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SMOOTHING STABILITY ROUGHNESS OF FRACTAL BOUNDARIES USING REINFORCEMENT LEARNING

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Abstract: We describe in this paper a new approach to the identification of the stable regions of nonlinear systems, using cell mapping equipped with measures of fractal dimension and those from rough set theory. The proposed fractal-rough set approach divides the state space into cells, finds out the chaotic region using cell to cell mapping technique and classifies the cells according to the fractal dimension of each cell. Assigning the fractal dimension to each cell in the state space, cells are then classified as the members of lower approximation, upper approximation or boundary region of the stable region with the help of rough set theory. Rough sets with fractal dimension as their attributes are used to model the uncertainty on the stable region which is treated as a set of cells in this paper. This uncertainty is then smoothed by a reinforcement learning algorithm. Our approach is applied to the stability of a dynamical system with finger shaped boundary region. *Copyright© 2002 IFAC*

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1. INTRODUCTION

In analyzing nonlinear dynamical systems, one is often interested in the local behaviour of the system around its equilibrium states. The local system behaviour information around equilibrium point and the determination of the global stable domain are critical in order to conclude on the control strategies and to assign the relevant parameters for the system controller.

In some cases, frequently the stability boundary of a domain of operation in the state space of a system is fractal and the precise identification of the stable region imbedded in that domain is nearly impossible. Since there exists an integral dimension of the region with a boundary with fractal dimension. This uncertainty in the identification renders the used classical approaches very inefficient.

In order to overcome the problem of fractal boundaries a probabilistic approach instead of a precise prediction is suggested in (Hsu, 1980a, b; 1987) where domains of attractions are generated using cell to cell mapping in examining the stability of nonlinear systems. As a result, such domains of attraction based on cell to cell mapping technique is far from being precise on the boundary of the region: having possible unstable elements, the boundary region of the domain of attraction introduces an uncertainty into the stability region.

In this paper, a new approach combining the fractal theory and the rough set theory in order to define the stability regions of nonlinear systems in a more precise manner is introduced. Our approach consists of three main steps, which are:

- conducting global stability analyses by cell to cell mapping,
- determining the boundary region of the attraction domain using rough set theory and fractal dimension concept (Section 3),
- smoothing uncertainty in the boundary region via a learning algorithm (Section 4).

Our approach uses Hsu's cell to cell mapping technique for the global stability analyses. The state space is partitioned into grids of same size determining cells. The midpoint of the cells are used as initial point of every integration step and all the points in the cell are treated the same as the centre point and are mapped into another cell according to the result of the integration process of the cellular transformation. After a number of integration processes that takes the centre of every cell as its initial point, the stable region is determined as a set of cells with a precision inversely proportional to the cell size. Thus, in order to have a true picture of stability for a dynamical system, the cells should be infinitely small which means every point in state space will then be treated as a cell and the cell to cell mapping technique becomes infeasible computationaly as a point to point mapping process.

The novelty of our approach is two fold where first the fractal dimension of each cell in the stable region is analysed and used as an attribute of a rough set which describes the uncertainty in the outcome of cell to cell mapping technique due to cellular irregularities. The fractal dimension, assigned as an attribute of the rough set in this study, is a common measure of dimension and especially applied to the chaotic attractors. Fractal dimension has been the focus of a multitude of leading works (Farmer, *et al*., 1983; Hsu, *et al*., 1994). Rough set theory is proposed by Zdzislaw Pawlak in (Pawlak, 1982, 1995, 1997) for modelling uncertainty and vagueness. The second contribution of our approach resides in the smoothing of the roughness in the cellular stability domain, modelled by rough set using reinforcement learning (Kaelbling, *et al.*, 1996).

We will introduce our approach on an illustrative example which will be presented in section 2.

