15th Triennial World Congress, Barcelona, Spain

DYNAMIC MODELLING AND CONFIGURATION STABILIZATION FOR AN X4-FLYER.

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Abstract: A model for the dynamics of a four rotor vertical take-off and landing (VTOL) vehicle known as an X4-flyer is proposed. The model incorporates the airframe and motor dynamics as well as aerodynamic and gyroscopic effects due to the rotors for quasi-stationary flight conditions. A novel control strategy is proposed for configuration stabilization of quasi-stationary flight conditions. The approach taken involves separating the rigid body (airframe) dynamics from the motor dynamics, developing separate control Lyapunov functions for the coupled systems and then bounding the perturbation error due to the interaction to obtain strong practical stability of the complete system.

Keywords: Mobile Robots, Unmanned Aerial Vehicle (UAV), control Lyapunov function.

1. INTRODUCTION

Recent advances in computer and sensing technology, and the associated reduction in cost of such systems, have made physical construction of autonomous mobile robotic systems possible at a reasonable price. Autonomous robotic cars and trucks have been under investigation for some years and, as well as the development of considerable body knowledge, there now several successful commercial systems. More recently, interest is growing in more complicated systems such as submarine vehicles (Egland *et al.*, 1996) and unmanned aerial vehicles (Koo *et al.*, 1998), because of their commercial possibilities. Fixed-wing unmanned aircraft are being routinely used for military and meteorological purposes and have been in service for years. The vehicle considered in this paper is an autonomous hovering system, capable of vertical take-off, landing and quasi-stationary (hover and near hover) flight conditions. In this paper, we propose a model for the dynamics of a four rotor vertical take-off and landing (VTOL) vehicle known as an X4-flyer. The model includes the airframe and motor dynamics as well as aerodynamic and gyroscopic effects due to the rotors. The fixed pitch, rigid rotors and four motor actuation leads to significant differences in the model proposed to other VTOL systems previously studied in the literature (Hauser *et al.*, 1992, Koo *et al.*, 1998). The control design proposed is based on separating the rigid body (airframe) dynamics from the motor dynamics, developing separate control Lyapunov functions for the coupled systems and then bounding the perturbation error due to the interaction to obtain strong practical stability of the complete system. The control strategy is novel in two ways. Firstly, the system dynamics are controlled in two separate dynamic systems corresponding to the rigid body dynamics and the motor dynamics. The separate system errors are combined into a single control Lyapunov function via a transient error bounding argument. This approach avoids the either the necessity of including a dynamic extension in the controller design (Koo *et al.*, 1998, Mahony *et al.*, 1999) or the need to use approximate linearization, saturated control or dynamic reduction (high gain) controllers (Frazzoli *et al.*, 2000, Teel 1996, Sepulchre *et al.*, 1997). The closed loop system is practically stable for trajectory tracking of the center of mass of the robotic vehicle. The second novel aspect of the control is the use of a quaternion representation of the rotation error in order to obtain a simple, smooth control design that contains only a single singularity in error space corresponding to an error of 180 degrees in the rotation. This is a considerable advance over earlier work by the authors and compliments recent work by Frazzoli et al.(Frazzoli *et al.*, 2000). The approach taken is based on earlier work by Egland et al.(Egeland *et al.*, 1996).

2. THE X4-FLYER MODEL

The X4-flyer [1] is a system consisting of four individual electrical fans attached to a rigid cross frame. It is an omnidirectional (vertical take-off and landing) VTOL vehicle ideally suited to stationary and quasi-stationary flight conditions. Control of an X4-flyer is achieved by differential control of the thrust generated by each electric fan. Up down motion is controlled by collectively increasing or decreasing the power of all four motors. Sideways motion is achieved by pitching in the desired direction and increasing collective thrust to overcome the tendency of the vehicle to side-slip towards the ground (cf. Figure 1). The rolling motion is achieved by increasing, for example, the power of the left rotor and decreasing that of the right rotor in proportion to preserve total collective thrust. By the same principle, differential control of the forward and rear rotors leads to control of the pitching motion of the vehicle. The 'yaw' control mechanism is more subtle. When a rotor turns, it has to overcome air resistance. The reactive force acts on the rotor in the direction opposite to the rotation of the rotor. In the X4-flyer both sets of front-rear and left-right rotors turn in opposite direction (cf. Figure 1). However as long as all rotors produce the same torque, more precisely produce the same reactive torque, which is mostly a function of speed of rotation and rotor blade pitch, the sum of all air resistances is zero and there is no horizontal rotation. If one set of rotors increase their speed, the induced torque will cause the X4-flyer to rotate in the direction of the induced torque. It is important to note that because of the "$\times$" arrangement, this operation has no effect on translation in x or y direction. The effect on up/down motion can be compensated by reducing the pitch or speed of the other diagonal pair.

