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ROBUST H_2 CONTROLLER DESIGN AND TUNING FOR THE ACC BENCHMARK PROBLEM AND A REAL-TIME APPLICATION

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Abstract: This paper presents an LMI-based method for the design and tuning of robust H_2 output feedback controllers, illustrated with two case studies. The conservatism of design is considerably reduced by a proposed scaling procedure, referred to as K-S iteration. The method is applied to the ACC benchmark problem and shown to outperform previously published solutions. The choice of tuning parameters and tuning strategies are discussed in detail. Real-time experiments on an experimental version of the benchmark problem demonstrate the practicality of the proposed approach.

Keywords: Uncertain linear systems, robust control, output feedback, linear matrix inequalities, LQG control, benchmark examples

1. INTRODUCTION

It is well known that most practical control problems are still being solved using PID control or lead-lag compensation, because of the simplicity of tunig the parameters of such a controller. So far, the impressive recent developments in the area of robust control theory have failed to make much impact on how real-life problems are solved, the main reason being the complexity of the design and tuning procedure. Another serious problem associated with robust design strategies is their inherent conservatism. Both issues are addressed in this paper.

The approach followed here is known as robust H_2 controller design: Given a linear system with uncertainty in the system parameters, find the linear, time-invariant controller that minimizes a worst-case H_2 performance index.

The robust H_2 problem has been studied in (Packard and Doyle, 1987) and (Stoorvogel, 1993). In (Petersen and McFarlane, 1994) bounds were proposed on the worst-case H_2 norm of a system subject to norm-bounded, time-varing uncertainties. In (Feron, 1997) it was shown that the computation of all these bounds on the H_2 performance can be reduced to a convex optimization problem involving linear matrix inequalities (LMI), which can be solved via efficient convex optimization techniques.

The main contribution of this paper is to propose a simple, LMI-based design and tuning strategy for robust H_2 controllers, and to demonstrate its efficiency with two illustrative case studies. This approach is based on a simplified version of the work in (Apkarian *et al.*, 1996) (chapter 4, page 88). There, two sets of scaling matrices were used to characterize the uncertainty, while the approach proposed in this paper uses only one scaling matrix. The choice of the scaling matrix determines the degree of conservatism of the design. Using convex optimisation to find the best scaling is not possible because this problem is non-convex in the output feedback case; to overcome this difficulty, an iterative procedure - referred to as *K-S iteration* - is proposed in this paper. The K-S iteration technique leads to an efficient design procedure where only up to three intuitive parameters are used for tuning the controller.

The paper is organised as follows. In section 2, the robust H_2 approach is briefly reviewed, and an iterative procedure is proposed for reducing conservatism. The proposed method is then applied to the ACC benchmark problem in section 3. A real-time implementation is presented in section 4, together with experimental results. Conclusions are drawn in section 5.

2. ROBUST H_2 CONTROL AND K-S ITERATION

Consider the control system shown in Figure 1. The generalised plant P has a state space representation

$$\begin{aligned} \dot{x} &= A_0 x + B_1 w_1 + B_2 w_2 + Bu \\ z_1 &= C_1 x \\ z_2 &= C_2 x + D_{2u} u \\ y &= Cx + D_{2w} w_2 \end{aligned} \quad (1)$$

Here (A_0, B, C) represents the physical plant with control input u and measured output y. Perturbation of the nominal plant dynamics (A_0) are expressed via fictitious inputs through B_1 and fictitious outputs through C_1: Introducing feedback $w_1 = \Delta z_1$, where the matrix $\Delta(t)$ represents perturbations and is assumed to satisfy $\|\Delta\| < 1$ at all times, leads to

$$\dot{x} = (A_0 + B_1 \Delta C_1)x + B_2 w_2 + Bu$$

The input w_2 is a white noise process with unit variance. If the matrices C_2, D_{2u}, B_2 and D_{2w} are chosen as

$$C_2 = \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix}, \qquad D_{2u} = \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix} \quad (2)$$

$$B_2 = [Q_e^{1/2} \;\; 0], \qquad D_{2w} = [0 \;\; R_e^{1/2}] \quad (3)$$

then

$$J = E\|z_2(t)\|_2^2 = E\left[\lim_{t\to\infty} \frac{1}{T}\int_0^T z_2^T z_2 \, dt\right] \quad (4)$$

represents a LQG cost function with the usual weight matrices Q, R and noise covariances Q_e, R_e.

