15th Triennial World Congress, Barcelona, Spain

A VARIATIONAL INEQUALITY FOR A CLASS OF MEASUREMENT FEEDBACK ALMOST-DISSIPATIVE CONTROL PROBLEMS

Peter M. Dower [*,1]

[*] *Department of Electronic Engineering, La Trobe University, Victoria 3086, Australia.*

Abstract: In this paper, an optimal measurement feedback control problem that yields an almost-dissipative closed loop system is considered. Using information state ideas and the definition of the optimal cost presented, a dynamic programming equation is derived. Incremental analysis yields a corresponding variational inequality (VI) which naturally generalizes the information state based partial differential equation (PDE) associated with measurement feedback nonlinear H-infinity control. In theory, this variational inequality can be used to synthesize an optimal measurement feedback controller which guarantees that the closed loop system almost satisfies a given dissipation property. This "almost-dissipation" property admits a weaker form of stability for the closed loop system, allowing presistence of excitation in the absence of disturbance inputs. Finally, certainty equivalence control is investigated as a special case of the results presented.

Keywords: Dissipative control, optimal control, measurement feedback, information state, dynamic programming, variational inequality, certainty equivalence.

1. INTRODUCTION

Dissipative systems theory (Willems, 1972; Hill and Moylan, 1976; Hill and Moylan, 1980) has wide ranging implications and applications in control theory. One of the most popular of these in recent times has been nonlinear H^∞ control.

As a design method for nonlinear robust control, nonlinear H^∞ control was first explored geometrically in (van der Schaft, 1992; Isidori and Astolfi, 1992). The more general information state approach of (Basar and Bernhard, 1995; Helton and James, 1999) has subsequently produced significant advances in the understanding of the measurement feedback control problem.

Information state control provides the theoretical tools for designing measurement feedback H^∞ controllers and, more generally, dissipative controllers for nonlinear systems. Although decoupling (or separation) of a measurement feedback dissipative control problem into traditional state estimation and state feedback problems is not in general possible, the information state controller overcomes this problem via the feedback of the *information state* instead (the information state is a function which evolves in time according to a partial differential equation (PDE) dependent on past plant measurements and applied controls). That is, the traditional measurement feedback problem is replaced by an equivalent information state feedback problem.

Information state control thus consists of a dynamic controller which maps past plant measurements and controls to present controls via the information state and an information state control policy. When connected in feedback with the nonlinear plant, the resulting information state controller yields a closed loop system with a prescribed dissipation property. In the H^∞ case, this means an L_2-gain bound from disturbances to outputs for the closed loop system. (This is equivalent to the closed loop being dissipative with an L_2 supply rate.)

[1] Partially supported by the Australian Research Council.

In this paper, the first step in desiging analogous controllers to yield "almost-dissipative" closed loop systems is considered. The important generalization here is that a prescribed dissipation property for the closed loop system is almost met, but not quite. This means that the attendant (typically asymptotic) stability of a dissipative closed loop system is weakened, allowing *practical* stability. Hence, trajectories of the controlled plant may converge to some neigbourhood of the origin.

Almost-dissipation is defined in this paper by including an offset in the supply rate used to define the conventional dissipation property, in the same way as in other practical properties such as input-to-state practical stability (Z.P. Jiang, 1994) and power gain (practical L_2-gain) analysis (Dower and James, 1998). Using this notion of almost-dissipation, an optimization problem is defined and the corresponding dynamic programming equation derived. Incrementally, this equation is shown to correspond to a variational inequality (VI) which naturally generalizes the information state based PDE of (Helton and James, 1999). In the special case of certainty equivalence, an explicit solution of this VI is provided.

All omitted proofs will appear in a later article (Dower, 2002).

2. PRELIMINARIES

We consider nonlinear plants G of the form

$$\begin{aligned} \dot{x}(t) &= f(x(t),u(t),w(t)),\\ y(t) &= g(x(t),w(t)),\\ z(t) &= h(x(t),u(t)), \end{aligned} \tag{1}$$

where f, g and h are zero at zero. Here $x(t) \in \mathbf{R}^n$ is the state, $w(t) \in \mathbf{R}^s$ is the disturbance, $u(t) \in \mathbf{R}^m$ is the control, $y(t) \in \mathbf{R}^p$ is the measurement and $z(t) \in \mathbf{R}^r$ is the performance measure.

