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HOW TO REDISTRIBUTE ENERGY BETWEEN DIFFERENT LINKS OF THE PENDUBOT

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Abstract: This paper considers a new stabilization problem for the Pendubot. Namely how to construct and stabilize via feedback the following trajectory: the first (controlled) link of the Pendubot remains at rest having a given angle with horizontal, while the second (freely moving) link duplicates a given motion of the 1-d.o.f. plane pendulum. Such a stabilization could be also seen as re-distributing the energy of the Pendubot between actuated and non-actuated parts, or saving the energy in the motion of the non-actuated part. The main result of the paper is the description of a wide family of the state feedback controllers, which solve the problem. In addition, the delicate issue of the convergence's rate of the closed loop system solutions to the desired trajectory is discussed in details.

Keywords: passivity, feedback transformation, the Pendubot, constructing periodic motion via feedback

1. INTRODUCTION

Controlling an underactuated nonlinear system is inherently difficult problem. There are a few analytical methods which are able to tackle such a problem, most of them are based on structural properties of the system. The reader, for example, can check the papers (Ortega *et al.*, n.d.) and (Bloch *et al.*, 2000), where conserved quantities and symmetries of the system play a major role in constructing a stabilizing controller for an equilibrium. Another example, where conserved quantities are important, is related to the problem of stabilization of some particular subset of the state space, possibly different from an equilibrium. The reader can check, for example, the results of (Fradkov, 1996; Åström and Furuta, 2000; Shiriaev *et al.*, 2001), where a stabilization of *homoclinic curves* of the plane pendulum was made; and the results of (Ludvigsen *et al.*, 1999; Albouy and Praly, 2000), where a stabilization of the stable manifold of the spherical pendulum (a 2-dimensional subset of the 4-dimensional state space), was done.

This paper is concerned with an two-link underactuated robot called the Pendubot. It has an actuator at the shoulder (link 1) and no actuator at the elbow (link 2). One of standard control problems related to the Pendubot is a stabilization of one of its equilibria. Among other papers the reader can check (Spong and Block, 1995; Fantoni *et al.*, 2000; Kolesnichenko and Shiriaev, 2002), where some methods for stabilizing the upper equilibrium are suggested.

Another interesting control problem related to the Pendubot is a construction and orbital stabilization of periodic motions via feedback. This paper is aimed at constructing and stabilizing

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a particular family of periodic motions for the Pendubot. Namely, the desired motions are: *The first link is controlled to be at rest having a given value of angle with the horizontal, while the second uncontrolled link duplicates some periodic motion of the freely moving plane pendulum.*

There is a intuitive physical reasoning for choosing these periodic motions to be stabilized. Indeed, such a problem reflects an intention, first to store some energy in the system motion, and then, to deliver all the stored energy to the underactuated subsystem. Fortunately the underactuated subsystem of the Pendubot is just a pendulum, then this stored energy will correspond to periodic motion.

This strategy of storing energy could be useful in the case when a controlling device has a limited power. Then it is reasonable to save some energy in the system before the main control law is implemented. For example, in the case of swing up problem for the Pendubot with law-power actuator on the first link and the heavy non-actuated second link, we suggest first to store appropriate energy in rotation of the second link. When the rotations achieve some level, this stored energy could help for implementation a swing up controller.

The main contribution of the paper is the description of a wide range of state feedback controllers, which solve the problem. Then we have discussed the properties of the controllers to provide an *exponential rate* of convergence. The paper is organized as follows. The problem statement with some preliminaries are given in the Section 2. The main results are presented in the Section 3, while some simulations and conclusions are drawn in Sections 4 and 5.