The problem considered in this paper is to find a strictly proper controller $K(s)$ with state space realisation

$$\begin{aligned} \dot{\zeta}(t) &= A_K \zeta(t) + B_K y(t) \\ u(t) &= C_K \zeta(t) \end{aligned} \quad (5)$$

such that the LQG cost is guaranteed to be less than a given value $J \leq \nu^2$ in all admissible operating conditions, i.e. for all $\|\Delta\| < 1$.

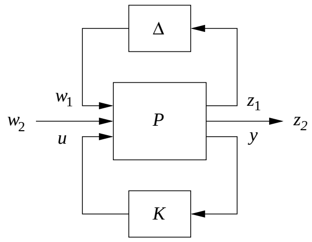


Fig. 1. Generalized plant

This problem can be expressed in the form of linear matrix inequalities as follows. Consider the closed-loop system

$$\begin{bmatrix} \dot{x} \\ \dot{\zeta} \end{bmatrix} = \bar{A}\begin{bmatrix} x \\ \zeta \end{bmatrix} + [\bar{B}_1 \;\; \bar{B}_2]\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \bar{C}_1 \\ \bar{C}_2 \end{bmatrix}\begin{bmatrix} x \\ \zeta \end{bmatrix}$$

where

$$\bar{A} = \begin{bmatrix} A_0 & BC_K \\ B_K C & A_K \end{bmatrix}, \; \bar{B}_1 = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \; \bar{B}_2 = \begin{bmatrix} B_2 \\ B_K D_{2w} \end{bmatrix}$$

$$\bar{C}_1 = [C_1 \;\; 0], \quad \bar{C}_2 = [C_2 \;\; D_{2u} C_K]$$

In the control system in Figure 1, the performance index satisfies $J \leq \nu^2$ for all $\|\Delta\| < 1$, if there exist a positive definite matrix P and a matrix W such that

$$\text{trace}\, W < \nu^2$$

and

$$\begin{bmatrix} P\bar{A}^T + \bar{A}P & P\bar{C}_2^T & P\bar{C}_1^T & \bar{B}_1 S \\ \bar{C}_2 P & -I & 0 & 0 \\ \bar{C}_1 P & 0 & -S & 0 \\ S\bar{B}_1^T & 0 & 0 & -S \end{bmatrix} < 0, \; \begin{bmatrix} W & \bar{B}_2^T \\ \bar{B}_2 & P \end{bmatrix} > 0 \quad (6)$$

For a proof see the full version of this paper (Farag and Werner, 2001). Here the matrix S is a suitable positive definite, symmetric scaling matrix that satisfies $S\Delta = \Delta S$ for all admissible Δ; the choice of S is discussed below (when Δ is square and diagonal, S is diagonal).

Conservatism of Design and K-S Iteration

The closed-loop matrices $\bar{A}$, $\bar{B}_2$ and $\bar{C}_2$ in (6) depend on the controller matrices A_K, B_K, C_K. Due to the presence of the product terms $\bar{A}P$ and $\bar{C}_2 P$, (6) cannot be solved as an LMI problem for the controller, because it is nonlinear in the controller matrices and the matrix variable P. However, a linearizing change of variables, proposed in (Chilali and Gahinet, 1996), can be used to transform (6) into an LMI problem that can be solved with efficient LMI solvers.