We assume that g is invertible in the sense that there exists a function $g^\# : \mathbf{R}^p \times \mathbf{R}^n \times \mathbf{R}^s \to \mathbf{R}^s$ such that for any triple $(y,x,w) = (y(t),x(t),w(t)) \in \mathbf{R}^p \times \mathbf{R}^n \times \mathbf{R}^s$ with $y(\cdot)$, $x(\cdot)$ and $w(\cdot)$ satisfying system (1), there exists a $v \in \mathbf{R}^s$ such that

$$w = g^\#(y,x,v), \tag{2}$$

where for any $x \in \mathbf{R}^n$,

$$\begin{aligned} y &= g(x, g^\#(y,x,v)),\\ w &= g^\#(g(x,w),x,w). \end{aligned}$$

Remark 2.1. Invertibility of function g is utilized in the H^∞ case (Helton and James, 1999) so that the optimal control problem of interest can be expressed as an optimization over controls and measurements rather than controls and disturbances. See (Helton and James, 1999) for details relevant to "reversing arrows" in that case.

3. ALMOST-DISSIPATIVE SYSTEMS

System (1) is almost-dissipative (or practically dissipative) with respect to supply rate $r : \mathbf{R}^s \times \mathbf{R}^r \to \mathbf{R}$ if there exists a locally bounded nonnegative function $V : \mathbf{R}^n \to \mathbf{R}$ and a real nonnegative offset λ such that

$$V(x_\circ) + \int_0^T r(w(s),z(s))ds \;\geq\; V(x(T)) - \lambda T \tag{3}$$

for all initial states $x_\circ \in \mathbf{R}^n$, all disturbances $w \in \mathscr{W}[0,T]$ and all time horizons $T \geq 0$. Here, $\mathscr{W}[0,T]$ is the space of inputs for which the integral in (3) is finite. We assume the following:

(A1) The supply rate satisfies the inequality $r(0,z) \leq 0$ for all $z \in \mathbf{R}^r$.

Note that in the dissipative ($\lambda = 0$) case, this corresponds to energy liberation in the absence of disturbances.

The following result links almost-dissipation with the corresponding input/output property.

Theorem 3.1. A system is almost-dissipative with offset λ iff there exists a locally bounded nonnegative function $\beta : \mathbf{R}^n \to \mathbf{R}$ such that

$$\int_0^T [-r(w(s),z(s))]ds \;\leq\; \beta(x_\circ) + \lambda T \tag{4}$$

for all $x_\circ \in \mathbf{R}^n$, all $w \in \mathscr{W}[0,T]$ and all $T \geq 0$.

4. THE INFORMATION STATE AND THE OPTIMAL CONTROL PROBLEM

Using the notion of *information state* (Helton and James, 1999), a cost function is defined in terms of the supply rate, offset and controller. Results linking this cost function to the almost-dissipative systems property are presented.

The information state $p_t^{u,y}(x)$ (Helton and James, 1999) captures the worst possible integrated cost for all trajectories of system (1) given the final state x, consistent with the obtained measurements $y \in \mathscr{Y}[0,t]$ (here, $\mathscr{Y}[0,t]$ is the space of all obtainable measurements on $[0,t]$). It is often referred to as the "cost to come". Formally,

$$p_t^{u,y}(x) = \sup_{w \in \mathscr{W}[0,t]} \left\{ p_\circ(\xi(0)) + \int_0^t [-r(w(s),z(s))]ds \right.$$

$$: \dot{\xi}(s) = f(\xi(s), u(s), w(s)),$$
$$g(\xi(s), w(s)) = y(s) \ \forall s \in [0,t]$$
$$\left.\xi(t) = x\right\}. \qquad (5)$$

Here $p_\circ : \mathbf{R}^n \to \mathbf{R} \cup \{-\infty\}$ is the initial information state. This integral equation (5) can be reformulated under suitable differentiability conditions as a PDE as proved in (Helton and James, 1999). In particular,

$$\begin{aligned}\frac{\partial p_t^{u,y}}{\partial t}(x) \;=\; &\sup_{w \in \mathbf{R}^s} \{-\nabla_x p_t^{u,y}(x) \cdot f(x,u,w) \\ &\qquad -r(w, h(x,w)) : g(x,w) = y\} \\ =:\; &F(\nabla_x p_t^{u,y}, u, y)(x), \end{aligned} \qquad (6)$$

where f, g and h are the system functions given in (1) and r is the supply rate.

