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POLYNOMIAL SPECTRAL FACTORIZATION WITH COMPLEX COEFFICIENTS [1]

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Abstract: Conceptual and numerical issues related to the spectral factorization of polynomials and polynomial matrices with complex coefficients are studied in this report. Such investigation is motivated by the demand for reliable algorithms and CAD tools capable of solving latest signal processing problems involving complex polynomials (Ahlen and Sternad, 1993). Basic concepts of the *real* polynomial spectral facorization theory are inspected first, and their generalization and necessary modification for complex polynomials then follows. Efficient numerical methods which are known to work in the real case are then revisited and their applicability for complex coefficients is considered. As an immediate result of this research, the powerful algorithms proposed in this paper have given rise to several routines implemented in the Polynomial Toolbox for Matlab (Kwakernaak and Sebek, 1999) and addressing the spectral factorization problem. *Copyright ©2002 IFAC*

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1. INTRODUCTION

The polynomial spectral factorization is one of the basic tools in the algebraic theory of linear control (Kailath, 1980) (Kucera, 1991). All quadratic-optimal design problems, such as Wiener filtering, linear-quadratic controller, Kalman filter, LQG compensator, being solved via the algebraic approach involve polynomial spectral factorization as the crucial computational step (Kailath, 1980) (Kucera, 1991).

Within the control community there is not so much need to take polynomial matrices with *complex* coefficients into account in fact. All the design problems mentioned above involve real transfer functions only if applied to plant or process control. That is why the great majority of research results on polynomial spectral factorization, including numerical algorithms, concern just the real case as a rule.

However, quite recently the polynomial design methods found a new great field of application outside the control area: the algebraic approach to control systems design has been used successfully in mobile communications (Sternad and Ahlen, 1996), (Ahlen and Sternad, 1993). In contrast to the control systems synthesis, polynomials and polynomial matrices with *complex* coefficients are typically required to encompass the overall mobile communication channel.

Up to the authors' knowledge, no systematic research has been undertaken to cover the problematic of equations in the ring of polynomials and polynomial matrices with complex coefficients. Recently this gap has been partly removed in the case of linear equations, see (Henrion, *et al.*, 1999), when the problematic of linear symmetric polynomial equations in the discrete time case

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was considered. In this paper we investigate the situation in the case of quadratic problems involving complex polynomials. Hence the present work and the mentioned paper (Henrion, *et al.*, 1999) can be thought of as two complementary reports contributing to the new research in theory and algorithms of complex matrix polynomial equations. Besides their theoretical value, results of such research find direct practical application in modern software tools for polynomials and polynomial matrices (Kwakernaak and Sebek, 1999) dedicated to cope with advanced signal processing and control design problems.

2. COMPLEX COEFFICIENTS IN MOBILE COMMUNICATIONS

Basically, there are two reasons for going into complex computations in signals. First, it often appears convenient to code the transmitted bit-stream into complex numbers and send their real and imaginary parts using the same frequency range. Such a way, the bandwidth reserved for the channel is more efficiently exploited. However, even if the transmitted symbols are real, the amplitude response of the overall communication channel can become asymmetrical around the carrier frequency if interference or multipath propagation occurs, giving rise to a complex transfer function in the baseband representation. Many

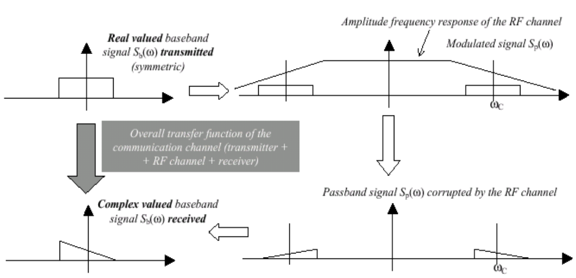

Figure 2: Scheme of the complete communication channel. Since the overall transfer function of the communication channel maps real-valued signals onto complex signals, it necessarily features complex coefficients.

results on applying polynomial design methods in filters and equalizers for mobile communications have been achieved by the Signals and Systems Group at the University of Uppsala, Sweeden. Their algorithms based on polynomial approach for LQ optimal feedforward filters and LQ optimal decision feedback equalizers have been applied by the Ericsson company in their mobile phones and other gadgets. An interested reader is referred to http://www.signal.uu.se/Publications/ for detailed description of particular procedures. As one observes, the crucial computational parts of all procedures are the linear Diophantine equations and mainly *complex* polynomial spectral factorizations.