2. PROBLEM STATEMENT AND PRELIMINARIES

Under the standard assumptions the equations of the Pendubot are

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau, \qquad (1)$$

where $q = [q_1, q_2]^T \in S^1 \times S^1$, q_1 is the angle that link 1 has with horizontal, q_2 is the angle that the link 2 makes with link 1;

$$M(q) = \begin{bmatrix} \theta_1 + \theta_2 + 2\theta_3 \cos q_2 & \theta_2 + \theta_3 \cos q_2 \\ \theta_2 + \theta_3 \cos q_2 & \theta_2 \end{bmatrix}, \qquad (2)$$

where

$$\theta_1 = m_1 l_{c_1}^2 + m_2 l_1^2 + I_1, \quad \theta_2 = m_2 l_{c_2}^2 + I_2,$$
$$\theta_3 = m_2 l_1 l_{c_2},$$

m_1, m_2 are the masses of the link 1 and the link 2; l_1, l_2 are the lengths of the link 1 and the link 2; l_{c_1} is the distance to the center mass of the link 1 from the suspension point, l_{c_2} is the distance to the center mass of the link 2 from the suspension point; I_1, I_2 are the moments of inertia of the link 1 and the link 2 about their centroids;

$$C(q,\dot{q}) = \theta_3 \sin q_2 \begin{bmatrix} -\dot{q}_2 & -\dot{q}_2 - \dot{q}_1 \\ \dot{q}_1 & 0 \end{bmatrix},$$

$$G(q) = \begin{bmatrix} \theta_4 g \cos q_1 + \theta_5 g \cos(q_1 + q_2) \\ \theta_5 g \cos(q_1 + q_2) \end{bmatrix}, \ \tau = \begin{bmatrix} \tau_1 \\ 0 \end{bmatrix}$$

where τ_1 is the control input, and

$$\theta_4 = m_1 l_{c_1} + m_2 l_1 \quad \theta_5 = m_2 l_{c_2}.$$

The total energy of the Pendubot is

$$E(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + \Pi(q) = \frac{1}{2}\dot{q}^T M(q)\dot{q} +$$
$$+\theta_4 g\,(\sin q_1 + 1) + \theta_5 g\,(\sin(q_1 + q_2) + 1) \quad (3)$$

and it could attain any values from the interval $[0, +\infty)$. Let us formulate the control problem: *Given an angle* $q_1^d = \frac{\pi}{2}$ *and a motion* $[q_2^d(t), \dot{q}_2^d(t)]$ *such that*

$$\ddot{q}_2^d(t) = \frac{\theta_5 g}{\theta_2} \sin q_2^d(t), \quad \forall\, t \geq 0, \qquad (4)$$

the objective is to define a feedback controller, which makes the trajectory

$$\begin{aligned} q_1(t) &= q_1^d(t) = \frac{\pi}{2}, & \dot{q}_1(t) &= \dot{q}_1^d(t) = 0 \\ q_2(t) &= q_2^d(t), & \dot{q}_2(t) &= \dot{q}_2^d(t) \end{aligned} \qquad (5)$$

invariant with respect to closed loop system vector field. In addition, it is of interest to make this periodic solution orbitally asymptotically stable. ■

Let us comment the problem:

(1) The value of the angle q_1^d is chosen $\pi/2$ just for simplicity. It can be any except two critical values $q_1^d = 0, \pi$, but this will just complicate expressions;
(2) Besides an exceptional case, when $[q_2^d(t), \dot{q}_2^d(t)]$ corresponds to the homoclinic curve of the system (4), the motion $[q_2^d(t), \dot{q}_2^d(t)]$ can be locally described via an appropriate value of the energy E_p of the system (4), where

$$E_p(q_2, \dot{q}_2) = \frac{\theta_2}{2}\dot{q}_2^2 + \theta_5 g(1 + \cos q_2) + 2\theta_4 g. (6)$$

(3) The case of homoclinic curves $[q_2^d(t), \dot{q}_2^d(t)]$ of the system (4) was previously considered in (Fantoni *et al.*, 2000; Kolesnichenko and Shiriaev, 2002), where stabilization of homoclinic curves was used for stabilizing the upper equilibrium of the Pendubot. The results of the papers (Fantoni *et al.*, 2000; Kolesnichenko and Shiriaev, 2002) explicitly show how to construct stabilizing controller,

but both papers do not provide or even discuss the rate of convergence issue. Coming to the physical implementation of the controller it is not enough to ensure (orbital) asymptotic stability, but rather get exponential convergence. This is one of the subjects of the current paper. Below we extend the stabilization scheme suggested in (Fantoni *et al.*, 2000; Kolesnichenko and Shiriaev, 2002) and discuss the rate of convergence in details.