On the other hand, the matrix inequality (6) is only a sufficient condition for the worst case bound on the performance. The resulting conservatism can be reduced by a suitable choice of the scaling matrix S. The form of (6) suggests to treat S as a matrix variable and solve an LMI problem to find the scaling that yields the best worst-case performance. Unfortunately, the linearizing transformation introduces a term that is nonlinear in S and the controller variables. To overcome this problem, we propose the following iterative technique, for more details, see (Farag and Werner, 2001)

K-step: Assume $S = I$ and solve

$$\min_{K(s),P} \ \mathrm{trace}\, W \quad \text{subject to the linearized form of (6)}$$

S-step: Using the controller obtained in the 'K-step', solve

$$\min_{P,S} \mathrm{trace}\, W \quad \text{subject to} \quad (6)$$

Go back to the 'K-step' and repeat with S obtained in the 'S-step' until no further drop in trace W is observed.

3. THE ACC BENCHMARK PROBLEM

This problem was proposed as a benchmark problem for robust control at the American Control Conference 1990 (Wie and Bernstein, 1990). Two bodies with masses m_1 and m_2 are connected by a spring with stiffness k, as shown in Fig. 2. A state space model of the system is

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k/m_1 & k/m_2 & 0 & 0 \\ k/m_2 & -k/m_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/m_1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/m_1 & 0 \\ 0 & 1/m_2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix},$$

$$y = x_2 + v \tag{7}$$

For this position control system three robust design problems were posed in (Wie and Bernstein, 1990). The problem considered here is the union of problems 1 and 2 in the original reference; the same problem was also studied in (Thompson, 1995) and can be summarized as follows.

Design Problem: For a unit impulse disturbance exerted either on body 1 or 2, the controlled output x_2 must have a settling time of no more than 15 sec for the nominal system ($m_1 = m_2 = k = 1$; the settling time t_s is defined by $|x_2| < 0.1 \forall t \geq t_s$). The closed-loop system should be stable for $0.5 \leq k \leq 2.0$ and $m_1 = m_2 = 1$, and show robustness against variations of m_1 and m_2.

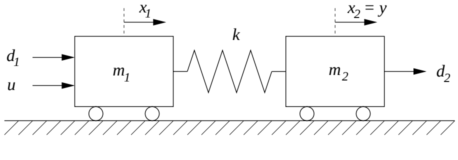


Fig. 2. Two-mass-spring system

Since k is only known to be in the range $0.5 \leq k \leq 2.0$, it is required to construct matrices B_1, C_1 such that the uncertain matrix A in the state space model (7) can be expressed as

$$A = A_0 + B_1 \Delta C_1 \quad \text{with} \quad -1 < \Delta < 1 \tag{8}$$

From (7), a straightforward choice is

$$A_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}, \quad B_1 = \delta \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} -1 & 1 & 0 & 0 \end{bmatrix} \tag{9}$$

The sytem matrix A_0 corresponds to the nominal stiffness $k_0 = 1$. A tuning parameter $\delta > 0$ has been introduced that can be used to scale the uncertainty: when $\delta = 0$ the representation (8) is reduced to the nominal model, and larger values of δ mean that a larger range of uncertain parameters is covered.

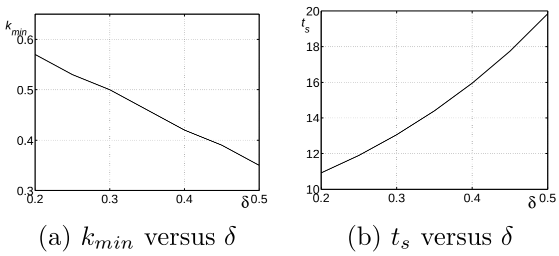
A quadratic performance index is included in the model by choosing the matrices C_2, D_{2u}, B_2 and D_{2w} in (1) according to (3), with

$$Q = qI, \quad Q_e = q_e I, \quad R = 1, \quad R_e = 1$$

This representation leaves the designer with three tuning parameters q, q_e and δ. Once values for these parameters have been chosen, the K-S iteration procedure presented in the previous section can be applied to compute the controller.

Tuning Parameters

The influence of δ on performance and robustness is shown in Figure 3. As expected, the price to be paid for improving the robustness of the system (decreasing k_{min}, the minimum spring constant for which the system is stable) is a loss of performance (i.e. larger values for t_s).