3. BASIC CONCEPTS

Common terms and concepts of the polynomial spectral factorization theory are discussed in this section and extended for the complex coefficients case. Both continuous- and discrete-time polynomials are considered in scalar and matrix case.

3.1 *Conjugated Operators*

In the problems of signal processing and control design with quadratic criteria, the conjugation operator comes into play in addition to standard operations such as sum and product.

3.1.1. *Continuous Time Case* For a scalar real polynomial $p(s) = p_0 + p_1 s + p_2 s^2 + \cdots + p_n s^n$ in the differential operator s it follows that the adjoint $p^\star(s)$ of p equals $p^\star(s) = p_0 - p_1 s + p_2 s^2 - \cdots + (-1)^n p_n s^n = p(-s)$ (Kailath, 1980) (Kucera, 1991).

Generalization for matrix and complex coefficients is now quite natural.

The matrix transpose plays the role of conjugated operator if the space of real vectors with traditionally defined scalar product and matrix-vector multiplication is considered.

Complex numbers can be handled alike. Given the space C of complex numbers with scalar product defined by $\langle a, b\rangle = \overline{a}b$ where $\overline{x}$ denotes the complex conjugate, and with common complex multiplication, then $\langle ax, y\rangle = \langle x, \overline{a}y\rangle$ holds.

Combining the results of previous paragraphs, the final formulas read:

$$p^\star(s) = p_0 - p_1 s + \cdots + (-1)^n p_n s^n = p(-s) \quad \text{(scalar real)}$$
$$P^\star(s) = P_0^T - P_1^T s + \cdots + (-1)^n P_n^T s^n = P^T(-s) \quad \text{(matrix real)}$$
$$p^\star(s) = \overline{p_0} - \overline{p_1} s + \cdots + (-1)^n \overline{p_n} s^n = \overline{p(\overline{-s})} \quad \text{(scalar complex)}$$
$$P^\star(s) = \overline{P_0}^T - \overline{P_1}^T s + \cdots + (-1)^n \overline{P_n}^T s^n = \overline{P^T(-\bar{s})} \quad \text{(matrix complex)}$$

3.1.2. *Discrete Time Case* Similar steps can be directly carried out with discrete-time polynomials in the forward-shift operator z:

$$p^\star(z) = p_0 + p_1 z^{-1} + \cdots + p_n z^{-n} = p(z^{-1}) \quad \text{(scalar real)}$$
$$P^\star(z) = P_0^T + P_1^T z^{-1} + \cdots + P_n^T z^{-n} = P^T(z^{-1}) \quad \text{(matrix real)}$$
$$p^\star(z) = \overline{p_0} + \overline{p_1} z^{-1} + \cdots + \overline{p_n} z^{-n} = \overline{p(\overline{z}^{-1})} \quad \text{(sc. complex)}$$
$$P^\star(z) = \overline{P_0}^T + \overline{P_1}^T z^{-1} + \cdots + \overline{P_n}^T z^{-n} = \overline{P^T(\bar{z}^{-1})} \quad \text{(mat.complex)}$$

3.2 *Symmetry*

A polynomial or a polynomial matrix p is said to be symmetric if it equals its adjoint $p^\star$ as it is defined above.

3.2.1. *Continuous Time Case* A scalar polynomial $p(s) = p_0 + p_1 s + \cdots p_d s^d$ is continuous-time symmetric if $p_i = \bar{p_i}$ for $i = 0, 2, 4, etc.$ and $p_i = -\bar{p_i}$ for $i = 1, 3, 5, etc.$ In other words, $p(s)$ is symmetric if its even part $p_e = p_0 + p_2 s^2 + \cdots$ is real and odd part $p_o = p_1 + p_3 s^2 + \cdots$ purely imaginary. If in addition only real polynomials are accepted, p has to consist of even degrees terms only.