2. ANALYSIS OF AN EXAMPLE

We consider the same system as in (Hsu, 1987) in order to demonstrate our approach on the probabilistic approach of Hsu, in the case of fractal boundaries. The system is chosen because of the very simple nature of its nonlinearity and its well understood global characteristics. The dynamic model of the system is represented as discrete state equations as:

$$\begin{aligned} x_1[n+1] &= (1-m)x_2[n] + \left(2-2m+m^2\right)x_1^2[n] \\ x_2[n+1] &= -(1-m)x_1[n] \end{aligned} \qquad (1)$$

where $m=0.1$. The system has a stable spiral point at (0,0) and a saddle point at (1,-0.9).

The 2D state space is further divided into cells of dimension *hxh* . A two dimensional cell is designated by its integer valued Z_1 and Z_2 components and a point x_i belongs the cell Z_i if

$$(Z_i - 1/2)h \le x_i < (Z_i + 1/2)h \qquad (2)$$

where the interval size for both coordinates is taken as *h*.

In order to determine the attraction domain of the system, system equations given in Eq.1. are iterated 43 times in (Hsu, 1987). However, in order to be safe, these equations are iterated 100 times for each cell in the region of interest taken as $-3<x_1<3$ *and* $-3<x_2<3$. At each integration step, starting from cell **Z**(n), first, **x**(n) the centre point of **Z**(n), is put into the integration process as the initial state and the point **x**(n+1) results from the integration. **Z**(n+1), the image cell of **Z**(n), is the cell in which **x**(n+1) lies. The next integration is performed by taking the centre point of **Z**(n+1) as the initial point and this process is iterated for 100 integration steps.

The system is examined over a state space described by *201x201* cells meaning that the x_1 and the x_2 states are divided into *201* intervals. The interval length *h* is selected as *0.03* for both states that is to say in x1 and x2 directions respectively. The region is thus divided into *40401* regular cells. The domain of attraction found in *100* step is shown in Figure 1. In the figure white areas denote the unstable region while the black ones stand for the domain of attraction for the stable spiral point at *(0,0)*.

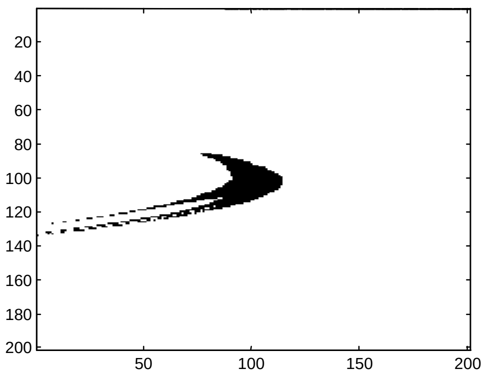

Fig.1. A domain of attraction for division 201x201

As seen in the figure, the domain of attraction fails to determine the finger shape regions, in spite of the fact that the cell size is very low*(0.003)*. In the cell mapping technique stable points are classified as unstable, while the unstable points are classified as stable in the boundary region. This fact can be seen by comparing Figure 1. with Figure 2. which is constructed by using lower cell size*(0.004)*.

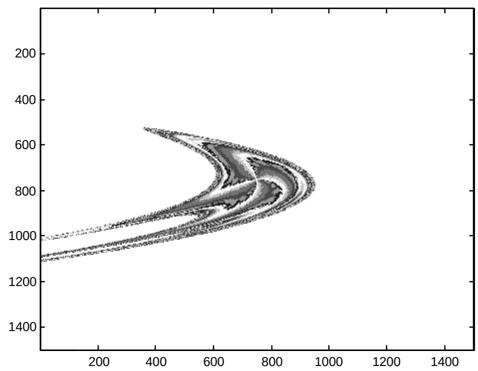

Fig.2. A domain of attraction for division 1500x1500

The comparison demonstrates that smaller the cell size; the more precise gets the domain of attraction.

Consequently one natural way of having a totally precise domain of attraction is to employ an infinite number of cells with infinitely small dimensions in the cell mapping. This approach is numerically impossible since it needs infinite iterations, which means infinite time. Another solution is to handle uncertainty by modelling it.

The following section deals basically with the uncertainty modelling of the control space, introducing the representation of its roughness by Rough set theory extended in this work by the fractal dimension concept.