The following dynamic model of an X4-flyer is presented for the simple case where the rotors are fixed pitch, rigid rotors and thrust control is obtained through control of the torque to the motors. Let $\mathcal{I} = \{E_x, E_y, E_z\}$ denote a right-hand inertial frame such that E_z denotes the vertical direction downwards into the earth. Let the vector $\xi = (x, y, z)$ denote the position of the centre of mass of the airframe in the frame $\mathcal{I}$ relative to a fixed origin $0 \in \mathcal{I}$. Let $\mathcal{A} = \{E_1^a, E_2^a, E_3^a\}$ be a (right-hand) body fixed frame for the airframe. The orientation of the rigid body is given by a rotation $R : \mathcal{A} \to \mathcal{I}$, where $R \in SO(3)$ is an orthogonal rotation matrix.

[1] The authors propose the term 'X4-flyer' as a simple, highly descriptive name that will apply to a wide range of four rotor flying robots.

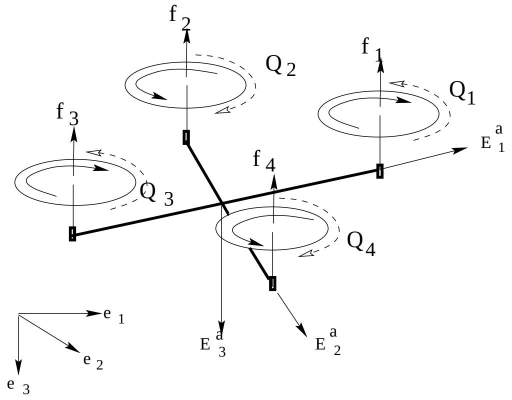

Fig. 1. The four rotors hover system with Force and Torque Control.

Let $v \in \mathcal{I}$ denote the linear velocity expressed in the inertial frame and $\Omega \in \mathcal{A}$ denote the angular velocity of the airframe expressed in the body fixed frame. Let m denote the mass of the rigid object and $\mathbf{I} \in \mathbb{R}^{3\times3}$ denote the constant inertia matrix around the centre of mass (expressed in the body fixed frame $\mathcal{A}$). Newton's equations of motion yield the following dynamic model for the motion of the airframe:

$$\dot{\xi} = v \tag{1}$$

$$m\dot{v} = mge_3 + RF \tag{2}$$

$$\dot{R} = R\text{sk}(\Omega), \tag{3}$$

$$\mathbf{I}\dot{\Omega} = -\Omega \times \mathbf{I}\Omega + \Gamma. \tag{4}$$

The notation $\text{sk}(\Omega)$ denotes the skew-symmetric matrix such that $\text{sk}(\Omega)v = \Omega \times v$ for the vector cross-product $\times$ and any vector $v \in \mathbb{R}^3$. The vector $F \in \mathcal{A}$ combines the principal non-conservative forces applied to the X4-flyer airframe including thrusts (generated by the rotors cf. Figure 1) and drag terms associated with the rotors downwash on the airframe. The torque $\Gamma \in \mathcal{A}$ is derived from differential thrust associated with pairs of rotors along with aerodynamic effects and gyroscopic effects.

Due to the rigid rotor constraint the dynamics of each rotor disk around its axis of rotation can be treated as a decoupled system in the generalized variable ϖ_i denoting angular velocity of a rotor around its axis. The torque exerted by each electrical motor is denoted τ_i. The motor torque is opposed by an aerodynamic drag Q_i. Newton's equations are

$$\mathbf{I}_r\dot{\varpi}_i = \tau_i - Q_i \tag{5}$$

where $\mathbf{I}_r$ is the moment of inertia of a rotor around its axis.

The lift generated by a rotor in free air may be modelled as

$$f_i := -b\varpi_i^2 e_3$$

where $b > 0$ is a proportionality constant depending on the density of air, the cube of the radius of the rotor blades, the number of blades, the chord length of the blades, the lift constant (linking angle of attack of the blade airfoil to the lift generated), the drag constant (associated with the airframe) and the geometry of the wake [2]. For quasi-stationary manoeuvres in free, still air it is a reasonable assumption that the scalar $b > 0$ is indeed a constant.

