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INPUT TRACKING FOR STABLE LINEAR SYSTEMS

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Abstract: In this paper, the input-output behavior of a linear stable system is studied in some detail. Some results are presented to make more precise the folklore statement that outputs follow inputs for stable linear systems. Based on these results, it is also discussed how to choose an output and how to fuse a number of sensor outputs, to best track the input in stationarity.

Keywords: Tracking, sensor fusion, linear systems.

1. INTRODUCTION

The folklore of linear system theory maintains that for a stable linear system the output of the system tracks the input. As is generally the case for such results they are almost true but trying to make a precise statement is difficult if not impossible. In this paper we examine this statement in some detail in the case that the input is the output of a linear system whose eigenvalues satisfy $\lambda + \bar{\lambda} \geq 0$.

Our motivation to further study this classical problem lies in that the results we obtain can be applied to other important problems such as optimal input tracking and sensor fusion. It is natural that before we can develop a procedure to choose an output or a combination of sensors that best tracks an input, we need to understand when a given output tracks the input.

In the second part of the paper, we will discuss a sensor fusion problem. There has been a vast literature on sensor fusion, see for example, the papers in (*Proceedings of the IEEE*, 1997) and the references therein. However, treatment of the problem from the input tracking point of view has to our knowledge not been addressed. The following example shows the practical relevance of the issues we will address in the paper.

Example: Consider a mobile system that consists of a base vehicle and a platform mounted on top of the base. One can use, for example, inclinometers and gyros to measure the attitude of the platform. However, it is known that the readings of these sensors may be affected by the external forces and it is desirable to compensate the effects due to these forces. Now the question is if we can use the odometer and speedometer in the base to estimate the force. Naturally, if we do not know anything about the force, the only way to measure it would be to use an accelerometer, which is a sensor of fairly high complexity. In this paper, we will illustrate that if we can make some assumptions on the force, it is possible to estimate the acceleration by displacement and velocity. More precisely, consider the dynamics

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = a_1 x_1 + a_2 x_2 + bu.$$

If we know u is generated by an exo-system

$$\dot{w} = \Gamma w$$
$$u = qw,$$

then we will discuss how to find an optimal combination $c_1x_1 + c_2x_2$ to estimate u.

This paper is organized as follows. In section 2, we use a classical example by Desoer to demonstrate the research issues. In section 3, we discuss the problem of how a given output tracks an input in stationarity. In sections 4 and 5, we discuss the problem of how to choose an output, or a combination of sensors, to optimally track an input in stationarity.

2. AN EXAMPLE FROM DESOER

This example is taken from (Desoer, 1970) and evidently comes from linear circuit theory. Suppose that we have a second order linear system that is known to be stable. We assume that the transfer function is of the form

$$G(s) = \frac{as+b}{es^2+fs+g}$$

We excite the system with sinusoidal input of the form $u(t) = \sin \omega t$. Then the Laplace transform of the output is

$$\frac{as+b}{es^2+fs+g}\frac{\omega}{s^2+\omega^2}.$$

A little algebra then decomposes the product into a sum of the form

$$\frac{xs+y}{es^2+fs+g} + \frac{As+B\omega}{s^2+\omega^2}$$

where

$$A = \frac{((g-e\omega^2)a - fb)\omega}{(g-ew^2)^2 + f^2\omega^2}$$

and

$$B = \frac{a\omega^2 f + (g-e\omega^2)b}{(g-ew^2)^2+f^2\omega^2}.$$

Now we have assumed that the system is stable so the piece of the sum with denominator es^2+fs+g decays to zero and the steady state response is given by

$$\frac{As+B\omega}{s^2+\omega^2}.$$

Hence

$$y(t) = A\cos\omega t + B\sin\omega t,$$

which is equivalent to

$$y(t) = \sqrt{A^2+B^2}\sin(\omega t+\beta),$$

where $\sin\beta = \frac{A}{\sqrt{A^2+B^2}}$. Thus β is a phase shift and $\sqrt{A^2+B^2}$ is a change in amplitude. This shows that apart from amplification and phase lag the output tracks the input perfectly.

On the other hand, if we choose the output in such a way that

$$a = f, \quad b = g - e\omega^2,$$

then we have

$$y(t) = u(t)$$

in steady state. We will consider more general formulations of these two problems in the following sections.

