ldots & 0 & & * & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
* & 0 & \ldots & 0 & * & 0 & \ldots & 0 & & * & 0 & \ldots & 0 & + & 0 & \ldots & 0 \\
\hline
* & 0 & \ldots & 0 & * & + & \ldots & 0 & & * & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ldots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
* & 0 & \ldots & 0 & * & 0 & \ldots & + & & * & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
* & 0 & \ldots & 0 & * & 0 & \ldots & 0 & & * & 0 & \ldots & 0 & 0 & + & \ldots & 0 \\
\hline
 & & \vdots & & & & \vdots & & \ddots & & & \vdots & & \vdots & \vdots & \vdots & \vdots \\
\hline
* & 0 & \ldots & 0 & * & 0 & \ldots & 0 & & * & + & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ldots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
* & 0 & \ldots & 0 & * & 0 & \ldots & 0 & & * & 0 & \ldots & + & 0 & 0 & \ldots & 0 \\
* & 0 & \ldots & 0 & * & 0 & \ldots & 0 & & * & 0 & \ldots & 0 & 0 & 0 & \ldots & +
\end{array}\right] \hat{B}$$

From this result a characterization of single-input positive system was given in (Valcher, 1996).

Theorem 8. The single-input positive invariant system $(A, b) \geq 0$ is reachable if and only if there exists permutation matrix P such that

$$[P^T AP | P^T g] = \left[\begin{array}{ccccc|c}
* & + & & & & 0 \\
* & & + & & & 0 \\
\vdots & & & \ddots & & \vdots \\
* & & & & + & 0 \\
* & & \cdots & & 0 & +
\end{array}\right]$$

In theorem 1 of (Caccetta and Rumchev, 1998) was proved that this permutation matrix is unique. The above proposition was reformulated in the following graph-theoretic form.

Theorem 9. (see (Caccetta and Rumchev, 1998)) The pair $(A,b) \geq 0$ is reachable if and only if b is an i_1-monomial and the digraph $D(A)$ of A is a union of an i_1-monomial path of length $n-1$ that spans all the vertices, i. e., an i_1-monomial spanning path, and possibly, arcs $\{(i_n,i),\ i=1,\ldots,n\}$

When considering systems without nonnegative restrictions, we know that the reachability and complete controllability properties are transferred under similar transformations. However, in the positive case, these properties can be transferred only under special matrices for preserving the positive restrictions.

Theorem 10. (see (Romero, 2001) and (Bru *et al.*, 2001a)) Let $(F,G) \geq 0$ be a reachable positive invariant system similar to $(\hat{F},\hat{G}) \geq 0$, where $\hat{F} = M^{-1}FM$ and $\hat{G} = M^{-1}G$. Then, $(\hat{F},\hat{G}) \geq 0$ positively reachable if and only if M is a monomial matrix

The above property has represented an important advantage in this area because has allowed to obtain general canonical forms of the reachable and controllable positive invariant systems. In addition that property, give rise to *open questions* such as the study of invariants of a system.

Given a positive invariant system (2), in (Bru *et al.*, 2000a) canonical forms of reachability and controllability properties were obtained. These are based on characterizations of the digraph of A that was held previously.

Theorem 11. Given a positive invariant system (2). Then, (A,B) is reachable if and only if there exist permutation matrices P and Q such that the matrix $[P^TAP|P^TBQ]$ has the following structure

$$[P^TAP|P^TBQ] =$$

$$\left[\begin{array}{c|c|c|c|c||c|c|c|c|c} \mathscr{C} & O & O & O & \Delta_{n_{R_1}} & O & O & O & \mathscr{G}_{\mathscr{C}} & \\ \hline O & \mathscr{B} & O & \Sigma & \Delta_{n_{R_2}} & O & O & O & O & \\ \hline O & O & \mathscr{A}' & \Delta_{n_B} & \Delta_{n_{R_3}} & O & O & \mathscr{G}_{\mathscr{A}'} & \mathscr{G}_{\mathscr{A}'_C} & \mathscr{G}_R \\ \hline O & O & O & \mathscr{A}_{n_B} & \Delta_{n_{R_4}} & O & \mathscr{G}_{n_B} & O & O & \\ \hline O & O & O & O & \mathscr{A}_{n_R} & \mathscr{G}_{n_R} & O & O & O & \end{array}\right] \quad (8)$$