The reactive torque (due to rotor drag) generated by a rotor in free air may be modelled as

$$Q_i := \kappa \varpi_i^2$$

The constant κ depends once again on the factors mentioned above for rotor thrust and particularly on the pitch angle of the rotor blades.

The thrust applied to the X4-flyer airframe is

$$T = \sum_{i=1}^{4} |f_i| = b \left(\sum_{i=1}^{4} \varpi_i^2 \right) \tag{6}$$

Recalling the discussion preceding the model Eqn's 1-4, the aerodynamic torque inputs applied to the X4-flyer structure using the combination of the produced forces and air resistances are $\tau_a = (\tau_a^1, \tau_a^2, \tau_a^3)$

$$\begin{aligned}
\tau_a^1 &= db\left(\varpi_2^2 - \varpi_4^2\right) \\
\tau_a^2 &= db\left(\varpi_1^2 - \varpi_3^2\right) \\
\tau_a^3 &= \kappa\left(\varpi_2^2 + \varpi_4^2 - \varpi_1^2 - \varpi_3^2\right)
\end{aligned}$$

where d represents the displacement of the rotors with respect to the centre of mass of the X-4 flyer.

The final torque contribution to the X4-flyer dynamics comes from gyroscopic effects. Each rotor may be thought of as a rigid disk rotating around the axis e_3 in the body-fixed-frame with angular velocity ϖ_i. The axis of rotation of the rotor is itself moving with the angular velocity of the airframe. This leads to the following gyroscopic torques applied to the airframe

$$G_a = -\sum_{i=1}^{4} I_r(\Omega \times e_3)\varpi_i.$$

Based on the above discussion the following model is proposed:

$$\dot{\xi} = v \tag{7}$$

$$\dot{v} = ge_3 - \frac{1}{m}TRe_3 \tag{8}$$

$$\dot{R} = R\mathrm{sk}(\Omega), \tag{9}$$

$$\mathbf{I}\dot{\Omega} = -\Omega \times \mathbf{I}\Omega + G_a + \tau_a. \tag{10}$$

$$\mathbf{I}_r\dot{\varpi}_i = \tau_i - \kappa\varpi_i^2. \tag{11}$$

The dynamic equations may be thought of in two parts; firstly the rigid body dynamics of the airframe Eqn's 7-10 with inputs $(T, \tau_a^1, \tau_a^2, \tau_a^3)$ and secondly the Eq. 11 that links the motor torque inputs τ_i to the rigid body forces and torques via the mapping

$$\begin{pmatrix} T \\ \tau_a^1 \\ \tau_a^2 \\ \tau_a^3 \end{pmatrix} = \begin{pmatrix} -b & -b & -b & -b \\ 0 & db & 0 & -db \\ db & 0 & -db & 0 \\ k & -k & k & -k \end{pmatrix} \begin{pmatrix} \varpi_1^2 \\ \varpi_2^2 \\ \varpi_3^2 \\ \varpi_4^2 \end{pmatrix} = A \begin{pmatrix} \varpi_1^2 \\ \varpi_2^2 \\ \varpi_3^2 \\ \varpi_4^2 \end{pmatrix} \tag{12}$$

It is easily verified that the matrix $A \in \mathbb{R}^{4\times 4}$ defined above is full rank for $b, k, d > 0$.

[2] For a detailed discussion of the aerodynamic model of a helicopter rotor the reader is referred to any standard text on helicopter modelling (cf. for example (Prouty 1995)). A condensed discussion is given in (Mahony and Hamel 2001).

3. CONTROL DESIGN METHODOLOGY

In this section a backstepping control design is provided for the model Eqn's 7-11 proposed in the previous section.