3. AUTONOMOUS LINEAR SYSTEMS

In this section we consider Desoer's example in the more general case of an arbitrary finite dimensional linear system with input u taken to be the output of an unstable linear system.

Consider a stable, controllable and observable SISO linear system:

$$\begin{aligned}\dot{x} &= Ax + bu \\ y &= cx\end{aligned} \tag{1}$$

where $x \in R^n$ and $\sigma(A) \in C^-$.

The problem that we will study is the following.

Problem 1: *When does there exist functions $a(t)$ and $m(t)$ such that for arbitrary ϵ and for all sufficiently large t,*

(1) $\sup_t |y(t) - a(t)u(t+m(t))| < \epsilon$,
(2) $a(t)$ is bounded and
(3) $m(t)$ is bounded.

We will consider the case when the input u is generated by the following exogenous system:

$$\begin{aligned}\dot{w} &= \Gamma w, \quad w(0) = w_0 \\ u &= qw\end{aligned} \tag{2}$$

where $w \in R^m$ and $\sigma(\Gamma) \in \bar{C}^+$. This exo-system can generally have a block diagonal Jordan realization

$$\begin{aligned} q &= \begin{pmatrix} q_1 & q_2 & \dots q_M \end{pmatrix} \\ \Gamma &= \mathrm{diag}(\Gamma_1, \Gamma_2, \dots, \Gamma_\mathrm{m}) \end{aligned} \tag{3}$$

where each $q_m = \begin{pmatrix} 1 & 0 \dots 0 \end{pmatrix}$ is a first unit vector of length $\dim(\Gamma_m)$ and the Jordan blocks correspond to polynomial, exponential, and sinusoidal functions. The output of the exo-system becomes

$$u(t) = \sum_{m=1}^{M} q_m e^{\Gamma_m t} w_{0_m}.$$

Such exo-systems can generate, for example, step functions, ramp functions, polynomials, exponentials, sinusoidals, and combinations of such functions.

Remark: In Laplace domain the output of (1) is

$$Y(s) = G(s)U(s)$$

where $G(s) = c(sI - A)^{-1}B = \frac{b(s)}{a(s)}$, and $U(s) = q(sI - \Gamma)^{-1}w_0 = \frac{f(s)}{g(s)}$. The stationary behavior is as in Desoer's example determined by the unstable poles in the following partial fractions expansion.

$$Y(s) = \frac{b(s)}{a(s)}\frac{f(s)}{g(s)} = \underbrace{\frac{\tilde{b}(s)}{a(s)}}_{\text{transient}} + \underbrace{\frac{\tilde{f}(s)}{g(s)}}_{\text{stationary}}$$

If we have a realization $\frac{\tilde{f}(s)}{g(s)} = h(sI - \Gamma)^{-1}r$ then our concern is whether there exists bounded functions $m(t), a(t)$, such that

$$he^{\Gamma t}r \approx a(t)qe^{\Gamma(t+m(t))}w_0$$

for large times. We will, using state space techniques, show that the above expression can be simplified since it is possible to use $r = w_0$ and then h is obtained from the solution of a special Lyapunov equation (note that (Γ, w_0) need not be controllable so it is by no means obvious that $\beta = w_0$ is a possible choice).

Proposition 1. Suppose A is a stable matrix, then all trajectories of $(x(t), w(t))$ tend asymptotically to the invariant subspace $S := \{(x, w) :\ x = \Pi w\}$, where Π is the solution of

$$A\Pi - \Pi\Gamma = -bq.$$

On the invariant subspace, we have

$$y(t) = c\Pi w(t).$$

Proof: We suspect that the proof of this proposition is in the literature but have not been able to locate a definitive version. We know however that there is a vast literature on the nonlinear case, see for example (Coddington and Levinson, 1984; Kelly, 1967). We include a simple proof for the sake of completeness.