This symmetry of coefficients yields also a symmetry of roots. Supposing a number r_i is a root of $p(s)$ and taking the symmetry of p into account, we receive

$$0 = p(r_i) = \overline{p(r_i)} = \overline{p^\star(r_i)} = p(-\bar{r_i}).$$

In other words, the roots of p can be divided into pairs $\{[r_i, -\bar{r_i}], i = 1, \ldots, d$. Each two roots associated in a certain pair have opposite real parts and the same imaginary part. Observe that for r_i on the imaginary axis $r_i = -\bar{r_i}$ holds and related couple degenerates to a single point. $p(s) = 1 + js$ is an instant of such a polynomial.

For matrix polynomials similar results can be achieved. A complex number r_i is said to be a (finite) root of a polynomial matrix P if $\operatorname{rank}(P(r_i)) < \operatorname{rank}(P(z)) = \max_z(\operatorname{rank}(P(z)))$. If A is nonsingular then the finite roots of A equal the roots of its determinant. Considering these facts instead of the nullity of $p(r_i)$ and taking the symmetry into account the symmetry of roots can be proved for polynomial matrices as well.

3.2.2. *Discrete Time Case* A two-sided polynomial $p(z) = \sum_{i=-d}^{d} p_i z^i$, where p_i are complex numbers, is discrete-time symmetric if and only if $p_i = \bar{p}_{-i}$. For the standard situation of p real we receive the condition $p_i = p_{-i}$.

The roots of a complex discrete time symmetric polynomial are distributed in pairs $\{[r_i, \bar{r}_i^{-1}], i = 1, \ldots, d\}$ [2]. Each two roots associated in a certain pair have reciprocal magnitudes and the same phase. For real polynomials we get quadruples $[r_i, r_i^{-1}, \bar{r_i}, \bar{r_i}^{-1}]$.

Nevertheless the circumstances that arise if some roots appear on the bound of stability region make the situation slightly more complicated now:
Theorem 1: If $p(z) = \sum_{i=-d}^{d} p_i z^i$ is a discrete time symmetric scalar polynomial then all its roots r_i with $|r_i| \neq 1$ are distributed in pairs $[r_i, \bar{r}_i^{-1}]$. In addition, $p(z)$ can also have an even number of roots [3] arbitrarily placed on the unit circle.

Proof: $p(z)$ can be expressed as $p(z) = z^{-d}\hat{p}(z)$ where $\hat{p}(z)$ is a one sided polynomial of even degree $2d$, also symmetric in certain sense. As a result both $\hat{p}$ and p must have an even number of roots in total. Those r_i's with $|r_i| \neq 1$ form pairs $[r_i, \bar{r}_i^{-1}]$ as it has been shown already. Hence the count of possibly remaining roots with magnitude equal to one has to be even as well.

The freedom in their location remains to be shown. In other words, existence of a two-sided polynomial symmetric in discrete time sense and having any set of $2k$ points placed on the unit complex circle as its roots must be proved. To do it we take an arbitrary pair of complex roots of unity $r_1 = e^{j\phi_1}$, $r_2 = e^{j\phi_2}$ and construct a polynomial $\hat{a}(z) = K(z - r_1)(z - r_2)$. One can check that for $K = e^{-j\frac{\phi_1+\phi_2}{2}}$ the expression $a(z) = z^{-1}\hat{a}(z)$ becomes a discrete time symmetric two sided polynomial for any combination of ϕ_1, ϕ_2. Applying this procedure for all particular $|r_i| = 1$ yields the result. ◇

3.3 *Definitness*

Next to the symmetry, definitness of polynomials at the boundary of the stability region is an important issue.

If $P(s)$ is a continuous time symmetric polynomial matrix, real or complex, then for the points $s = j\omega$ we get $P(s) = P^\star(s) = \overline{P^T(-\bar{s})} = \overline{P^T(s)}$. Hence $P(s)$ is Hermitian on the imaginary axis and as a result, its eigenvalues are always real. For this reason the terms of negative/positive (semi)definitness can be naturally introduced for complex polynomials as well.

The same conclusions hold in the discrete time case as well with purely imaginary points $s = j\omega$ replaced by $z = e^{j\omega}$ on the unit circle.

A necessary condition for P symmetric being positive/negative semidefinite follows.