Let $\xi_d(t)$ be the desired position trajectory. The dynamics associated with tracking such a trajectory fully determine two degrees of freedom (pitch and roll) in the attitude of the airframe. The yaw of the airframe must be separately assigned. There is no 'correct' way in which this assignment should be made. In this paper we use the classical 'yaw', 'pitch' and 'roll' Euler angles (ϕ, θ, ψ) commonly used in aerodynamic applications (Murray *et al.*, 1994). Although these angles are not globally defined they provide a suitable local representation for all quasi-stationary manoeuvrers undertaken by an X4-flyer. The yaw angle trajectory is specified directly in terms of the angle $\phi_d(t)$. The relationship between the Euler angles used and the rotation matrix is [3]

$$R = \begin{pmatrix} c_\theta c_\phi & s_\psi s_\theta c_\phi - c_\psi s_\phi & c_\psi s_\theta c_\phi + s_\psi s_\phi \\ c_\theta s_\phi & s_\psi s_\theta s_\phi + c_\psi c_\phi & c_\psi s_\theta s_\phi - s_\psi c_\phi \\ -s_\theta & s_\psi c_\theta & c_\psi c_\theta \end{pmatrix}. \tag{13}$$

The trajectory tracking control problem considered is:

Find a smooth static state feedback $(\tau_1, \tau_2, \tau_3, \tau_4)$ depending only on the measurable states $(\xi, \dot{\xi}, R, \Omega)$, the angular velocity of each rotor (ϖ_i) and arbitrarily many derivatives of the smooth trajectory $(\xi_d(t), \phi_d(t))$ such that the tracking error $(\xi(t) - \xi_d(t), \phi(t) - \phi_d(t))$ is asymptotically stable.

Define

$$\begin{aligned}
\delta_1 :=& \xi(t) - \xi_d(t) \\
\delta_2 :=& \frac{1}{k_1}(v - \dot{\xi}_d) + \delta_1
\end{aligned} \tag{14}$$

where k_1 is a positive constant. Let S_1 be the first storage function for the backstepping procedure. It is chosen for the full linear dynamics Eqn's 7-8

$$S_1 = \frac{1}{2}|\delta_1|^2 + \frac{1}{2}|\delta_2|^2. \tag{15}$$

Taking the time derivative of S_1 and substituting for Eq. 8 yields

$$\frac{d}{dt}S_1 = -k_1|\delta_1|^2 + k_1|\delta_2|^2 + \frac{1}{k_1}\delta_2^T(-\frac{1}{m}TRe_3 + ge_3 - \ddot{\xi}_d), \tag{16}$$

From the point of view of a classical backstepping control design the vectorial term $(1/m)TRe_3$ is the virtual input to this stage of the backstepping design. To apply this approach in full generality it is necessary to dynamically extend the thrust input T in order that the vectorial virtual control assigned (for Eq. 16) can be cascaded through the attitude dynamics (Mahony *et al.*, 1999 and Frazzoli *et al.*, 2000). In the case of the X4-flyer, the

[3] The following shorthand notation for trigonometric function is used:

$$c_\beta := \cos(\beta), \quad s_\beta := \sin(\beta), \quad t_\beta := \tan(\beta).$$

highly coupled nature of the motor dynamics that generate both torque and thrust mean that this approach is not recommended. Alternatively, the vectorial input can be split into its magnitude T, that is linked directly to the motor torques via Eqn's 11 and 12, and its direction Re_3, that defines two degrees of freedom in the airframe attitude dynamics Eqn's 9 and 10. Assigning the thrust T immediately and then controlling the attitude dynamics leads to a design approach similar to those proposed for the VTOL (Teel 1996, Sepulchre *et al.*, 1997). Such control strategies lead to time scale separation between the attitude and linear dynamics of the airframe dynamics and require significant control response in the thrust input T. Once again the coupled nature of the motor dynamics indicate that this approach is not advised. The approach taken in the present paper discards the concept of exact linearization or classical backstepping and falls back on a control Lyapunov function design for the full dynamics. A backstepping design for the rigid-body dynamics of the airframe is undertaken, however, we do not attempt to directly cancel the effect of the virtual control error in the rigid-body dynamics within the backstepping control design itself. Rather, these errors are left as perturbations to the rigid-body dynamics. The errors introduced into the control design are linear in an error criterion that forms the basis of a second control Lyapunov design for the motor dynamics Eq. 11.

Applying classical backstepping one would assign a virtual vectorial control for $(1/m)TRe_3$

$$(TRe_3)_d := mge_3 - m\ddot{\xi}_d + mk_1(k_1+k_2)\delta_2, \quad k_2 > 0 \tag{17}$$

Here $(TRe_3)_d$ denotes the desired vectorial control input. Taking the norm of the right hand side of Eq. 17 leads to

$$T_d = |mge_3 - m\ddot{\xi}_d + mk_1(k_1+k_2)\delta_2|. \tag{18}$$

The desired rotation matrix R_d

$$R_d e_3 := \frac{1}{T_d}(TRe_3)_d \tag{19}$$

is obtained by solving for (ϕ, θ, ψ) using Eq. 13 and subject to the constraint given by the specification of $\phi_d(t)$ [4].