We will establish that there exists a matrix Π so that the set

$$S = \{(x, w) :\ x = \Pi w\}$$

is invariant under the action of the linear system in the following sense. Let

$$u = qw$$

then in order to show S is invariant we only need to show $\dot{x} = \Pi\dot{w}$. Namely,

$$\Pi\Gamma w = A\Pi w + bqw \tag{4}$$

which gives us the Lyapunov equation

$$\Pi\Gamma - A\Pi = bq. \tag{5}$$

Now the eigenvalues of Γ are in the right half plane and the eigenvalues of $-A$ are in the strict right half plane and hence no sum of eigenvalues is zero. Thus there exists a unique solution Π to the equation, (Gantmacher, 1998).

Now consider the linear system defined by (1) and (2):

$$\begin{pmatrix} \dot{x} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} A & bq \\ 0 & \Gamma \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix}.$$

Using the coordinate change

$$\bar{x} = x - \Pi w$$

we have,

$$\begin{pmatrix} \dot{\bar{x}} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & \Gamma \end{pmatrix} \begin{pmatrix} \bar{x} \\ w \end{pmatrix},$$

which obviously implies $\bar{x}(t)$ tends to zero for all initial conditions. Q.E.D

Using the matrix Π we have that the output of the linear system in the steady-state can be represented as

$$y_s = c\Pi w.$$

Now we can answer Problem 1.

Theorem 2. Suppose the input is generated by (2), then Problem 1 has a solution $a(t)$, $m(t)$ if and only if there exists a solution $a(t)$, $m(t)$ to the equation

$$(c\Pi - a(t)qe^{\Gamma m(t)})e^{\Gamma t}w_0 = 0, \tag{6}$$

for large t.

Proof: It is easy to see that to solve Problem 1, we just need to have on the manifold $x = \Pi w$ that

$$y(t) = a(t)u(t + m(t)).$$

for some bounded $a(t)$ and $m(t)$. Since

$$y_s(t) = c\Pi w(t),$$

we must have, if Problem 1 is solved

$$c\Pi w(t) - a(t)qw(t + m(t)) = 0.$$

The rest follows. Q.E.D

Even when Problem 1 does not have a solution, the following result tells us that for such a linear system, it can tracks the input very "tightly" anyway by using an observer.

Proposition 3. Let the system

$$\dot{w} = \Gamma w, \quad u = qw$$

be observable and no eigenvalue of Γ is a transmission zero of (1). Then the system on the invariant subspace

$$\dot{w} = \Gamma w, \quad y = c\Pi w$$

is also observable.

Proof: We first need to establish that under the hypotheses, the composite system

$$\begin{pmatrix} \dot{x} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} A & bq \\ 0 & \Gamma \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} \tag{7}$$
$$y = cx$$

is observable. Proofs of similar results can be found, for example, in (Chen, 1984). We include a simple proof for the sake of completeness.

Define

$$H(s) = \begin{pmatrix} sI - A & -bq \\ 0 & sI - \Gamma \\ c & 0 \end{pmatrix}.$$

By Hautus test we know that the system is observable if

$$rank(H(s)) = n + m \quad \forall s.$$

If s is not an eigenvalue of Γ, it is easy to see that $rank(H(s)) = n + m$ since (c, A) is observable. Now suppose s is an eigenvalue of Γ,

$$H(s) = \begin{pmatrix} sI - A & b & 0 \\ 0 & 0 & I_m \\ c & 0 & 0 \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & q \\ 0 & sI - \Gamma \end{pmatrix}.$$

If s is not a transmission zero of (1), then the first matrix on the right-hand side has rank $n + 1 + m$ and the second has rank $n + m$ since (q, Γ) is observable. By Sylvester's inequality, we have

$$rank(H(s)) \geq n+1+m+n+m-(n+m+1) = n+m.$$

Therefore $rank(H(s)) = n + m$.

Now as we did before, we do a coordinate change $\bar{x} = x - \Pi w$. Then (7) becomes

$$\begin{pmatrix} \dot{\bar{x}} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & \Gamma \end{pmatrix} \begin{pmatrix} \bar{x} \\ w \end{pmatrix}$$
$$y = c\bar{x} + c\Pi w$$

It is straight forward to see that

$$((c, c\Pi), \begin{pmatrix} A & 0 \\ 0 & \Gamma \end{pmatrix})$$

is observable implies $(c\Pi, \Gamma)$ is so too. Q.E.D.