Theorem 2: If a complex nonsingular polynomial matrix, symmetric in the discrete or continuous time sense respectively, is positive or negative semidefinite on the stability region bound, then its roots lying on this boundary have even multiplicities.

Proof: For scalars the result follows directly. The concerned symmetric polynomial p attains only real values on the bound of related stability area $\mathcal{B}$. If it has no roots on $\mathcal{B}$ then it does not achieve

[2] A consequence of this observation is that all roots of $p(z)$ are nonzero.

[3] Including their multiplicities

zero as its value on $\mathcal{B}$ and must be either negative or positive there through its continuity. However, if a root r_i of p appears on $\mathcal{B}$ then the polynomial has to behave as an even power in the neighborhood of r_i (eg. $(s-r_i)^2, (s-r_i)^4, \ldots$ in the continuous time case) in order to achieve only either positive or negative values around this point.

For a nonsingular polynomial matrix A the problem can be converted to the above scalar case by considering its determinant $a = \det(A)$ which is a symmetric scalar polynomial. $\diamond$

4. SPECTRAL FACTORIZATION AND COMPLEX COEFFICIENTS

Given a symmetric polynomial matrix P, the spectral factorization problem lies in finding such a stable polynomial A that $A^\star A = P$.

It is a well known fact in the real case that the spectral factor exists if and only if P is positive semidefinite on the bound of stability region (Kailath, 1980) (Kucera, 1991) (Jezek and Kucera, 1985). However, this necessary and sufficient condition for the spectral factor existence appears to hold in the complex case as well. Namely, if the spectral factor exists, one can write $A^\star A = P$. Substituting a point p on the bound of stability, we receive $\overline{A^T(p)}A(p) = P(p)$ in both continuous and discrete time cases. Obviously, $P(p)$ is positive semidefinite for all such p's. On the contrary, if P symmetric is positive semidefinite on the stability region bound, it has either none or coupled roots situated on this bound according to the Theorem 2. Based on this fact, existence of spectral factor can be deduced in the same way as for a real polynomial matrix, see (Youla, 1961) for instance.

For real scalar polynomials, the spectral factor is unique up to the sign in the continuous time case. For polynomial matrices, only multiplying the spectral factor from the left by a constant orthogonal matrix preserves the desired factor properties.

For discrete time real polynomials, the uniqueness is somewhat more complicated. Observe that $(\pm z^n)^\star(\pm z^n) = 1$ for every integer n. Moreover, no other polynomials exist fulfilling this equation. To show this, suppose $a(z)$ being a polynomial of degree d satisfying $p(z) = a^\star(z)a(z) = 1$. Comparing particular degrees we arrive at a set of equations $p_k = p_{-k} = \sum_{i=0}^{n-k} a_i a_{k+i} = \delta(k), k = 0, 1, \ldots, d$ $\delta(k) = 1$ for $k = 0, \delta(k) = 0$ otherwise. Starting with $k = n$ and proceeding backwards, we end up with all but one a_i's equal to zero and the remaining entry being plus or minus one.

Generalizations for the matrix case follows: the matrix spectral factor being multiplied from the right by $O\mathrm{diag}(z^{n_i})$ keeps its required properties. Here O is an orthogonal constant matrix of appropriate size, it means $O^T O = I$ where I stands for a unity matrix.

Nevertheless, if we require in addition the set of roots of p consist of the union of roots of $A(z)$ and $A^\star(z)$, no additional factors of the z^{n_i} type are acceptable and the uniqueness relations in the continuous and discrete time cases become the same.

Involving complex entries does not bring any surprising issues, just the orthogonality of complex constant matrices has to be considered - a complex constant matrix O is said to be orthogonal if $\overline{O^T} O = I$. For scalars, multiplying the factor by any complex root of unity is possible if we relax the realness condition.

Among the set of spectral factors, one with some desirable properties can be chosen. Typically triangularity of either leading or trailing coefficient is required in combination with its realness and the positiveness of its diagonal entries in the real case. The factor thus obtained is unique.