Substituting for Eqn's 18-19 one obtains

$$\begin{aligned}\dot{\delta}_2 = &- k_1\delta_1 - k_2\delta_2 - \frac{1}{mk_1}(\tilde{R} - I)T_d R_d e_3 \\ &- \frac{1}{mk_1}\tilde{T}Re_3\end{aligned} \tag{20}$$

where

$$\tilde{T} = T - T_d, \quad \tilde{R} = RR_d^T \in SO(3).$$

From the above discussion the dynamics of the first storage function S_1 (Eq. 16) can be bounded as

$$\begin{aligned}\frac{d}{dt}S_1 \leq &- k_1|\delta_1|^2 - k_2|\delta_2|^2 \\ &+ \frac{|T_d|}{mk_1}|\delta_2|\,|\tilde{R} - I| + \frac{|\tilde{T}|}{mk_1}|\delta_2|\end{aligned} \tag{21}$$

[4] This only one of a number of possibilities for fully determining R_d. The key point is that R_d is fully defined by the vectorial constraint on $R_d e_3$ combined with some additional constraint that fixes the yaw parameter.

Note that the error terms $|\delta_2|$, $|\tilde{T}|$ and $|\tilde{R} - I|$ enter bilinearly into the last two terms of this expression.

The next stage of the control design involves controlling the attitude dynamics such that the error $(\tilde{R} - I)$ is minimized. Designing controllers to stabilize attitude dynamics has been an awkward problem in recent papers (Koo *et al.*, 1998, Mahony *et al.*, 1999, Frazzoli *et al.*, 2000). The key problem comes in finding an elegant method of representing the attitude of the system that does not suffer from singularities and leads to a simple control design. In this paper we employ a quaternion representation of the rotation in order to obtain a globally defined smooth static control for the attitude dynamics with a single singularity corresponding to an attitude error of 180 degrees. The attitude deviation $\tilde{R}$ is parameterized by a rotation $\tilde{\gamma}$ around the unit vector $\tilde{k}$. Using Rodrigues' formula one has (Murray *et al.*, 1994)

$$\tilde{R} = I + \sin(\tilde{\gamma})\text{sk}(\tilde{k}) + (1 - \cos(\tilde{\gamma}))\text{sk}(\tilde{k})^2$$

The quaternions describing the deviation $\tilde{R}$ are given by (Egeland *et al.*, 1996):

$$\tilde{\eta} := \sin\frac{\tilde{\gamma}}{2}\tilde{k}, \quad \tilde{\eta}_0 := \cos\frac{\tilde{\gamma}}{2}$$

which are subject to the constraint:

$$|\tilde{\eta}|^2 + \tilde{\eta}_0^2 = 1 \tag{22}$$

The deviation matrix $\tilde{R}$ is then defined as follows:

$$\tilde{R} = (\tilde{\eta}_0^2 - |\tilde{\eta}|^2)I + 2\tilde{\eta}\tilde{\eta}^T + 2\tilde{\eta}_0\text{sk}(\tilde{\eta}) \tag{23}$$

The attitude control objective is achieved when $\tilde{R} = I$. It is easy to see from Eqn's 22-23 that this is equivalent to $\eta = 0$ and $\tilde{\eta}_0 = 1$. Indeed, it may be verified that

$$|\tilde{R} - I|_F = \sqrt{\text{tr}((\tilde{R} - I)^T(\tilde{R} - I))} = 2\sqrt{2}|\tilde{\eta}| \tag{24}$$

Based on this result the attitude control objective used is to drive $\tilde{\eta}$ to zero. Deriving (η, η_0) yields (Murray *et al.*, 1994, pg. 74)

$$\dot{\tilde{\eta}} = \frac{1}{2}(\tilde{\eta}_0 I + \text{sk}(\tilde{\eta}))\tilde{\Omega}, \quad \dot{\tilde{\eta}}_0 = -\frac{1}{2}\tilde{\eta}^T\tilde{\Omega} \tag{25}$$

where $\tilde{\Omega}$ defines the error angular velocity

$$\tilde{\Omega} = R_d(\Omega - \Omega_d) \tag{26}$$

and Ω_d represents the desired angular velocity. It is defined by in Appendix A.