(a) k_{min} versus δ (b) t_s versus δ

Fig. 3. Influence of δ on robustness and performance($q = q_e = 1$)

Figure 4 illustrates how variation of δ determines the trade-off between control effort and robustness. The maximum control input is plotted against k_{min} when δ is varied (stability is more difficult to achieve at k_{min} than at k_{max}). Curves are shown for different values of q and q_e; as expected,

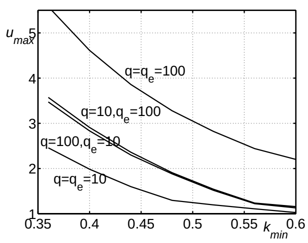
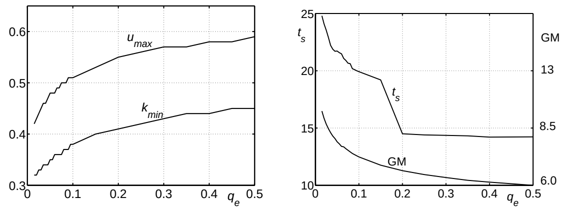


Fig. 4. Control effort versus robustness

larger values of q and q_e lead to a higher control effort. However, the plots also suggest that q and q_e have a similar effect on the system response: the curve for $q = 100,\ q_e = 10$ is almost the same as that for $q = 10,\ q_e = 100$. Therefore in the tuning process described below only q_e was used as tuning parameter, and starting from initial values, q was adjusted proportionally.

Figure 5 shows the effect of q_e on performance and robustness. A larger value of q_e leads to a faster response, but at the same time to less robustness and a larger control effort.



(a) u_{max} and k_{min} (b) t_s and GM

Fig. 5. Influence of q_e on speed and robustness

Design and Tuning of a Robust H_2 Controller

The first step in the tuning procedure is to find suitable starting values for the three tuning parameters $q,\ q_e$ and δ. Initially, the system response was evaluated with $q,\ q_e$ and δ taking combinations of an initial range of values. This test can be carried out by running a simple Matlab routine; the initial values were $\{0.01, 0.1, 1.0, 10, 100\}$ for q and q_e, and $\{0, 0.1, 0.2, 0.3, 0.4, 0.5\}$ for δ.

It was found that the gain margin and the phase margin requirements can be satisfied only when $q, q_e < 0.1$ and $\delta < 0.2$. The choice of $q = 0.01,\ q_e = 0.1,\ \delta = 0.1$ results in a controller

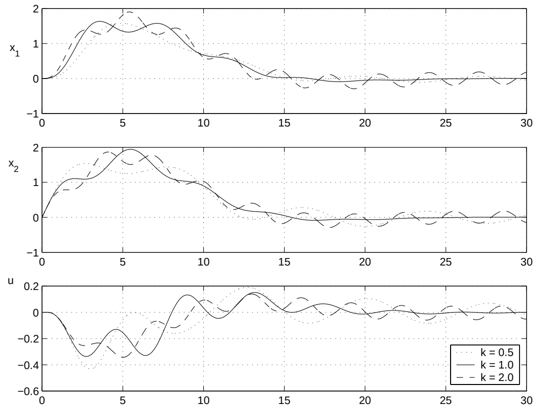


Fig. 6. Impulse applied at mass 2

that satisfies all requirements except the settling time requirement.

To achieve a faster response, q_e was slightly increased. The controller obtained with $q = 0.02,\ q_e = 0.2$ and $\delta = 0.1$ is $K(s) =$

$$\frac{0.0615(s - 8.0097)(s + 2.2270)(s + 0.1326)}{(s^2 + 0.7379s + 2.4270)(s^2 + 1.5976s + 0.8105)} \tag{10}$$

This controller satisfies all requirements of the benchmark problem. In (Thompson, 1995), a scoring scheme was proposed to evaluate and compare the performance of different controllers. The achieved performance measures are shown in Table 1, and compared with the performance achieved in (Thompson, 1995) and a collection of controllers presented in (Special Issue, 1992), including the three best designs. It is clear that the proposed robust H_2 design outperforms all other controllers. Moreover, the design procedure is simple and it would be straightforward to re-tune the controller to trade speed of response against robustness, or both against control effort. Figure 6 shows the response to a disturbance impulse at mass 2, with spring constants 0.5, 1.0 and 2.0.