where

- $\mathscr{C}$ and $\mathscr{B}$ are block diagonal matrices with irreducible cyclically blocks, that is,

$$\begin{bmatrix} 0 & + & & \\ 0 & & + & \\ \vdots & & & \ddots \\ 0 & & & & + \\ + & & \cdots & & 0 \end{bmatrix} \quad (9)$$

- $\Sigma = diag\,[\Phi,\ldots,\Phi]$ and $\Delta_{n_B} = \Delta_{n_{R_j}} = diag\,[\Psi,\ldots,\Psi]$, $j=1,2,3,4$ with

$$\Phi = \begin{bmatrix} 0 & 0 & & \\ 0 & & 0 & \\ \vdots & & & \ddots \\ 0 & & & & 0 \\ + & & \cdots & & 0 \end{bmatrix}, \Psi = \begin{bmatrix} * & 0 & & \\ * & & 0 & \\ \vdots & & & \ddots \\ * & & & & 0 \\ * & & \cdots & & 0 \end{bmatrix}. \quad (10)$$

- $\mathscr{A}'$ is a block upper triangular matrix as follows

$$\mathscr{A}' = \left[\begin{array}{c|c|c|c|c} \mathscr{A}_1 & \Delta & \cdots & \Delta & \Delta \\ \hline O & \mathscr{A}_2 & \cdots & \Delta & \Delta \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline O & O & \cdots & \mathscr{A}_{n-2} & \Delta \\ \hline O & O & \cdots & O & \mathscr{A}_{n-1} \end{array}\right], \quad (11)$$

  where each $\mathscr{A}_j$, for $j=0,1,\ldots,n-1$ is a block diagonal matrix, where each block is

$$\begin{bmatrix} 0 & + & & \\ 0 & & + & \\ \vdots & & & \ddots \\ 0 & & & & + \\ 0 & & \cdots & & 0 \end{bmatrix}. \quad (12)$$

  and the matrices Δ are in the same way that the matrices Δ_{n_B}.
- $\mathscr{A}_{n_B}$ is a block diagonal matrix with blocks in the same that (12).
- $\mathscr{A}_{n_R}$ is a block matrix where all off-diagonal blocks are Ψ and the blocks in the diagonal are given by

$$\begin{bmatrix} * & + & & \\ * & & + & \\ \vdots & & & \ddots \\ * & & & & + \\ * & & \cdots & & 0 \end{bmatrix}.$$

- $\mathscr{G}_{\mathscr{A}'_C}$ is a nonnegative matrix with at least one positive entry in each one of their columns. $\mathscr{G}_C$, $\mathscr{G}_{A'}$, $\mathscr{G}_{n_B}$ y $\mathscr{G}_{n_R}$ are in the same way,

$$\begin{bmatrix} O & O & \cdots & \mathscr{M} \\ \vdots & \vdots & & \vdots \\ O & \mathscr{M} & \cdots & O \\ \mathscr{M} & O & \cdots & O \end{bmatrix} \quad (13)$$

  where the blocks $\mathscr{M}$ is formed by a unique column of the type col$\,[0\ 0\ \ldots 0\ +]$.
- Finally, the submatrix $\mathscr{G}_R$ contains the remaining columns of the matrix B ordered by the permutation matrices P and Q.

From this canonical form of reachability and adding the condition of nilpotence of the state matrix the following canonical form of controllability was obtained in (Bru *et al.*, 2000a).