The virtual control $\tilde{\Omega}^v$ is defined to ensure the following storage function decreases,

$$W_1 := \frac{1}{2}|\tilde{\eta}|^2$$

Set

$$\tilde{\Omega}^v = -2k_\eta\tilde{\eta}_0\tilde{\eta}$$

With the above choice one has

$$\dot{W}_1 \leq -k_\eta\tilde{\eta}_0^2|\tilde{\eta}|^2 + \tilde{\eta}_0 k_\eta\tilde{\eta}^T\varepsilon + (k_1 + k_2)\frac{|\tilde{T}|}{T_d}|\tilde{\eta}|$$

where the error approximation follows from the error introduced in the definition of X in Eq. 36 and ε represents the final error used in the process of backstepping

$$\varepsilon := \frac{1}{2k_\eta}\tilde{\Omega} + \tilde{\eta}_0\tilde{\eta}. \tag{27}$$

to simplify the control design a control input linearization of equation Eq. 10 is undertaken. Define

$$w := -\mathbf{I}^{-1}\Omega \times \mathbf{I}\Omega + \mathbf{I}^{-1}G_a + \mathbf{I}^{-1}\tau_a^d \qquad (28)$$

where τ_a^d is the desired torque input. Since $\mathbf{I}$ is full rank then this is certainly a bijective control input transformation between τ_a^d and w. With this choice Eq. 10 becomes

$$\dot{\tilde{\Omega}} = \tilde{w} + R_d\mathbf{I}^{-1}\tilde{\tau}, \qquad (29)$$

where

$$\tilde{w} = R_d(w-\dot{\Omega}_d)+R_d\text{sk}(\Omega_d)R_d^T\tilde{\Omega}, \ \text{ and, } \ \tilde{\tau} = \tau_a-\tau_a^d.$$

The torque error $\tilde{\tau}$ acts as a perturbation error in the control Lyapunov function for the rigid body dynamics and will be used as a basic error signal for design of the control Lyapunov function for the motor dynamics. The 'error' input $\tilde{w}$ may be arbitrarily assigned via its dependence on w.

Set

$$\begin{aligned}\tilde{w} :=& 2k_\eta\left(k_\eta\tilde{\eta}_0^3\tilde{\eta} - \frac{1}{2}\tilde{\eta}_0\text{sk}(\tilde{\eta})\tilde{\Omega} - k_\eta\tilde{\eta}_0^2\varepsilon + \frac{1}{2}\tilde{\eta}\tilde{\eta}^T\Omega\right)\\ &- 2k_\eta^2\tilde{\eta}_0\tilde{\eta} - 2k_\eta k_\varepsilon\tilde{\eta}_0^2\varepsilon \qquad (30)\end{aligned}$$

Define S_2 the storage function for the attitude deviation

$$S_2 = \frac{1}{2}|\tilde{\eta}|^2 + \frac{1}{2}|\varepsilon|^2. \qquad (31)$$

Taking the derivative of S_2, substituting for the derivative of ε and taking care to identify all terms that are due to the error approximation made in Eqn's 36 and 29 leads to

$$\begin{aligned}\dot{S}_2 \leq& -k_\eta\tilde{\eta}_0^2|\tilde{\eta}|^2 - k_\varepsilon\tilde{\eta}_0^2|\varepsilon|^2\\ &+ (k_1+k_2)\frac{|\tilde{T}|}{T_d}(|\tilde{\eta}_0||\tilde{\varepsilon}| + |\tilde{\eta}|) + \frac{|\mathbf{I}^{-1}|}{2k_\eta}|\varepsilon|\,|\tilde{\tau}| \qquad (32)\end{aligned}$$

This completes the control design for the attitude dynamics. Observe that all the error terms in this expression are bilinear in the error variables. Moreover, the final two terms in the expression depend on the control errors $|\tilde{T}|$ and $|\tilde{\tau}|$.