4. INPUT TRACKING

We will now discuss how Proposition 1 can be used to determine an appropriate output in order to track the input in stationarity. In the same way as in Theorem 2 it follows that the output tracks the input if the vector c is chosen such that

$$(c\Pi - q)e^{\Gamma t}w_0 = 0 \tag{8}$$

This is clearly the case if $c\Pi = q$. In other words, if Π has full column rank then it is possible to design an output c for perfect input tracking in stationarity. It is possible to show that under our assumptions Π has full column rank if $\dim(A) \geq \dim(\Gamma)$.

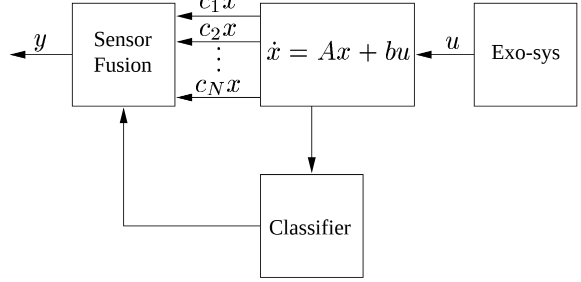

Fig. 1. Sensor fusion set-up.

The exo-system will in many applications have significantly larger dimension than the linear system (1) and then there only exists a solution to (8) for special choices of initial condition w_0 of the exo-system. In the next section we discuss a strategy for fusing the output of a number of sensors in order to minimize the steady state tracking error.

5. SENSOR FUSION FOR INPUT TRACKING

We will here consider a special sensor fusion problem where we try to minimize the tracking error by appropriately combining the outputs of a number of sensors. The complete system is given in Figure 1. A sensor in our terminology means a particular choice of c_k matrix. If the state space model represents physical variables, then typically each c_k corresponds to one state variable. The idea is that the sensor fusion block should determine a linear combination of the sensor signals such that the output

$$y = \sum_{k=1}^{N} \alpha_k c_k x \tag{9}$$

tracks the input u in stationarity. We will here discuss how this sensor fusion idea works for the case when the input is generated by an observable exo-system of the form (2). The objective of the two new blocks is the following.

The Classifier Block. The task of the classifier block in Figure 1 is to determine what Jordan blocks are active in the generation of the input u, i.e. it determines a set $\mathcal{M} \subset \{1, \ldots, M\}$ of indices such that the input can be represented as

$$u(t) = \sum_{m \in \mathcal{M}} q_m e^{\Gamma_m t} w_{0_m}.$$

We will see below that this information sometimes is enough to obtain perfect tracking. However, it is generally important to use as much information on the vector w_0 as possible in order to obtain better tracking. For example, if we in addition to $\mathcal{M}$ also obtain an estimate $\hat{w}_{0_\mathcal{M}} = \left[\hat{w}_{0_{m_1}}, \ldots, \hat{w}_{0_{m_n}}\right]$ of the initial condition then our ability to reconstruct the input improves. Even qualitative information such as the relative amplitude of the various blocks is useful.

The Sensor Fusion Block. This block takes as input the classification $\mathcal{M}$ and maps it to a vector α that minimizes the steady state tracking error for the output (9) according to some cost criterion. We will discuss this in more detail below where we also give necessary and sufficient conditions for obtaining perfect tracking. In more sophisticated schemes we may also use an estimate $\hat{w}_{0_\mathcal{M}}$ of the initial condition of the exo-system. This may give better tracking, however at the price of more complex classifier and sensor fusion blocks. Note that this scheme will be independent of the state space realization and the convergence to the steady state solution depends on the spectrum of A.

A main practical motivation for our sensor fusion scheme is due to the limited communication and computation resources in many embedded systems, such as mobile robotic systems. There is a need to develop "cheap" sensing algorithms. The central idea in our scheme is to optimally combine the existing sensors for state variables to measure the external signals. This optimization can be done off-line and then the sensor fusion block only needs to use a table look-up to decide the parameter vector α. The only remaining issue is how to design the online classifier.

The most natural way from a systems point of view is perhaps to use a dynamical observer (in discrete time) to identify qualitatively the initial condition (or the active Γ blocks) and then shut it down. This is possible since the state of the exo-system is observable from the output from any sensor such that (c_k, A) is observable, see Theorem 3. However, this approach could be computationally expensive even if we only run it once in a while.