3. FRACTAL/ROUGH SET APPROACH TO UNCERTAINTY MODELLING IN STATE SPACE

In this approach, rough set theory is employed to classify the regions with different stability characteristics. Rough set proposed by Zdzislaw Pawlak is a mathematical approach for modeling vagueness in uncertainty. The rough set theory is based on the idea that any object of the universe is associated with some kind of information. Objects characterized by the same information are indiscernible and assigned to same uncertain set.

Sets containing indiscernible objects are called elementary sets and the union of these kind of sets are called the crisp sets while the other sets are called rough sets. As a result, each rough set has a boundary region where set elements cannot be classified either as set members or complementary set members. Boundary region elements are the ones that cannot be classified by employing the available knowledge. In rough set approach, two precise concepts called the lower and the upper approximations of a rough set, are exploited to replace the vagueness. The lower approximation consists the elements that are surely the elements of the set and the upper set consists the elements that are possibly elements of the set:

The ***lower approximation*** of a set *X* is described by the domain (*U*) objects *x* which are known "with certainty" to belong to the subset of interest according to the attribute *B*.

$$\underline{B}(X) = \{x \in U : B(x) \subseteq X\} \qquad (3)$$

The ***upper approximation*** of a set *X* containing objects *x* which "possibly" belong to the subset of interest with respect to the attribute *B*.

$$\overline{B}(X) = \{x \in U : B(x) \cap X \neq f\} \qquad (4)$$

The ***boundary region*** of a rough set is a region of uncertainty where the set elements of that region are not known to be inside or outside the set "with certainty" with respect to the attribute *B*.

$$BN_B(X) = \overline{B}(X) - \underline{B}(X) \qquad (5)$$

From this brief review, it becomes natural that state space obtained by cell mapping can be modeled using rough set theory. We model the state space obtained by cell mapping as a rough set of cells where the lower approximation consists of stable cells and the boundary region of the set consists partially stable cells, while the other cells are unstable and not members of that set. *Stable cells* are the ones where all points in them are stable, *unstable cells* contain only unstable points and partially stable cells are those that contain both stable and unstable points in one cell rendering it possibly stable thus vague in stability. *Partially stable* cells form the boundary region of the set.

To differentiate the cells according to such a classification, we thought of using the satbility vagueness information in a partially stable cell as an irregularity measure. The fractal dimension of each cell is this measure-based information that we found suitable for such a classification and that we assign as an attribute of the rough set. We use the definition of capacity in the computation the fractal dimension. The capacity of a set is defined as

$$d_c = \lim_{e \to 0} \frac{\log N(e)}{\log(1/e)} \qquad (6)$$

where *N(e)* is the minimum number of cubes of size ***e*** needed to cover the set.

The cell is divided up into rectangular subcells of size ***e*** and, the number of subcells *N (e)* found to be stable by cell to cell mapping technique as in Fig.3 is counted for every cell. Decreasing the subcell size ***e***, the process is iterated. Finishing this process, the log *N (e)* is plotted versus the log *(1/e)* for every cell and the slope of the plot is the fractal dimension in the limit as ***e*** goes to zero.

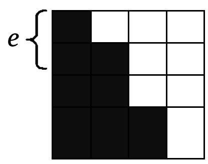

Fig.3. A cell divided up into 16 subcells (black = stable)

At this point, another attribute is assigned to the rough set defined. This second attribute is the ratio of stable subcells to all subcells within a cell where stable subcells are the ones that are found to be stable using the cell to cell mapping technique. When the subcell size goes to zero, this ratio becomes the probability of being in a safe stable region when one chooses a point in a cell. We call this attribute the *stability number (Sn)*. It follows naturally from our definition that a stable cell has a *Sn=1* and an unstable cell has *Sn=0*.

These two attributes contain different information. From the definitions, *Sn* gives the ratio of the stable region in a cell, while d_c (capacity dimension) shows if this ratio changes for different cell sizes chosen to find *Sn*, in other words if the cell (stable part of the related cell) is fractal or not.

In our approach a rough set contains stable cells of integer dimension and of *Sn=1* and uncertain cells identified as possibly stable, that have a fractal dimension and an *Sn* number ≠0 and ≠1. If a cell is unstable it does not contain any stable subcells and consequently its fractal dimension is undefined and *Sn*=0.