The control design for the motor actuators are based on minimizing the control errors $|\tilde{T}|$ and $|\tilde{\tau}|$ for the rigid body dynamics. To simplify the notation set

$$\begin{aligned}u &= (T, \tau_a^1, \tau_a^2, \tau_a^3),\\ u_d &= (T_d, (\tau_a^1)_d, (\tau_a^2)_d, (\tau_a^3)_d)\end{aligned}$$

and $\tilde{u} = u - u_d$. Furthermore, set $\varpi = (\varpi_1, \ldots, \varpi_4)$, $\varpi^2 = (\varpi_1^2, \ldots, \varpi_4^2)$, $\tau_m = (\tau_1, \ldots, \tau_4)$ and $\mathbf{I}_R = \text{diag}(\mathbf{I}_r, \ldots, \mathbf{I}_r)$. Then Eq. 11 can be written in block form

$$\mathbf{I}_R\dot{\varpi} = \tau_m - \kappa\varpi^2$$

Note that $\tilde{u} = A(\varpi^2 - A^{-1}u_d)$. Taking the derivative of $\tilde{u}$ it follows that

$$\dot{\tilde{u}} = A(2\varpi^T(\tau_m - \kappa A^{-1}u) - A^{-1}\dot{u}_d).$$

Set

$$\tau_m := \kappa A^{-1}u + \varpi^{-1}A^{-1}\dot{u}_d - \varpi^{-1}k_u\tilde{u}, \ \ k_u > 0,$$

where $\varpi^{-1} = (\varpi_1^{-1}, .., \varpi_4^{-1})$. This leads directly to

$$\dot{\tilde{u}} = -k_u\tilde{u}, \qquad (33)$$

and guarantees that the error $\tilde{u}$ converges exponentially to zero.

Theorem 3.1. Consider the system dynamics defined by Eqn's 7-11 along with the control inputs proposed in the body of the paper. Assume that there are constants $a, b > 0$ such that $|T_d| > a$ and $|\tilde{\eta}_0| > b$. Let σ be an upper bound on desired acceleration

$$|\ddot{\xi}_d| \leq \sigma$$

Let k_1, k_2 be two positive control gains (cf. Eqn's 14 and 17). Set

$$k = k_1 + k_2, \ \epsilon = \frac{1}{k_1}k_\varepsilon b^2(k_2k_\eta b^2k_1^2 - 8(g+\sigma)^2)$$

and choose the remaining control gains to satisfy

$$\begin{aligned}&k_\eta > \frac{8(g+\sigma)^2}{k_1^2k_2b^2},\\ &k_\varepsilon > 0,\\ &k_u > \frac{m^2}{\epsilon}\Big(k_2k_1^2(k_\eta + k^2k_\varepsilon) + 4(g+\sigma)^2(2\sqrt{2}kk_\varepsilon\\ &\qquad\qquad +k_\eta k_\varepsilon - 2k^2)\Big).\end{aligned}$$

Then, for any initial condition such that the initial value of the Lyapunov function

$$V(0) = S_1(0)+S_2(0)+\frac{1}{2}|\tilde{u}(0)|^2 < \left(\frac{g+\sigma}{k_1(k_1+k_2)}\right)^2, \qquad (34)$$

the Lyapunov function is bounded for all time

$$V(t) < \left(\frac{g+\sigma}{k_1(k_1+k_2)}\right)^2$$

and is asymtotically stable. The tracking error $(\xi(t) - \xi_d(t))$ is locally exponentially stable.

Proof 3.2. Let

$$e = (|\delta_1|, |\delta_2|, |\tilde{\eta}|, |\varepsilon|, |\tilde{T}|, |\tilde{\tau}|)^T \in \mathbb{R}^6$$

be a vector of absolute errors of the backstepping errors. Recalling Eqn's 21, 24, 32 and 33, it may be directly verified that the derivative of the Lyapunov function V is bounded by

$$\dot{V} \leq -e^TAe$$

where

$$A := \begin{pmatrix} k_1 & 0 & 0 & 0 & 0 & 0\\ 0 & k_2 & -\frac{\sqrt{2}|T_d|}{mk_1} & 0 & -\frac{1}{2mk_1} & 0\\ 0 & -\frac{\sqrt{2}|T_d|}{mk_1} & k_\eta\tilde{\eta}_0^2 & 0 & -\frac{k}{2|T_d|} & 0\\ 0 & 0 & 0 & k_\epsilon\tilde{\eta}_0^2 & -\frac{k|\tilde{\eta}_0|}{2|T_d|} & -\frac{|\mathbf{I}^{-1}|}{4k_\eta}\\ 0 & -\frac{1}{2mk_1} & -\frac{k}{2|T_d|} & -\frac{k|\tilde{\eta}_0|}{2|T_d|} & k_u & 0\\ 0 & 0 & 0 & -\frac{|\mathbf{I}^{-1}|}{4k_\eta} & 0 & k_u \end{pmatrix}$$

The quadratic expression $-e^TAe$ is guaranteed negative definite if and only if the symmetric matrix A is positive definite. This is true if and only if the principal minors of A are positive.