Denote $E_0 = E_p\left(q_2^d(t), \dot{q}_2^d(t)\right)$ and introduce the function

$$V(q,\dot{q}) = \frac{k_1}{2}(E-E_0)^2 + \frac{k_2}{2}\dot{q}_1^2 + \\ + \frac{k_3}{2}\left(q_1 - \frac{\pi}{2}\right)^2 + k_4(E-E_0)\left(q_1-\frac{\pi}{2}\right), \tag{7}$$

where k_1–k_4 are some constants. It is readily seen that the function V equals to zero exactly on the subset V_0 of the state space, where the desired trajectory (5) lives. To be the Lyapunov function candidate V should be positive around V_0. The reader can easily check that in this case the parameters k_1–k_4 satisfy to

$$k_1 > 0,\ k_2 > 0,\ k_3 > 0,\ k_1 \cdot k_3 > k_4^2. \tag{8}$$

Taking the time derivative of V along the solutions of the system (1), one has

$$\dot{V} = \dot{q}_1\left[\tau_1\left(k_1(E-E_0) + k_2[0\,1]M(q)^{-1}\begin{bmatrix}0\\1\end{bmatrix} + \\ + k_4\left(q_1-\frac{\pi}{2}\right)\right) + H(q,\dot{q})\right], \tag{9}$$

where

$$[0\ 1]M(q)^{-1}\begin{bmatrix}0\\1\end{bmatrix} = \frac{\theta_2}{\theta_1\theta_2 - \theta_3^2\cos^2 q_2}$$

and

$$H(q,\dot{q}) = k_3\left(q_1-\tfrac{\pi}{2}\right) + k_4(E-E_0) + \tfrac{k_2}{\theta_1\theta_2-\theta_3^2\cos^2 q_2} \\ \times\left(\theta_2\theta_3\sin q_2(\dot{q}_1+\dot{q}_2)^2 + \theta_3^2\cos q_2 \sin q_2 \dot{q}_1^2 - \\ \theta_2\theta_4 g\cos q_1 + \theta_3\theta_5 g\cos q_2\cos(q_1+q_2)\right).$$

To ensure sign semi-definiteness of $\dot{V}$ one can try to solve the following equation with respect to the control variable τ_1

$$\tau_1\left(k_1(E-E_0) + \tfrac{k_2\theta_2}{\theta_1\theta_2-\theta_3^2\cos^2 q_2} + k_4\left(q_1-\tfrac{\pi}{2}\right)\right) \\ + H(q,\dot{q}) = -\phi(\dot{q}_1). \tag{10}$$

Here $\phi(x)$ is any smooth function, $x^T\phi(x) > 0$ $\forall x \neq 0$. The next statement gives simple sufficient condition for solvability of the equation (10)

Lemma 1. If $k_1 > 0$, $k_2 > 0$ and k_4 satisfy the inequality

$$k_2 > \theta_1 \cdot (k_1 \cdot E_0 + |k_4| \cdot 2\pi), \tag{11}$$

then the value of the function

$$k_1(E-E_0) + \tfrac{k_2\theta_2}{\theta_1\theta_2-\theta_3^2\cos^2 q_2} + k_4\left(q_1-\tfrac{\pi}{2}\right) \tag{12}$$

is strictly positive for any $(q,\dot{q}) \in S^2 \times R^2$, and the control variable τ_1 can be found from (10). ■

Proof. The energy E of the Pendubot is nonnegative function, therefore if the function

$$-k_1\cdot E_0 + k_2\frac{\theta_2}{\theta_1\theta_2-\theta_3^2\cos^2 q_2} + k_4\left(q_1-\frac{\pi}{2}\right)$$

is positive then the function (12) is positive too. The last expression one can rewrite as

$$k_2\frac{\theta_2}{\theta_1\theta_2-\theta_3^2\cos^2 q_2} > k_1\cdot E_0 + k_4\left(\frac{\pi}{2}-q_1\right).$$

One can easilty check that

$$\min_{q_2\in S^1}\frac{\theta_2}{\theta_1\theta_2-\theta_3^2\cos^2 q_2} = \frac{1}{\theta_1}, \\ \max_{q_1\in S^1} k_4\left(\frac{\pi}{2}-q_1\right) \leq |k_4|2\pi.$$

Therefore, the last inequality holds provided that the inequality (11) is valid. ■

As a result, if the parameters k_i are chosen to satisfy (11), then the controller τ_1 determined via (10), makes the time derivative of the Lyapunov function candidate V sign semi-definite

$$\dot{V}(t) = -\dot{q}_1(t) \times \phi\left(\dot{q}_1(t)\right). \tag{13}$$

Using the standard terminology, the globally defined feedback transformation (10) makes the transformed system passive with the storage function V.