The representation (1) of the model uncertainty with matrices (B_1, C_1) is not unique, for example with a scaling factor δ the same parameter

Design	Reference (equation)	PM (deg)	GM (dB)	t_s (sec)	u_{max}	$k_{min} - k_{max}$	p_m	Score
Requirement		30	6.0	15	1	0.5 - 2.0	0.30	
Robust H_2	this paper (10)	32	6.6	14.5	0.55	0.410 - 3.1	0.48	8.5
Classical/H_2	(Thompson, 1995) (19)	35	6.0	14.5	0.759	0.450 - 2.800	0.41	7.3
H_∞	(Wie *et al.*, 1992) (40)	34	6.1	15.2	0.573	0.440 - 3.900	0.45	6.4
Pole placement	(Lilja and Astrom, 1992) next after (5)	24	3.7	28.9	0.549	0.230 - ∞	0.37	0.7
μ-synthesis	(Braatz and Morari, 1992) (29-32)	27	2.8	14.1	0.953	0.580 - 2.500	0.37	-0.1
Minimax LQG	(Mills and Bryson, 1992) (37)	19	3.4	18.1	0.691	0.690 - 1.400	0.18	-7.2
QFT	(Jayasuriya *et al.*, 1992) (8)	11	2.5	4.7	345.9	0.180 - 1.300	0.09	-19.0

Table 1. Controller performance

uncertainty can be expressed by $(\delta B_1, \delta^{-1}C_1)$. It turns out that when solving (6) with S fixed, the minimum achievable value of the performance index J depends on the selection of the matrices B_1, C_1. To illustrate this point, consider two possible scalings of the matrices B_1, C_1. Let B_0, C_0 be given by:

$$B_0^T = [0\ 0\ 1\ -1]^T, \quad C_0 = \begin{bmatrix} 0 & 0 & -1 & 1 \end{bmatrix}$$

and consider the following different ways of scaling B_1, C_1

$$\text{case 1: } B_1 = \delta B_0; \quad C_1 = C_0$$
$$\text{case 2: } B_1 = B_0; \quad C_1 = \delta C_0$$

Case 1 is identical with the choice of the tuning parameter δ in (9). The first line in table 2 (iteration 0) gives the values of J when (6) is solved with setting $S = 1$ (in case 2 the start value $S = 0.1$ was used because the problem is infeasible for $S = 1$).

The point to observe here is that for the same values of δ, q_e and q, solving (6) without K-S iteration results in two different controllers with different perfomance. On the other hand, two steps of K-S iteration lead to the scaling S and the controller that yield the minimum cost $J_{min} \approx 73.5$ in both cases. Note that the optimal scaling is different for each case.

It.	case 1		case 2	
	S	J	S	J
0	1.0	73.6	0.1	628
1	1.17	73.49	0.0109	73.65
2	1.19	73.49	0.0119	73.55

Table 2. Cost variation with number of iterations (at $q = 0.2$, $q_e = .1$, $\delta = 0.1$)

4. REAL-TIME CONTROL OF A RECTILINEAR PLANT

The electromechanical plant used to represent the ACC benchmark problem, is the rectilinear mechanism shown in Figure 7, manufactured by ECP systems. The actuator is a brushless DC servo motor, acting on the first mass. The signal used for feedback is the position of the second mass, measured with a high resolution encoder. Three springs are available with different stiffness. To realise the configuration of the ACC benchmark problem, the second spring - connected to a third mass - was removed. The Matlab Real-Time Windows Target interface was used for real-time controller implementation (with sampling time 2 ms).

The original regulator problem was modified into tracking a step change in the reference position.