For many practical systems, it is perhaps more realistic to design the classifier based on other sensors that sense the interaction of the system with the environment (such as laser scanners and video cameras), or/and on the nature of application the system is operated for. In this way, typically only a range of the Γ blocks (such as a frequency range) can be identified.

Perfect Steady-state Tracking. Assume we are given K sensors $c_1, \ldots, c_K$. We derive a sufficient (and in a sense also necessary) condition for obtaining perfect steady state tracking using these sensors. We have the following result.

Proposition 4. Suppose $\mathcal{M} = \{m_1, \ldots, m_n\} \subset \{1, \ldots, M\}$ are the active Jordan blocks. Then we can obtain perfect tracking if

$$q_\mathcal{M}^T \in \mathrm{Im}(\Pi_\mathcal{M}^T C^T)$$

where $A\Pi_\mathcal{M} - \Pi_\mathcal{M}\Gamma_\mathcal{M} = -bq_\mathcal{M}$ and

$$\begin{aligned} \Gamma_\mathcal{M} &= \mathrm{diag}(\Gamma_{m_1}, \ldots, \Gamma_{m_n}) \\ q_\mathcal{M} &= (q_{m_1}, \cdots, q_{m_n}) \\ C^T &= \begin{bmatrix} c_1^T & \ldots & c_K^T \end{bmatrix} \end{aligned} \tag{10}$$

Proof: The steady state output will be $y = \alpha C \Pi_\mathcal{M} w_\mathcal{M}(t)$, where $w_\mathcal{M}(t) = q_\mathcal{M} e^{\Gamma_\mathcal{M} t} w_{0_\mathcal{M}}$. Hence, we obtain perfect tracking since our assumption implies that there exists a solution to $\alpha C \Pi_\mathcal{M} = q_\mathcal{M}$. Note that this condition is necessary if $w_{0_\mathcal{M}}$ is allowed to take any value. Q.E.D.

If the condition of the proposition holds then we normally want to find the vector α with the minimum nonzero coefficients such that

$$\alpha C \Pi_\mathcal{M} = q_\mathcal{M}$$

in order to minimize the number of sensors used. This can be done off-line and then the sensor fusion block only need to use a table look-up to decide the vector α.

Approximate Tracking. It will often happen that we have too few sensors or too poor knowledge of the exo-system to obtain perfect tracking. In such cases we need to optimize the sensor fusion in order to get best tracking in some average sense. We here consider two cases

Case 1: If $\mathcal{M} = \{m_1, \ldots, m_n\}$ and the condition in Proposition 4 does not hold then we let the sensor fusion be determined by the solution to the least squares problem

$$\min_\alpha |\alpha C \Pi_\mathcal{M} - q_\mathcal{M}|$$

where we use the same notation as in Proposition 4. This optimization can be done in advance and the sensor fusion can be implemented as a table look-up.

Assume now that the classifier has obtained more information on the exo-system in terms of an estimate of the initial conditions $\hat{w}_{0_\mathcal{M}}$. This can be used to design better sensor fusion parameters by minimizing a cost criterion that takes this information into account. One example is the cost

$$\min_\alpha \int_0^\infty |(\alpha C \Pi_\mathcal{M} - q_\mathcal{M}) e^{(\Gamma_\mathcal{M} - \lambda I)t} \hat{w}_{0_\mathcal{M}}|^2 dt \tag{11}$$

where $\lambda > \rho(\Gamma_\mathcal{M})$, the spectral radius of $\Gamma_\mathcal{M}$. This optimization problem is equivalent to the weighted least squares problem $\min_\alpha |\alpha C \Pi_\mathcal{M} - q_\mathcal{M}|_Q$, where $|x|_Q = x^T Q x$, and $Q \geq 0$ is the solution to the Lyapunov equation

$$(\Gamma_\mathcal{M} - \lambda I)Q + Q(\Gamma_\mathcal{M} - \lambda I)^T = -\hat{w}_{0_\mathcal{M}} \hat{w}_{0_\mathcal{M}}^T.$$