5. ALGORITHMS FOR COMPLEX SPECTRAL FACTORIZATION

We present two numerical algorithms for complex spectral factorization in this section. While the former one relies on discrete Fourier transform (DFT) techniques, the latter method is based on Newton iterative scheme. Both methods grow up from existing numerical routines developed for real polynomial spectral factorization problem.

5.1 *DFT and Spectral Factorization*

Quite recently a new fast numerical algorithm based on DFT and addressing the real scalar discrete time spectral factorization problem has been published in (Hromcik, *et al.*, 2001) by the authors of this paper.

In the sequel we review the method in the case of real symmetric polynomials positive on the unit circle as it was presented in (Hromcik, *et al.*, 2001). Considerations concerning involving complex coefficients then follow.

5.1.1. *Real Case* The spectral factorization problem presents solving equation $a(z)a(z^{-1}) = p(z)$ with the stability constraint on $a(z)$. In order to solve this equation, logarithm is applied.

The polynomial $p(z)$ to be factored is positive for $|z| = 1$ by assumption. As a result, $p(z)$ is analytic and nonzero in $1 - \varepsilon < |z| < 1 + \varepsilon$ for some sufficiently small ε, and the factor $a(z)$ is analytic and nonzero in $1 - \varepsilon < |z|$ including $z = \infty$. The single valued branches of the logarithms then exist: $\ln a(z) = y(z),\ \ \ln p(z) = n(z)$. Here $n(z)$, obtained from the given $p(z)$, is also analytic in $1 - \varepsilon < |z| < 1 + \varepsilon$ and can be expressed as a symmetric (infinite) power series $n(z) = \cdots + n_1 z + n_0 + n_1 z^{-1} + \cdots$. It can be directly decomposed, $n(z) = y(z) + y(z^{-1})$ with power series $y(z^{-1}) = y_0 + y_1 z^{-1} + \cdots = \frac{n_0}{2} + n_1 z^{-1} + \cdots$, analytic for $1 - \varepsilon < |z|$. Finally, the spectral factor $a(z)$ is recovered as $a(z) = \mathrm{e}^{y(z^{-1})} = a_0 + a_1 z^{-1} + \cdots$. Since $y(z)$ is analytic in $1 - \varepsilon < |z|$, so is $a(z)$ and hence it can be expanded according to (2). Moreover, as a result of exponential function, $a(z)$ is nonzero in $1 - \varepsilon < |z|$. In other words, it has all its zeros inside the unit disc and is therefore Schur stable. Note also that $a(z)$ has to be a (finite) polynomial (due to the uniqueness of the solution to the problem which is known to be a polynomial) though $y(z)$ is an infinite power series.

Based on these considerations, numerical implementation of the procedure follows. For mutual conversion between time and frequency domains, the DFT performed via the FFT algorithm is used. The resulting procedure is described step-by-step in (Hromcik, *et al.*, 2001).

5.1.2. *Complex Coefficients Case* Almost the whole procedure can be kept as it is for complex entries as well. Nevertheless, involving complex coefficients at the input brings slight modifications that have to be considered. Inspecting particular steps of the algorithm gives rise to the following observations (refer to (Hromcik, *et al.*, 2001) for details):

Step 2 If the vector of coefficients $\boldsymbol{p}$ is not real, $\boldsymbol{P}$ is not symmetric. Nevertheless, its realness is preserved owing to the symmetry of $\boldsymbol{p}$. Moreover $P_i > 0$ holds by assumption ($p(z)$ is nonnegative on the unit circle).

Step 3 For any X complex,

$$\ln(X) = \ln(|X|e^{j\phi}) = Y^0 + j2k\pi$$

holds where $Y^0 = \ln|X| + j\phi$ is the main value of $\ln(X)$, $\phi \in (-\pi, \pi)$, and k is an integer. This formula inserted to the presented algorithm gives rise to $N_i = \ln(M_i) = N_i^0 + j2k\pi$. Nevertheless, the periodic term when passed through the steps 5-6 vanishes in the step 7 for any k since $\exp(j2k\pi) = 1$ holds. Hence the choice of particular branch of $\ln(X)$ is not an issue. Taking the main real values of the logarithms seems to be a reasonable choice in general. (Hromcik, *et al.*, 2001).