3. MAIN RESULTS

If the parameters k_1, k_2 and k_4 are chosen to satisfy to (11), then the controller determined by (10) is globally defined. This makes possible to find all ω-limit points of the closed loop system.

Lemma 2. Consider the Pendubot together with the controller defined by (10) where the coefficients k_1–k_4 satisfying the inequalities (8) and (11). Then the ω-limit set of the closed loop system consists of the set V_0 and a number equilibria with the coordinates (q_1^*, q_2^*) defined as a solution of the equations

$$\frac{k_3\left(q_1^*-\frac{\pi}{2}\right)+k_4\{\theta_4 g(1+\sin q_1^*)+\theta_5 g - E_0\}}{k_1\left\{E_0-\theta_4 g(1+\sin q_1^*)-\theta_5 g\right\}+k_4\left(\frac{\pi}{2}-q_1^*\right)} = \theta_4 g\cos q_1^* \tag{14}$$

$$q_1^* + q_2^* = \left\{\frac{\pi}{2}\ \text{or}\ -\frac{\pi}{2}\right\} \bmod 2\pi. \tag{15}$$

■

The proof of Lemma 2 is omitted due to lack of space. The next step in the controller design for the Pendubot is to determine the range of the parameters k_i, which satisfy the constraints (8), (11), and which guarantee that any additional equilibrium – the solution of the equations (14)–(15), if exists, is hyperbolic. It could be done for example by taking linear approximation of the closed loop system around this equilibrium.

In any case, it is obvious that number of solution of the equations (14)–(15) is finite (or maybe empty). Therefore, Lemma 2 guarantees at least that the set V_0 is asymptotically stable, but possibly not global. **This implies that the periodic motion (5) of the closed loop system is asymptotically orbitally stable.** To implement the derived controller (10), one would appreciate more information about the asymptotic orbital stability of the constructed periodic trajectory. Particularly, the cases, when this stability is *exponential*, are of great interest. The conclusion about the convergence made before is based on the analysis of the ω-limit sets of the closed loop system. Unfortunately, this method does not provide any quantitative measures of the transition, it rather shows sets, which may attract solutions.

To perform the analysis of convergence's rate, we suggest to linearize the closed loop system around the periodic solution (5).

Lemma 3. Consider the Pendubot with the feedback controller τ_1 defined via (10). Then the linear approximation of the closed loop system around the periodic solution (5) is a linear periodic system

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \alpha_1(t) & \alpha_2(t) & \alpha_3(t) & \alpha_4(t) \\ \beta_1(t) & \beta_2(t) & \beta_3(t) & \beta_4(t) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}, \quad (16)$$

where

$$\begin{aligned}
\alpha_1(t) &= -\tfrac{k_4}{k_2}\theta_3 \sin q_2^d(t) \left(\tfrac{\theta_5}{\theta_2} g \cos q_2^d(t) - \left(\dot q_2^d(t)\right)^2\right) - \\
&\quad -\tfrac{k_3}{k_2} + \gamma \cdot \theta_5 g \sin q_2^d(t) \\
\alpha_2(t) &= \gamma \cdot \theta_5 g \sin q_2^d(t) \\
\alpha_3(t) &= -\tfrac{\phi'(0)}{k_2} - \gamma \cdot \left(\theta_2 + \theta_3 \cos q_2^d(t)\right) \dot q_2^d(t) \\
\alpha_4(t) &= -\gamma \cdot \theta_2 \dot q_2^d(t) \\
\beta_1(t) &= -\tfrac{\theta_2 + \theta_3 \cos q_2^d(t)}{\theta_2} \cdot \alpha_1(t) + \tfrac{\theta_5 g \cos q_2^d(t)}{\theta_2} \\
\beta_2(t) &= -\tfrac{\theta_2 + \theta_3 \cos q_2^d(t)}{\theta_2} \cdot \alpha_2(t) + \tfrac{\theta_5 g \cos q_2(t)}{\theta_2} \\
\beta_3(t) &= -\tfrac{\theta_2 + \theta_3 \cos q_2^d(t)}{\theta_2} \cdot \alpha_3(t) \\
\beta_4(t) &= -\tfrac{\theta_2 + \theta_3 \cos q_2^d(t)}{\theta_2} \cdot \alpha_4(t) \\
\gamma &= \tfrac{k_4}{k_2} + \tfrac{k_1}{k_2}\theta_3 \sin q_2^d(t) \left(\tfrac{\theta_5}{\theta_2} g \cos q_2^d(t) - \left(\dot q_2^d(t)\right)^2\right)
\end{aligned}$$