The first two principal minors are positive definite due to the choice of $k_1, k_2 > 0$. The third principal minor is positive if

$$k_\eta > \frac{2|T_d|^2}{m^2 k_1^2 k_2 |\tilde{\eta}_0|^2}.$$

Recalling Eq. 18 and applying the bounds in the theorem statement one has

$$T_d \leq m(g+\sigma) + m k_1 (k_1 + k_2)|\delta_2|$$

Using the bound Eq. 34, it follows that

$$|T_d| < 2m(g+\sigma).$$

In addition applying the bound $|\tilde{\eta}_0| > b$ it follows that choosing k_η superior to the bound given in the theorem statement ensures that the third minor of A is positive definite. Once this choice is made it is clear that the fourth minor is also positive definite for $k_\epsilon > 0$.

The final two minors are of interest since they involve the interaction terms associated with the approximations made in the virtual control inputs during the control design. Thus, unlike the case for classical backstepping designs, the dynamics of $\tilde{T}$ and $\tilde{\tau}$ interact with all stages of the error dynamics. However, since the control gain k_u used at this stage is independent of any further calculations it may be chosen arbitrarily and indeed may even be chosen time varying to avoid robustness problems in the asymptotic limit of the control design. A more complete discussion of the potential of the proposed control design is beyond the scope of this paper. Straightforward but tedious calculations show that choosing k_u superior to the bound given in the theorem statement ensures the the final two minors of A are positive definite. The proof follows by applying Lyapunov's direct method.

Remark 3.3. The theorem statement includes simplifying bounds $|T_d| > a$ and $|\tilde{\eta}_0| > b$ as well as a bound $|\ddot{\xi}_d| \leq \sigma$. In practice, this covers all situations in which one wishes to apply the control design. It is the authors opinion that these theoretical bounds can be significantly relaxed, however, such an analysis is beyond the scope of the present paper.

APPENDIX A

Consider the kinematics of the desired attitude R_d

$$\dot{R}_d = R_d \text{sk}(\Omega_d) \tag{35}$$

From Eqn's 3 and 35, it follows

$$\dot{\tilde{R}} = \tilde{R}\text{sk}(\tilde{\Omega})$$

Deriving the expression of $R_d e_3$ (Eq. 19) one obtains

$$\text{sk}(\Omega_d) e_3 = R_d^T X, \tag{36}$$

Where X is defined as known part of $\frac{d}{dt}(R_d e_3)$. The derivative is not exactly known due to the error term $\tilde{T}$ in Eq. 20. Direct calculation leads to the error bound

$$|X - \frac{d}{dt}(R_d e_3)| \leq (k_1 + k_2)\frac{|\tilde{T}|}{|T_d|}.$$

Recalling that $\text{sk}(\Omega_d)e_3 = -\text{sk}(e_3)\Omega_d$ one has

$$\Omega_d^1 := -e_2^T R_d X, \quad \Omega_d^2 = e_1^T R_d X. \tag{37}$$

This process determines the first two components of the desired angular velocity Ω_d. To determine Ω_d^3 it is necessary to recall the kinematic relationship between the Euler angles and the angular velocity of a rigid body (cf. for example (Murray *et al.*, 1994)). One has

$$\begin{pmatrix} \dot{\phi}_d \\ \dot{\theta}_d \\ \dot{\psi}_d \end{pmatrix} = \frac{1}{c_\theta} \begin{pmatrix} 0 & s_\psi & c_\psi \\ 0 & c_\theta c_\psi & -c_\theta s_\psi \\ c_\theta & s_\theta s_\psi & s_\theta c_\psi \end{pmatrix} \Omega_d.$$

Solving for Ω_d^3 in terms of $\dot{\phi}_d$ using the first row of this equation leads to

$$\Omega_d^3 = \frac{c_\theta}{c_\psi}\dot{\phi}_d - t_\psi \Omega_d^2$$

where Ω_d^2 is given by Eq. 37.

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