Step 5 Despite the fact that neither of the vectors $\boldsymbol{P}$ and $\boldsymbol{N}$ are symmetric, the coefficient vector $\boldsymbol{n}$ features symmetry. Hence splitting $\boldsymbol{n}$ makes sense in the complex case as well.

All remaining steps are independent on the real or imaginary nature of the coefficients.

5.2 *Newton-Raphson Iterations*

First results on the relationship of spectral factorization and the theory of Newton's method in general Banach spaces are dated back to the 60's (Wilson, 1969). The early works by Vostrý concerned scalar discrete-time symmetric polynomials. Generalizations for continuous time symmetry followed (Vostry, 1975). Finally the method was proved to work for matrix symmetric polynomials as well in the paper (Jezek and Kucera, 1985).

However, all these reports take care about polynomials with real coefficients only and the behaviour of the Newton's iterations for complex polynomials is an open problem in fact. Hence, in the rest of this section we pay attention to this case.

5.2.1. *Newton's Method* Solving the spectral factorization problem is equivalent to finding a solution to the equation $f(A) = A^\star A - P = 0$ under the constraint of stability. Applying the Newton's scheme, considering $df(A) = A^\star dA + (dA)^\star A$, and replacing dA by $A_i - A_{i+1}$, we come to the formula

$$A_i^\star A_{i+1} + A_{i+1}^\star A_i = P + A_i^\star A_i \qquad (1)$$

for the succeeding iteration A_{i+1}. Moreover, stability and uniqueness of successive A_i's is guaranteed in the real case provided the initial A_0 is stable and triangularity of either leading or constant coefficient of all A_i's is required.

Nevertheless, despite only real matrices have been considered in the mentioned papers, the proof itself relies on the theory of complex valued functions. For this reason it is not surprising that including complex numbers does not change its main ideas in principle and the method remains valuable also if complex polynomials are involved.

Hence the only remaining issue is to find the solution to the symmetric polynomial equation with complex entries. As we have mentioned already, this problem was resolved successfully for the discrete time symmetry when a reliable algorithm based on Sylvester matrices was proposed in (Henrion, *et al.*, 1999).

We focus on complex continuous time polynomials in the sequel and develop a numerical method for related symmetric equation. The polynomial problem is rephrased in terms of Sylvester constant matrices so that powerful numerical tools of linear algebra could be utilized directly. Thanks to this the resulting algorithm features both high efficiency and reliability. Besides its contribution to the complex spectral factorization problem, the proposed method is of its own significance. The symmetric equations belong to a wider class of linear polynomial Diophantine equations standing in the core of almost all polynomial design procedures for linear controllers and filters.

5.2.2. *Complex Continuous Time Symmetric Equation in Scalar Case - an Algorithm* For $a(s)$ and $b(s)$ scalar the concerned equation $a^{\star}(s)x(s)+a(s)x^{\star}(s)=b(s)$ reads

$$(\bar{a}_0-\bar{a}_1 s+\bar{a}_2 s^2-\cdots+(-1)^{\delta a}\bar{a}_{\delta a}s^{\delta a})(x_0+x_1 s+\cdots+x_{\delta x}s^{\delta x})+$$
$$+(a_0+a_1 s+\cdots+a_{\delta a}s^{\delta a})(\bar{x}_0-\bar{x}_1 s+\bar{x}_2 s^2-\cdots+(-1)^{\delta x}\bar{x}_{\delta x}s^{\delta x})=$$
$$=b_0+b_1 s+\cdots+b_{\delta b}s^{\delta b}.$$

By inspection, the considered polynomial equation is equivalent to the following set of constant linear matrices for coefficients of $x(s)$:

$$\underbrace{\begin{bmatrix} \bar{a}_0 & & 0 \\ -\bar{a}_1 & \bar{a}_0 & \\ \vdots & -\bar{a}_1 & \ddots & \bar{a}_0 \\ \bar{a}_\delta(-1)^\delta & & \ddots & -\bar{a}_1 \\ & \bar{a}_\delta(-1)^\delta & & \vdots \\ 0 & & \ddots & \bar{a}_\delta(-1)^\delta \end{bmatrix}}_{\overline{A_1}} \underbrace{\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_\delta \end{bmatrix}}_{X} +$$