In order to cost a given measurement feedback controller K on a finite time horizon, define

$$J_{p_\circ}(K;T) = \sup_{w \in \mathscr{W}[0,T]} \sup_{x_\circ \in \mathbf{R}^n} \{J_{p_\circ}(K;T,w,x_\circ)\} \qquad (7)$$

where

$$J_{p_\circ}(K;T,w,x_\circ) = p_\circ(x_\circ) + \int_0^T [-r(w(s), z(s))]ds. \qquad (8)$$

With regard to interpretation of this cost, note that the $p_\circ(x_\circ)$ term on the RHS of (7),(8) represents the worst case cost in steering the state to $x_\circ$ (i.e. cost to come), whilst the integral term represents the cost to follow on the interval $[0,T]$, with the state initialized at $x_\circ$. (Note that the cost is also worst case with respect to the choice of state $x_\circ$.)

A "reverse arrows" characterization of J is provided via the following definition (Helton and James, 1999):

$$\begin{aligned}\tilde{J}_{p_\circ}(K;T) = \sup_{y \in \mathscr{Y}[0,T]} &\{\langle p_T^{u,y} \rangle : u(s) = K(y(s)), \\ &s \in [0,T],\ p_\circ \text{ given}\}. \end{aligned} \qquad (9)$$

Here, $\langle p \rangle := \max_{x \in \mathbf{R}^n} \{p(x)\}$.

Lemma 4.1. (Helton and James, 1999) For all $T \geq 0$, $\tilde{J}_{p_\circ}(K;T) = J_{p_\circ}(K;T)$.

The remaining results in this section provide bounds on the cost J under various conditions. Interpretation of these bounds lead to a suitable definition of the optimal control problem.

Lemma 4.2. Consider system (1) and assume that (A1) holds. Then, the following properties hold:

(1) Given any controller K, the finite horizon cost $J_{p_\circ}(K;T)$ is nondecreasing in T.
(2) Given a controller K initialized with information state $p_\circ$ and any $y \in \mathscr{Y}[0,\tau]$ such that $u(s) = K(y)(s)$ is defined for all $s \in [0,\tau]$, then
$$J_{p_\circ}(K;T) \;\geq\; \langle p_T^{u,y} \rangle \qquad (10)$$
for all $y \in \mathscr{Y}[0,T]$ and all $T \in [0,\tau]$, where $p_\circ^{u,y} = p_\circ$.
(3) The closed loop system (G,K) is almost-dissipative with offset $\lambda \geq 0$ iff for all $T \geq 0$,
$$J_{-\beta}(K;T) \;\leq\; \lambda T \qquad (11)$$
for some locally bounded function $\beta : \mathbf{R}^n \to \mathbf{R}_{\geq 0}$.
(4) Suppose that (G,K) is almost-dissipative with offset λ. Then, there exists a function β_K such that
$$J_{p_\circ}(K;T) \;\leq\; \langle p_\circ + \beta_K \rangle + \lambda T. \qquad (12)$$

Essentially, Lemma 4.2 provides a list of growth conditions for the finite horizon cost (7). Indeed, all assertions of the Lemma point towards growth in the cost which may be (at most) linear in the time horizon T. Hence, any useful definition of time horizon independent cost associated with a given controller must account for this growth with respect to T. With this in mind, the worst case time horizon independent cost for controller K and offset λ is defined to be

$$J_{p_\circ,\lambda}(K) \;=\; \sup_{T \geq 0} \{J_{p_\circ}(K;T) - \lambda T\}. \qquad (13)$$

Equivalently, using Lemma 4.1,

$$\begin{aligned}J_{p_\circ,\lambda}(K) = \sup_{T \geq 0} \sup_{y \in \mathscr{Y}[0,T]} &\{\langle p_T^{u,y} \rangle - \lambda T \\ &: u(s) = K(y)(s), \\ &\ s \in [0,T],\ p_\circ \text{ given}\}. \end{aligned} \qquad (14)$$

This definition is worst case as it assumes that the bound (14) provided by Lemma 4.2 is tight. Using this definition of cost function, it is now possible to define the optimal control problem of interest:

Definition 4.3. (Optimal Almost-Dissipative Control Problem) Find the optimal measurement feedback controller K^* which minimizes the cost functional $J_{p_\circ,\lambda}(K)$ given by (13). That is, find K^* such that

$$J_{p_\circ,\lambda}(K^*) \;=\; \inf_K \left\{J_{p_\circ,\lambda}(K)\right\} \;=:\; W_\lambda(p_\circ), \qquad (15)$$

where W_λ denotes the optimal cost for achieving closed loop almost-dissipation with offset λ.