Theorem 12. Given a positive invariant system (2). Then, (A,B) is completelly controllable if and only if there exist permutation matrices P and Q such that the matrix $[P^TAP|P^TBQ]$ has the following structure

$$[P^TAP|P^TBQ] = \left[\, \mathscr{A}' \| \mathscr{G}_{\mathscr{A}'} | \mathscr{G}_R \,\right] \qquad (14)$$

where

- $\mathscr{A}'$ is a block upper triangular matrix as in (11).
- $\mathscr{G}_{A'}$ is a matrix of the type given in (13).
- Finally, in $\mathscr{G}_R$ remain the rests columns of the matrix B ordered depending on the permutation matrices P and Q.

The study of assignment-pole by means of feedbacks in the positive case can be developed in a more detailed manner using these canonical forms. This question is a *open problem*.

5. ESSENTIAL REACHABILITY AND CONTROLLABILITY

In (Coxson and Shapiro, 1987), it was pointed out that for some positive systems the set of states which are reachable in finite time with nonnegative inputs is not equal to $\mathbb{R}^n_+$, however, each one of the nonnegative states not reachable in a finite time can be limit of a sequence of nonnegative reachable states in a finite time. This fact motivated the introduction of new structural properties which were introduced in the invariant case in (Coxson and Shapiro, 1987) and in the periodic case in (Bru and Hernández, 1989).

Next, we give the following concepts for N-periodic systems.

Definition 5. A positive periodic discrete-time linear system (5) is said to be

(a) *essentially reachable at time s* if for every positive final state $x_f \gg 0$, there exists a nonnegative input sequence transferring the state of the system from the origin at time s, $x(s) = 0$ to x_f in a finite time. It is *essentially reachable* if it is essential reachable at time s, for all $s \in \mathbb{Z}$.
(b) *essentially (completely) controllable at time s* if for every pair of nonnegative states $x_0 \geq 0$ and $x_f \gg 0$ e there exist a nonnegative input sequence transferring the state of the system from x_0 at time s, $x(s) = x_0$, to x_f in a finite time. It is *essentially (completely) controllable* if it is essentially controllable at time s, for all $s \in \mathbb{Z}$.
(c) *Asymptotically zero-controllable at time s* if if for every nonnegative state $x_0 \geq 0$ there exist a nonnegative input sequence transferring the state of the system from x_0 at time s, $x(s) = x_0$, to origin. It is *asymptotically zero-controllable* if it is asymptotically zero-controllable at time s, for all $s \in \mathbb{Z}$.

In particular, for period $N = 1$, the above definitions correspond with the definitions of essentially reachable, essentially controllable and asymptotically zero-controllable for positive invariant systems 2 (see (Coxson and Shapiro, 1987) and (Valcher, 1996)).

A similar property of essential reachability is the excitability property. (Muratori and Rinaldi, 1991) have given a characterization of excitability of single–input systems, that is, reachability for only strictly positive state vectors x, with $x_i > 0$, for all $i = 1, 2, \ldots, n$. In addition, they relate this property to stability, as

Theorem 13. For a single–input positive invariant system (A,b), consider these three properties:

(a) (A,b) has a strictly positive non–trivial equilibria;
(b) (A,b) is excitable;
(c) (A,b) is stable.

Then, any pair of properties (a), (b) and (c) implies the third.

5.1 *Positive invariant discrete-time systems*

Consider an invariant system (2). In (Coxson and Shapiro, 1987) the above properties are related in the following way.

Theorem 14. (see (Coxson and Shapiro, 1987)) A positive periodic system (5) is essentially controllable if and only if it is asymptotically zero-controllable and essentially reachable.

The proposition 4 indicates that the structural property of essential controllability of a positive invariant system (2) depends on the spectrum of the state matrix. In (Coxson and Shapiro, 1987), it was proved that essential reachability is equivalent to reachability if the state matrix is irreducible and primitive. (Valcher, 1996) extended this result to the class of irreducible matrices (not necessarily primitive). Another interesting result is given in (Coxson and Shapiro, 1987) revealing the behavior of the columns of the reachability matrix $\mathfrak{R}_t(A,B)$ for the class of irreducible matrices and large t.