■

The proof of Lemma 3 is based on the standard routine calculations. As shown above, the closed loop system possesses the periodic solution

$$\begin{aligned} q_1(t) &= q_1^d(t) = \frac{\pi}{2}, & \dot q_1(t) &= \dot q_1^d(t) = 0 \\ q_2(t) &= q_2^d(t), & \dot q_2(t) &= \dot q_2^d(t). \end{aligned} \quad (17)$$

One can readily see that the linearized system (16) has the following periodic solution

$$\begin{aligned} y_1^d(t) &= \tfrac{d}{dt} q_1^d(t) = 0, \ y_3^d(t) = \tfrac{d}{dt}\dot q_1^d(t) = 0 \\ y_2^d(t) &= \tfrac{d}{dt} q_2^d(t), \ y_4^d(t) = \tfrac{d}{dt}\dot q_2^d(t) = \tfrac{\theta_5 g \sin q_2^d(t)}{\theta_2}. \end{aligned} \quad (18)$$

The last equality is due to (4). As known, an existence of the periodic solution (18) of the linear periodic system (16) implies that one of the multipliers, for example ρ_1, of (16) equals to 1. Furthermore, the orbital asymptotic stability of the nonlinear closed loop system will guarantee that the system (16) is at least stable, i. e. all other three multipliers ρ_2, ρ_3, ρ_4 belong to closed unit disc of the complex plane.

For the nonlinear closed loop system the rate of convergence to the solution (17) depends mainly on the location of these additional 3 multiplies of the linearized system (16). Indeed, if these multipliers are *within open* unit disc, then the convergence is *exponential*. Otherwise, if some of them belong to a boundary to the unit disc (i.e. have the magnitude 1), then the convergence will be poor.

Lemma 4. Denote T the period of the system (16). Then

$$\rho_2 \cdot \rho_3 \cdot \rho_4 = \exp\left\{-T\frac{\phi'(0)}{k_2}\right\} < 1, \quad (19)$$

i. e. the product of unknown multipliers of the system (16) is always less than 1. ■

Proof. Denote $Y(t)$ a matrizant of the system (16), i. e. it is a fundamental matrix with $Y(0) = I_4$. Then by the Liouville formula we have

$$\det Y(t) = \det Y(t_0) \exp\left\{\int_{t_0}^{t} \mathrm{tr} A(s) ds\right\}, \quad \forall t.$$

Here $A(s)$ is a matrix of the system (16), and tr$A$ designates a trace of the matrix. If T is a period of the system, then

$$\det Y(T) = \exp\left\{\int_0^T \mathrm{tr} A(s) ds\right\} = e^{\left\{\int_0^T -\frac{\phi(0)}{k_2} ds\right\}}.$$

From the other hand, the determinant of the monodromy matrix is equal to the product of all multipliers, i. e.

$$\det Y(T) = \rho_1 \cdot \rho_2 \cdot \rho_3 \cdot \rho_4.$$

To finish the proof one can compare the last two formulas taking into account $\rho_1 = 1$. ■

Take any solution $y(t)$ of (16), it is known that all solutions of (16) are bounded. Consider the infinite series of vectors $y(0)$, $y(T)$, $y(2T)$, ..., where T is a period of the system (16). Using standard arguments, one can deduce that the bounded sequence of vectors $\{y(kT)\}_{k=0}^{+\infty}$ has limit points. Denote Y_0 a family of solutions the linear system (16) with the origin from these limit points at $t = 0$.