This cannot be implemented using table look-up. *Case 2:* We will here consider the special case when we know that the exo-system can generate M sinusoidal signals but with some uncertainty in the exact location of the frequencies, i.e., $\omega_m - \Delta\omega \leq \omega \leq \omega_m + \Delta\omega$. If we assume that exactly one Jordan block is active at a time then a reasonable optimization problem for determining the sensor fusion is

$$\min_{\alpha} \int_{\omega_m - \Delta\omega}^{\omega_m + \Delta\omega} |\alpha C \Pi(\omega) - q|^2 d\omega. \qquad (12)$$

where C is defined as in (10), $A\Pi(\omega) - \Pi(\omega)\Gamma(\omega) = -bq$, $q = \begin{bmatrix} 1 & 0 \end{bmatrix}$, and

$$\Gamma(\omega) = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}.$$

The next theorem shows that (12) has a unique optimal solution when the number of sensors K is less than the dimension of A. Indeed, the optimal solution to (12) is $\alpha = YX^{-1}$, where

$$Y = q(\int_{\omega_m - \Delta\omega}^{\omega_m + \Delta\omega} \Pi(\omega)^T d\omega) C^T$$
$$X = C \left(\int_{\omega_m - \Delta\omega}^{\omega_m + \Delta\omega} \Pi(\omega)\Pi(\omega)^T d\omega \right) C^T$$

Theorem 5. The matrix

$$\int_{\omega_l}^{\omega_h} \Pi(\omega)\Pi(\omega)^T d\omega$$

where $w_l < w_h$, is strictly positive definite, hence (12) is strictly convex and a unique optimal solution exists.

Proof: Straight forward calculation shows

$$\Pi = A_\omega^{-1} A \left[(-b \ \omega A^{-1} b) q^T, (-\omega A^{-1} b \ -b) q^T \right].$$
$$\Pi\Pi^T = 2|q|^2 A_\omega^{-1} (Abb^T A^T + \omega^2 bb^T) A_\omega^{-T}.$$

where $A_\omega = \omega^2 I + A^2$. Now since

$$\int_{\omega_l}^{\omega_h} (\omega^2 I + A^2)^{-1} Abb^T A^T (\omega^2 I + A^2)^{-T} d\omega$$
$$\geq \frac{1}{2\omega_h} \int_{\sqrt{\omega_l}}^{\sqrt{\omega_h}} (sI + A^2)^{-1} Abb^T A^T (sI + A^2)^{-T} ds,$$

and $(-A^2, Ab)$ is obviously controllable, thus the above integral is positive definite by Theorem 5.7 in (Chen, 1984). Q.E.D

5.1 *An Example of optimal output design*

We will use the example in the introduction to illustrate our results. For this purpose, we will keep the input $u(t)$ simple. Now suppose the input is sinusoidal with a frequency in $[\omega_l, \omega_h]$. In this case the dimension of each Jordan block of Γ is same as that of A. For each $\omega_m \in [\omega_l, \omega_h]$, by solving the equation

$$A\Pi - \Gamma\Pi = -bq,$$

we obtain

$$\Pi = \frac{1}{d(\omega)} \begin{pmatrix} -(\omega_m^2 + a_1)b & \omega_m a_2 b \\ -\omega_m^2 a_2 b & -\omega_m(\omega_m^2 + a_1)b \end{pmatrix},$$

where $d(\omega) = (\omega_m^2 + a_1)^2 + a_2^2\omega_m^2$. If the classifier can determine the frequency, then the optimal sensor fusion would be

$$(\alpha_1 \ \ \alpha_2) = (-(\omega_m^2 + a_1) \ \ -a_2),$$

which is similar to what we obtained in the Desoer example (with $e = 1, f = -a_2, g = -a_1$, and $a = 0$).

If the classifier can only determine the range which the frequency is in, then the optimal sensor fusion would be

$$\alpha = (1 \ 0) \int_{\omega_l}^{\omega_h} \Pi(\omega)^T d\omega (\int_{\omega_l}^{\omega_h} \Pi(\omega)\Pi(\omega)^T d\omega)^{-1}.$$

6. CONCLUDING REMARKS

We have used state space techniques to obtain conditions for steady state input tracking for stable linear systems. It has been shown that these results can be used to devise a sensor fusion scheme for input tracking. It is interesting to consider extensions of this to multivariable and nonlinear systems. We will consider applications of these ideas to problems in autonomous mobile robotics in our future research.

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