In (Valcher, 1996), it was obtained a first approximation by means of the directed-graph theory of the essentially reachable property of a positive invariant system (2). That characterization was realized in terms of the communicating classes of the directed-graph of the state matrix.

Theorem 15. (see (Valcher, 1996)) Let $(A,B) \geq 0$. Then the following facts are equivalent:

(i) (A,B) is essentially reachable;
(ii) for every vertex $i \notin I(A,B)$, where $I(A,B)$ is the set of indices of all the monomial columns in the

reachability matrix $\Re_n(A,B)$, the following facts hold:

(a) i belongs to some closed communicating class C_{j_i} of D(A), which consists either of the single vertex i or of h_i vertices connected by a single cycle;

(b) there exists some column vector b_{τ_i} in B such that for every positive integer $t \geq n$ the block of components of $A^t b_{\tau_i}$ corresponding to C_{j_i}, that is, $\text{block}_{C_{j_i}}(F^t b_{\tau_i})$, constitutes a monomial vector. Moreover, each class $C \neq C_{j_i}$ such that $\text{block}_C(F^t b_{\tau_i}) > 0$, for some $t \in \mathbb{N}$, has a spectral radius not greater than the spectral radius of C_{j_i}, and if it coincides with the spectral radius of C_{j_i}, then the class C has access to C_{j_i};

(iii) for every $i \notin I(A,B)$ there exist $\tau_i \in \{1,2,\ldots,m\}$ and an integer h_i with $0 \leq c_i < h_i$ such that

$$\lim_{t\to\infty} \frac{A^{c_i+th_i} b_{\tau_i}}{||A^{c_i+th_i} b_{\tau_i}||_\infty} = e_i.$$

New combinatorial characterizations was given in (Bru *et al.*, 2000a) using the above theorem and hence, canonical forms of essentially reachable and controllable positive invariant system which are presented in the following result.

Theorem 16. Given a positive invariant system (2). Then, (A,B) is essentially reachable if and only if there exist permutation matrices P and Q such that the matrix $[P^T AP | P^T BQ]$ has the following structure

$$\left[\begin{array}{cccccccc|c} \mathscr{D}_2 & O & O & O & O & O & \Delta & \Delta & \\ O & \mathscr{C} & O & O & O & O & \Delta & \Delta & \\ O & O & \mathscr{B} & O & O & \Sigma & \Delta & \Delta & \\ O & O & O & \mathscr{D}_1 & O & O & \Sigma & \Delta & \\ O & O & O & O & \mathscr{A}' & \Delta & \Delta & \Delta & \mathscr{G} \\ O & O & O & O & O & \mathscr{A}_{n_B} & \Delta & \Delta & \\ O & O & O & O & O & O & \mathscr{A}_{n_{D_1}} & \Delta & \\ O & O & O & O & O & O & \Delta & \mathscr{A}_{n_R} & \end{array}\right] \quad (15)$$

where $\mathscr{C}$, $\mathscr{B}$, $\mathscr{A}'$, $\mathscr{A}_{n_B}$, $\mathscr{A}_{n_R}$, Δ and Σ have the same structure that in the case for the reachability property given in (8). Moreover, $\mathscr{A}_{n_{D_1}}$ is a block diagonal matrix with blocks given in (12) and $\mathscr{D}_1$ and $\mathscr{D}_2$ are block diagonal matrices with cyclically irreducible blocks given in (9).