In the example presented, the domain of attraction which is the stability region contains 1249 cells, which are either stable or possibly stable. This stability region is thus a rough set of stable cells in its lower approximation and possibly stable cells in its boundary. Cells from a portion of the rough set boundary together with their attributes are given in Table 1.

Table 1. Part of the rough set

Members	Z_1	Z_2	d_c	***Sn***
Cell 1	85	75	2	1
Cell 2	85	74	2.4456	0.95
Cell 3	110	76	2.3532	0.17

In this table three cells are shown. The first with an integer dimension of 2 and *Sn* of 1 is fully stable and the second and third are partially stable (all the points within the corresponding cell are not stable) with fractal dimensions of 2.4456 and 2.3532 respectively.

Examining all 40401 cells in the state space of the example system, it is found that there are 658 cells with S_n=1 and d_c=2 defining the fully stable cells; 39152 cells with S_n=0 and d_c is not defined defining the set of fully unstable cells; and 591 partially or possibly stable cells with $0<S_n<1$ and $d_c \neq 2$ but finite valued. From this piece of information, the following can be concluded:

- All the cells with indefinite dimension are unstable and not members of the rough set,
- The cells with dimension different than two are partially stable (some of the points in the cell are stable while the others not) and are members of the boundary region of the rough set,
- The cells with integer dimension two are stable and form the lower approximation of the rough set.

The lower approximation of the rough set, consisting of cells with integer dimension is given in Figure 4. and the boundary region, consisting of cells with fractal dimension is shown in Figure 5.

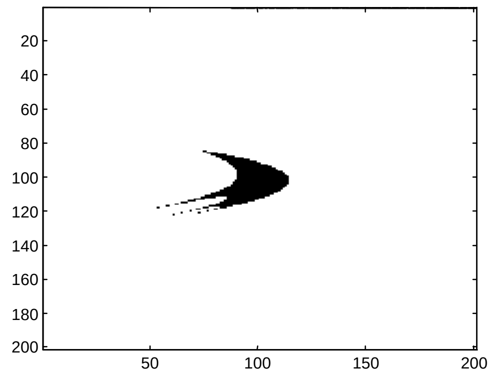

Fig. 4. Lower approximation of the rough set

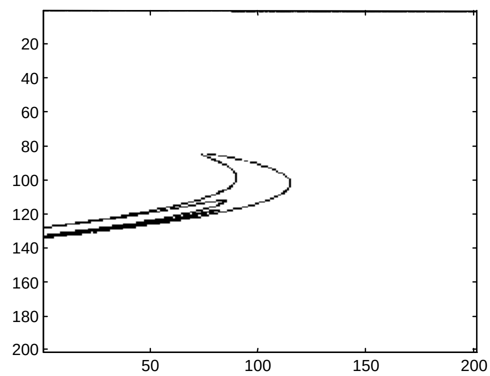

Fig. 5. Boundary region of the rough set

In the following section, a method enlarging the lower approximation by smoothing uncertainty in the boundary region via a learning algorithm is introduced. The smoothing process uses a reinforcement learning approach which is equivalent to an ironing process of the irregular stability boundary that the system guides itself by reinforcement signals. Those cells that can be ironed are then included in the lower approximation, consequently enlarging it.

4. SMOOTHING THE BOUNDARY REGION OF A ROUGH SET

Every cell in the state space is divided into 4, 9, 25 and 100 subcells to find the fractal dimension and 100-subcell structure is used to compute the stability number, S_n. A learning method to decrease the uncertainty in the boundary region and smooth the stability roughness there, is employed on the 100-subcell structure. The applied method is the *linear-reward inaction* type of reinforcement learning.