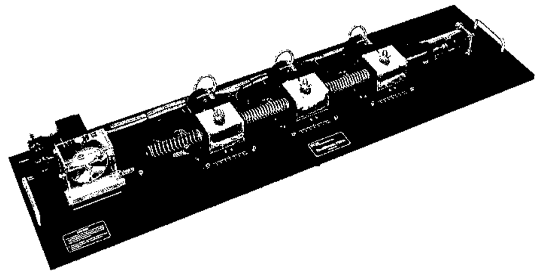


Fig. 7. Rectilinear plant model 210/210a by ECP systems

Therefore, the plant model is augmented with an integrator to include integral action in the controller. In a first step, the controller $K(s)$ is designed for robust stability of the closed-loop dynamics. A prefilter is then added to shape the response by reducing overshoot, for more details see (Farag and Werner, 2001).

Again the scaling parameter δ plays a central role when tuning the controller $K(s)$. In addition, in this real-time application it was found convenient to introduce a pole region constraint in the form of an upper bound x_r on the real part of the closed-loop poles. The parameter δ has a direct influence on the robustness against variation in the spring constant k, while tunig x_r has a direct influence on the rise and settling time. The penalty R on the control effort in the LQG cost (3) can be used to limit the control energy when it is needed.

The following strategy was used for tuning the controller.

- Based on the state space model, compute for a given value of x_r the value of δ for which the stability boundary is reached
- Implement the controller resulting from this choice of δ and check the response in real time. The parameter x_r can be used to make the response faster or slower
- If the actuator saturates (expected at fast response), the LQG weight R can be used to reduce control energy.

It turned out that the robustness requirements cannot be satisfied for $x_r < -5.5$. The design prameter R was needed for $x_r < -4$ to limit the control energy. Figure 8 shows the influence

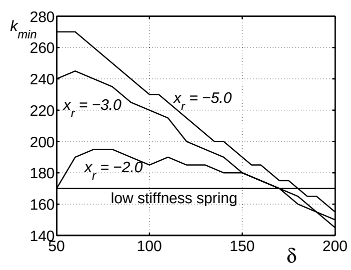


Fig. 8. The influence of δ and x_r on k_{min}

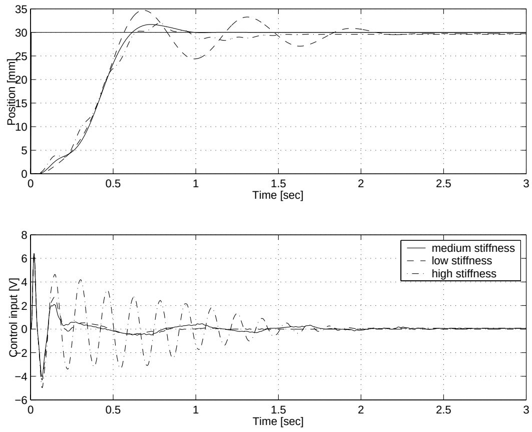


Fig. 9. Experimental results ($x_r = -4.5$)

of varying δ and x_r on k_{min}; as expected varying δ has not much effect when the system is too slow. The response when $x_r = -4.5$ is shown in Figure 9.

To overcome static friction and reduce the steady state position error, a large open-loop gain at low frequencies is required. Table 3 illustrates that increasing the magnitude of x_r leads not only to a faster response, but also to a higher low-frequency loop gain, and thus helps to achieve higher steady state accuracy.

δ	R	x_r	Settling time	u_{max}	loop gain at $\omega = 10^{-3}$ rad/s
180	1.0	-2.0	0.705	0.585	8.45×10^6
180	1.0	-3.0	0.695	0.580	8.69×10^6
180	1.0	-4.0	0.667	1.525	15.30×10^6
180	1.0	-4.5	0.547	6.414	25.50×10^6
185	10.0	-5.0	0.489	16.819	48.70×10^6

Table 3. Experimental results (medium stiffness)

5. CONCLUSIONS

A systematic design and tuning strategy for robust H_2 controllers has been proposed. The two case studies presented in this paper illustrate an important feature of this method: the design and tuning procedure can be reduced to the choice of three parameters, each with a clear impact on certain aspects of performance and robustness. This property can be exploited to develop semi-automatic design and tuning procedures for high-performance controllers. Once a controller for a given application has been designed and tuned, suitable 'tuning knobs' can be made available to users, in order to enable re-tuning of the control settings, if significant changes in the operating conditions make this necessary.

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