In addition,

$$\mathscr{G} = \left[\begin{array}{cccccc|c} O & O & O & O & O & \mathscr{G}_{\mathscr{D}_2} & \\ O & O & O & O & \mathscr{G}_{\mathscr{C}} & \mathscr{G}_{\mathscr{C}_{D_2}} & \\ O & O & O & O & O & \mathscr{G}_{\mathscr{B}_{D_2}} & \\ O & O & O & O & O & \mathscr{G}_{\mathscr{D}_{1D_2}} & \\ O & O & O & \mathscr{G}_{\mathscr{A}'} & \mathscr{G}_{\mathscr{A}'_C} & \mathscr{G}_{\mathscr{A}'_{D_2}} & \mathscr{G}_R \\ O & O & \mathscr{G}_{n_B} & O & O & \mathscr{G}_{n_{BD_2}} & \\ O & \mathscr{G}_{n_{D_1}} & O & O & O & \mathscr{G}_{n_{D_{1D_2}}} & \\ \mathscr{G}_{n_R} & O & O & O & O & \mathscr{G}_{n_{rD_2}} & \end{array}\right]$$

where $\mathscr{G}_{\mathscr{A}'_C}$, $\mathscr{G}_C$, $\mathscr{G}_{A'}$, $\mathscr{G}_{n_B}$ and $\mathscr{G}_{n_R}$ have the same structure given in (8) for the reachability property.

Moreover $\mathscr{G}_{D_2}$ is a block matrix which has the structure given in (13) and $\mathscr{G}_{\mathscr{C}_{D_2}}$, $\mathscr{G}_{\mathscr{B}_{D_2}}$, $\mathscr{G}_{\mathscr{D}_{1D_2}}$ $\mathscr{G}_{\mathscr{A}'_{D_2}}$, $\mathscr{G}_{n_{BD_2}}$, $\mathscr{G}_{n_{D_{1D_2}}}$, and $\mathscr{G}_{n_{rD_2}}$ are nonnegative matrices whose entries have different combinatorial restrictions. In addition, there are spectral conditions on the submatrices associated with the suitable communicating classes (see (Bru *et al.*, 2000a)).

There exist different restrictions on the nonzero entries of this canonical form. Moreover, adding to the above structure of the state matrix the stability condition, then a canonical form of essential controllable positive invariant system (2) is obtained . All this comments are explained in (Bru *et al.*, 2000a).

The precedent canonical forms, for single-input positive systems are similar to those given in (Valcher, 1996).

The preservation of the essential reachability property under feedbacks is a interesting *open problem* to analyze using diferent kinds of control, with nonnegative restriction or not.

5.2 *Positive periodic discrete-time systems*

Consider a positive N-periodic system. The set of states which are reachable at s, in finite time, with nonnegative inputs is given by $R_\infty(A(\cdot),B(\cdot),s)$. To obtain nonnegative states as limits of states in $R_\infty(A(\cdot),B(\cdot),s)$, the closure of this set is defined.

Definition 6. For each $s = 0,1,\ldots,N-1$:

(i) The *reachability cone at s* of the positive periodic system (5) is defined as the closure of $R_\infty(A(\cdot),B(\cdot),s)$, that is,

$$R(A(\cdot),B(\cdot),s) = \overline{R_\infty(A(\cdot),B(\cdot),s)}$$

(ii) The positive periodic system (5) is *essentially reachable at time s* when

$$R(A(\cdot),B(\cdot),s) = \mathbb{R}^n_+$$

(iii) The *reachable cone* of the positive invariant system (6) corresponding to index s is defined as the closure of $R_\infty(A_s,B_s)$, that is,

$$R(A_s,B_s) = \overline{R_\infty(A_s,B_s)}$$

(iv) The positive invariant system (6) is *essentially reachable* when

$$R(A_s,B_s) = \mathbb{R}^n_+$$

The next lemma gives the relationship between the reachable cones previously defined.

Lemma 4. (see (Bru and Hernández, 1989)) For each $s = 0, 1, \ldots, N-1$,

$$R(A(\cdot), B(\cdot), s) = R(A_s, B_s)$$

From the above definitions, the essential structural properties can be characterized in the following way.