In this method, the subcells in the boundary region are rewarded by their history of stability and their probability of being stable, which is S_n. Reward is decremented when an event of unstability occurs for a point in subcell "i". Taking numerous random points from each subcell and finding out if they are stable or not, S_n of each subcell is iterated by the following formula:

- When the point in the subcell "i" is stable,

$$S_n^i = S_n^i + a(1 - S_n^i) \quad (10)$$

- When the point in the subcell "i" is unstable,

$$S_n^i = S_n^i - aS_n^i \quad (11)$$

In our case, to find the S_n of a subcell, the above formula is iterated for 100 times with different 100 random points in the subcell and this process is repeated for every subcell of the boundary region. To provide a slow convergence rate and to make the algorithm fully convergent, a is chosen by trial and error as 0.05 and it is seen that, in our case, 100 iterations are enough for the *Sn* to converge from the simulations.

Applying the algorithm, the S_n of subcells, which can be classified as stable with a little probability of error, converge to 1, while the others converge to 0. The subcells with S_n similar to the that of cells in the lower approximation of the rough set (stable region), (in other words the subcells with S_n approximately equal to 1) are added to the lower approximation. This expansion of the lower approximation by possibly stable cells with high history of stability increases the level of knowledge about the characteristics of the stability region. Using this method, the stable region is enlarged based on a possibility measure related to *Sn* representing a stability history and the boundary region is diminished.

The rough set in our case has 658 lower approximation elements and 591 boundary elements which means that 52.69% of the upper approximation region is forming the lower approximation and 47.31% of the upper approximation region is forming the boundary region. Applying the smoothing via reinforcement learning, the lower approximation becomes 60% of the upper approximation. Including subcells with S_n>0.9, 15.45% of the boundary region is added to the stable region while the rest of the region converge to the unstable area for our present implementation.

In this study proposed method is applied to a 2D system, the method is applicable to higher order systems using the projections of high order systems on to the 2D space. Such a system is examined in (Kaygisiz, *et al.*, 2001).

5. CONCLUSION

In the novel approach introduced in this paper, we increase the richness of the information in the stability region of a nonlinear system using fractal/rough set representation of that region. The fractal/rough set model developed is a new approach brought to the area of the uncertainty modelling in systems with fractal, finger shaped boundary region. The approach introduces the fractal dimension of the elements in the rough set as a measure of stability roughness. The uncertainty represented as roughness is then minimized using reinforcement learning. Minimization of uncertainty in the stability region is done as smoothing ("ironing") of the irregularity of system stability in the boundary region.

We apply in this paper our approach to a nonlinear system with finger shaped boundary region, the smoothing performance has been found to be 15.45 %.

Recently, our work begun to focus on the robustness and performance measures of the fractal/rough set approach.

REFERENCES

Farmer D., Ott E., Yorke J., 'The Dimension of Chaotic Attractors', Physica 7D, 153-180, 1983

Hsu C.S., 'A Theory of Cell-to-Cell Mapping Dynamical Systems', Journal of Applied Mechanics Vol. 47, 931-939, 1980

Hsu C.S., 'An Unravelling Algorithm for Global Analysis of Dynamical Systems: An Application of Cell-to-Cell Mappings', Journal of Applied Mechanics Vol. 47, 940-948, 1980

Hsu C.S., 'Cell-to-Cell Mapping', Springer Verlag, New York, 1987

Hsu G.H., Ott E., Gregobi J., 'Strange Saddles and The Dimension of Their Invariant Manifolds', Physics Letters A, Volume 127, 199-204, 1994

Kaelbling L.P., Littman M.L., Moore A.W., 'Reinforcement Learning: A Survey', J. of Artificial Intelligence Research 4, 237-285, 1996

Kaygisiz H.B., Erkmen A.M., Erkmen I., 'Smoothing Stability Roughness of a Robot Arm Under Dynamic Load Using Reinforcement Learning', IROS 2001, Maui, Hawaii

Pawlak Z., ' Rough sets: Algebraic and topological approach', Institute of Computer Science Polish Academy of Sciences, 1982

Pawlak Z., ' Rough Sets, Rough Functions and Rough Calculus', Institute of Theoretical and Applied Informatics Polish Academy of Sciences,1995

Pawlak Z., ' Rough set approach to knowledge-based decision', European Journal of Operational Research 99, 48-57, 1997