Lemma 5. Suppose that a solution $y^*(t)$ of (16) belongs to the set Y_0. Then along this solution the function $y_3^*(t)$ is zero, and $y_1^*(t)$ is a some constant, i. e.

$$y_3^*(t) \equiv 0, \quad y_1^*(t) = C, \tag{20}$$

where C is some constant. ■

Proof. We know that along any solution of the nonlinear closed loop system the function V satisfies to the passivity relation (13). The reader can easily check that for the linearized system (16) an appropriate modification of (13) will take place with the term $-y_3^2(t) \times \phi'(0)$ in the right hand side of (13). So that $\phi'(0) > 0$, for any solution of (16) from Y_0 the value of the component $y_3^*(t)$ should be identically zero. Due to (16), $\dot{y}_1 = y_3$, therefore along the solution $y^*(t)$ the value of the component $y_1^*(t)$ is just a constant. ■

In other words, Lemma 5 shows that any solution of (16) will only wind around some particular solutions, which have to satisfy (20). The family of these *attractive* solutions is not empty, the trajectory (17) belongs to this family. If we are able to show that this family consists of just one trajectory (17) then we can conclude that the rate of convergence to (17) in the closed loop system is exponential. Otherwise, if there are several linear independent elements of this family then, apparently, the rate of convergence is not exponential.

Among all solutions of the system (16) the constraints (20) separate those $y(t) = [C, y_2(t), 0, y_4(t)]$ which satisfy simultaneously to the differential equation

$$\frac{d}{dt}\begin{bmatrix} y_2 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \beta_2(t) & \beta_4(t) \end{bmatrix}\begin{bmatrix} y_2 \\ y_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \beta_1(t) \end{bmatrix} C \tag{21}$$

and the algebraic relation

$$\alpha_1(t)C + \alpha_2(t)y_2(t) + \alpha_4(t)y_4(t) \equiv 0, \ \forall t. \tag{22}$$

Multiplying the identity (22) by

$$\frac{\theta_2 + \theta_3 \cos q_2^d(t)}{\theta_2}$$

and adding this product to the right hand side of the equation (21), one gets the new version of the equation (21)

$$\frac{d}{dt}\begin{bmatrix} y_2 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{\theta_5 g \cos q_2^d(t)}{\theta_2} & 0 \end{bmatrix}\begin{bmatrix} y_2 \\ y_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\theta_5 g \cos q_2^d(t)}{\theta_2} \end{bmatrix} C.$$

It is a quite interesting fact that the last system is totally independent on the parameters of the controller. By changing variables $z_1 := y_2 + C$, $z_2 := y_4$ we get the Hill's equation

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ p(t) & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{23}$$

with $p(t) = \frac{\theta_5 g}{\theta_2}\cos q_2^d(t)$. Furthermore, it is known that the equation (23) has one periodic solution. Then it is clear that this equation has the second linear independent periodic solution.

Let us summarize the arguments: *Suppose that we found two independent solutions of the Hill's equation (23), either analytically or via simulation. Then we have to substitute these solutions into identity (22) with the objective to identify those parameters k_i, if any, which make (22) valid only for the case: $C = 0$ and $z_1(t) = \dot{q}_2^d(t)$, $z_2(t) = \frac{\theta_5 g}{\theta_2}\sin q_2^d(t)$. If these parameters k_i, in addition, satisfy to the inequalities (8), (11), they will provide* exponential *convergence.* ■

4. SIMULATIONS

Here some experiments performed in Simulink, are collected. The Pendubot parameters were taken the same as in (Spong and Block, 1995). The value of E_0 was chosen two times larger then the energy of the Pendubot in its upper equilibrium. To observe the performance of the results we have chosen two sets of the controller's parameters:

$$K(1): \ k_1 = 1, \ k_2 = 1.5\theta_1(k_1 E + 2\pi k_3) = 2.2177, \quad k_3 = 1, \ k_4 = 0.3, \ \phi(x) = 3x$$

$$K(2): \ k_1 = 1, \ k_2 = 1.5\theta_1 k_1 E = 1.9918, \quad k_3 = 1, \ k_4 = 0, \ \phi(x) = x$$