Proposition 4. (see (Bru and Hernández, 1989)) The positive periodic system (5) is essentially positive controllable at s if and only if the transition matrix $\phi_A(s+N, s)$ is stable and $R(A(\cdot), B(\cdot), s) = \mathbb{R}^n_+$.

Moreover, given a positive periodic system (5), its structural properties remain totally determinated by the corresponding properties of the associated invariant systems (6).

Proposition 5. (see (Bru and Hernández, 1989)) The positive periodic system (5) is essentially controllable at s if and only if the positive invariant system (6) corresponding to index s is essentially controllable, for every $s = 0, 1, \ldots, N-1$.

Theorem 17. (see (Bru and Hernández, 1989)) Consider the positive system (5). If $A(k)$ is irreducible for all $k = 0, 1, \ldots, N-1$, and $A(k) \gg 0$ for some $k_0 \in \{0, 1, \ldots, N-1\}$, then $R(A(\cdot), B(\cdot), s) = \mathbb{R}^n_+$ if and only if for each $s = 0, 1, \ldots, N-1$, $R_\infty(A(\cdot), B(\cdot), s) = \mathbb{R}^n_+$.

Hence,

Theorem 18. (see (Bru and Hernández, 1989)) For each $s = 0, 1, \ldots, N-1$, the positive periodic system (5) satisfies that $R_\infty(F(\cdot), G(\cdot), s) = \mathbb{R}^n_+$ if and only if $R_{nN}(F(\cdot), G(\cdot), s) = \mathbb{R}^n_+$.

The characterization of the essential structural properties by means of the positive invariant cyclically augmented system (7) associated with the positive periodic system (5) are based on the following relationship of cones.

Proposition 6. (see (Bru *et al.*, 1997)) Let $x \in \mathbb{R}^{nN}_+$, where $x = \text{col}[x^1 x^2 \cdots x^N]$ and $x^j \in \mathbb{R}^n_+$. Then,

$$x \in R(A_e, B_e) \text{ if and only if}$$
$$x^j \in R(A(\cdot), B(\cdot), j),\ j = 1, \ldots, N.$$

Then, the following result follows.

Theorem 19. (see (Bru *et al.*, 1997)) Let a positive periodic system (5) and its associated cyclically augmented system (7). If A_e is a nonnegative and irreducible matrix and there exists some nonzero diagonal element of $\phi_A(N, 0)$ then $R_\infty(A(\cdot), B(\cdot), s) = \mathbb{R}^n_+$ if and only if $R(A(\cdot), B(\cdot), s) = \mathbb{R}^n_+$

This theorem is an extension of theorem 1 of (Coxson and Shapiro, 1987). Its proof is based on the theorem 2 of (Coxson and Shapiro, 1987) and in the following property.

Proposition 7. Let A_e be a nonnegative, irreducible matrix. If the trace of $\phi_A(N, 0)$ is nonzero, then A_e is a cyclic matrix of index N.

The authors of (see (Bru *et al.*, 1997)) proved that the conditions of the theorem 17 imply the conditions of theorem 19, and showed with an example that such conditions are not equivalent.

In (Romero, 2001) and (Bru *et al.*, 2000b) a broad study was done on essential properties of positive periodic systems by means of the directed-graph theory. Moreover, canonical forms of essential reachability and controllability were obtained. These results are in (Bru *et al.*, 2001a).

6. CONCLUDING REMARKS

In this paper, the standard and essential concepts of reachability and controllability properties of positive linear systems has been discussed. Results of both discrete–time and continuous–time positive linear systems has been studied using the algebraic and combinatorial point of views. Those results are given for invariant and periodic systems. However, it seems there are not any results for continuous periodic systems. Then, open problems on that topic and other open problems are presented. Further, canonical forms of reachability and controllability are displayed, either for invariant and periodic discrete–time positive linear systems.

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