$$+\underbrace{\begin{bmatrix} a_0 & & 0 \\ a_1 & -a_0 & \\ \vdots & -a_1 & \ddots & a_0(-1)^\delta \\ a_\delta & & \ddots & a_1(-1)^\delta \\ & -a_\delta & & \vdots \\ 0 & & \ddots & a_\delta(-1)^\delta \end{bmatrix}}_{A_2} \underbrace{\begin{bmatrix} \bar{x}_0 \\ \bar{x}_1 \\ \vdots \\ \bar{x}_\delta \end{bmatrix}}_{\overline{X}} = \underbrace{\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_\delta \end{bmatrix}}_{B}$$

Here δ is an integer such that $\delta \geq \max(\delta a, \delta b, \delta x)$. The terms a_i, b_i, and x_i respectively are zeros for $i \geq \delta a$, resp. $i \geq \delta b$, resp. $i \geq \delta x$.

This set can be rearranged as

$$\underbrace{\left(\begin{bmatrix} Re[A_1] & Im[A_1] \\ -Im[A_1] & Re[A_1] \end{bmatrix} + \begin{bmatrix} Re[A_2] & Im[A_2] \\ Im[A_2] & -Re[A_2] \end{bmatrix}\right)}_{A} \times$$
$$\times \underbrace{\begin{bmatrix} Re[X] \\ Im[X] \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} Re[B] \\ Im[B] \end{bmatrix}}_{B} \qquad (2)$$

If δ satisfies $\delta \geq \delta b$ then a solution to the above constant matrix equation exists. Particular coefficients of $x(s)$ can be directly distilled from the vector $\boldsymbol{X}$.

For complex polynomial matrices the problem of linear symmetric equations becomes far more complicated. Related issues are now the subject of further research.

6. CONCLUSION

The spectral factorization of symmetric polynomials with complex coefficients has been studied. Both scalar and matrix polynomials were considered as well as continuous and discrete time cases. Following the theoretical ideas two algorithms for real spectral factorization have been reviewed and adopted accordingly to work for complex polynomials too. These methods are based on discrete Fourier transform techniques and the Newton-Raphson iterative scheme respectively. The latter of the routines relies on the solution to the linear symmetric polynomial equation with complex coefficients. Since only results concerning the discrete-time case are known at the moment, a reliable numerical method for related complex continuous-time symmetric equation has been proposed in the paper.

7. REFERENCES

Ahlen, A. and Sternad, M. (1993). Optimal Filtering Problems. In: *Polynomial Methods in Optimal Control and Filtering* (Hunt, K.J. (Ed)), Chapter 5. Peter Peregrinus, London.

Sternad, M. and Ahlen, A. (1996). H_2 Design of Model-Based Nominal and Robust Discrete Time Filters. In: *A Polynomial Approach to H-2 and H-infinity Robust Control Design* (Grimble, M.J. and Kucera, V. (Eds)), Chapter 5. Springer-Verlag, Berlin.

Kailath, T. (1980). *Linear Systems.* Prentice Hall, New York.

Kučera, V. (1991). *Analysis and Design of Discrete Linear Control Systems.* Academia, Prague.

Youla, D.C. (1961). On the Fctorization of Rational Matrices. *IEEE Transactions on Information Theory*, **IT-17**, 732-739.

Hromčík, M., Ježek, J. and Šebek, M. (2001). New Algorithm for Spectral Factorization Based on FFT. In: *Proceedings of ECC'01*, 231-236. University of Porto, Porto.

Henrion, D., Ježek, J. and Šebek, M. (1999). Efficient Algorithms for Discrete-Time Symmetric Polynomial Equations with Complex Coefficients. In: *Proceedings of the IFAC World Congress*, 762-767. IFAC, Beijing.

Ježek, J. and Kučera, V. (1985). Efficient Algorithm for Matrix Spectral Factorization, *Automatica*, **29**, pp. 663-669.

Wilson, G.T. (1969). Factorization of the Covariance Generation Function of a Pure Moving Average Process, *SIAM Journal on Numerical Analysis*, **6**, 452-457.

Kwakernaak, H. and Šebek, M. (1999). PolyX Home Page. http://www.polyx.cz/.