Figures 1–2 shows the response of the closed loop system with the controller $K(1)$ for the initial conditions

$$q_1(0) = \frac{\pi}{2}, \ q_2(0) = \frac{\pi}{2}, \ \dot{q}_1(0) = 0.03, \ \dot{q}_2(t) = 12.$$

Figures 3–4 shows the response of the closed loop system with the controller $K(2)$ for the same initial conditions. We would like to attract the attention of the reader to Figures 2 and 4, where the behavior of the energies are depicted. As seen, a small discrepancy between the controllers $K(1)$ and $K(2)$ leads to the essential improvement.

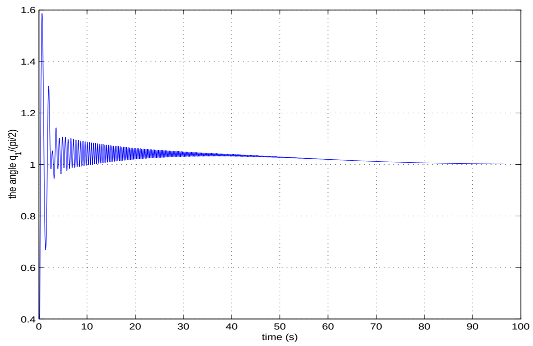

Fig. 1. The angle of the first link converges to $\frac{\pi}{2}$ under the controller K(1)

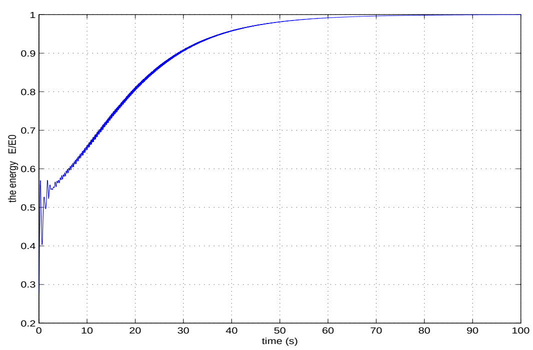

Fig. 2. The energy converges to E_0 under the controller K(1)

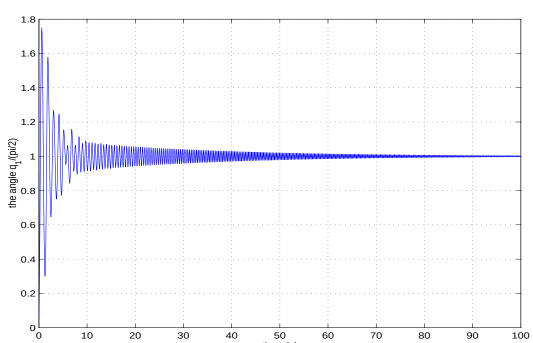

Fig. 3. The angle of the first link converges to $\frac{\pi}{2}$ under the controller K(2)

5. CONCLUSIONS

This paper is aimed at constructing and orbital stabilizing periodic trajectories for the two-link underactuated robot, called the Pendubot. These desired periodic trajectories, except some pathological cases, do not exist for the Pendubot without a control action. It is shown how to construct these motions via feedback. Then it is shown how to make these motions orbitally asymptotically stable. Furthermore, the linearized model of the closed loop system around these trajectories, is discussed. This issue seems to be important to improve rate of convergence, which sometimes could be poor.

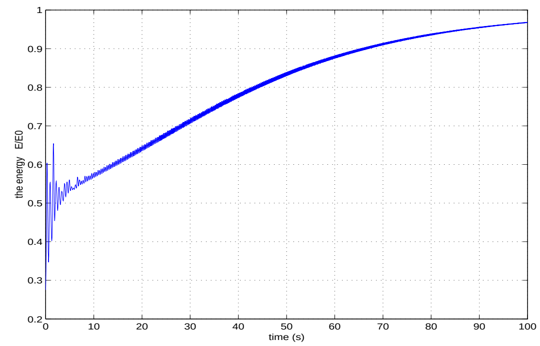

Fig. 4. The energy converges to E_0 under the controller K(2)

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