In order to find the controller K^*, the natural next step is to turn to dynamic programming.

5. DYNAMIC PROGRAMMING

The aim is to find a dynamic programming equation for W_λ. The following definitions and results are mostly technical, leading to the dynamic programming result of Theorem 5.3.

Controller K_δ is δ-optimal given $p_\circ$ if

$$W_\lambda(p_\circ) + \delta \ \geq \ J_{p_\circ,\lambda}(K_\delta). \tag{16}$$

Bounds on the optimal cost W_λ follow from Lemma 4.2, the definition (15) of the optimal cost W_λ, and the definition of δ-optimality.

Lemma 5.1. The optimal cost W_λ given by (15) satisfies the following properties:

(1) For any information state $p_\circ$,

$$W_\lambda(p_\circ) \ \geq \ \langle p_\circ \rangle. \tag{17}$$

(2) Given any controller K,

$$J_{p_\circ,\lambda}(K) \ \geq \ W_\lambda(p_t^{u,y}) - \lambda t \tag{18}$$

for any $y \in \mathscr{Y}[0,t]$ and any $t \geq 0$, where $u(s) = K(y)(s)$, $s \in [0,t]$.

(3) Let K_δ be a δ-optimal controller (16) for $W_\lambda(p_\circ)$. Then, for any $y \in \mathscr{Y}[0,t]$ and any $t \geq 0$,

$$W_\lambda(p_\circ) + \lambda t + \delta \ \geq \ W_\lambda(p_t^{u,y}) \tag{19}$$

where $u(s) = K_\delta(y)(s)$, $s \in [0,t]$.

This demonstrates that even for δ-optimal controllers, $W_\lambda(p_t^{u,y})$ "almost decreases" (i.e. may increase within the bound imposed by the λt term) along trajectories. This represents a departure from the H^∞ results of (Helton and James, 1999).

Using Lemma 5.1, the easier of the two dynamic programming inequalities can now be proved.

Lemma 5.2. For all $r \geq 0$, W_λ satisfies the inequality

$$\begin{aligned} W_\lambda(p_\circ) \geq \inf_K \sup_{T\geq 0} \sup_{y\in\mathscr{Y}[0,r\wedge T]} \big\{ & [W_\lambda(p_r^{u,y}) - \lambda r]\chi_{r<T} \\ & + [\langle p_T^{u,y}\rangle - \lambda T]\chi_{r\geq T} : u(s) = K(y)(s), \\ & s \in [0, r\wedge T],\ p_\circ \text{ given} \big\} \end{aligned} \tag{20}$$

where $r \wedge T := \min(r,T)$ and $\chi_b = \begin{cases} 1 & b \text{ is true} \\ 0 & b \text{ is false} \end{cases}$.

Using Lemma 5.2 and by proving the opposite inequality, we can now state a dynamic programming result for W_λ. Note that in the dissipative ($\lambda = 0$) case, the proof of this result would be identical to that in (Helton and James, 1999; James and Baras, 1996). The significant difference in the almost-dissipative ($\lambda > 0$) case is that a stopping time must be included in the dynamic programming equation. The proof of this result is presented in Appendix A.

Theorem 5.3. For all $r \geq 0$, W_λ satisfies the dynamic programming equation

$$\begin{aligned} W_\lambda(p_\circ) = \inf_K \sup_{T\geq 0} \sup_{y\in\mathscr{Y}[0,r\wedge T]} \big\{ & [W_\lambda(p_r^{u,y}) - \lambda r]\chi_{r<T} \\ & + [\langle p_T^{u,y}\rangle - \lambda T]\chi_{r\geq T} : u(s) = K(y)(s), \\ & s \in [0, r\wedge T],\ p_\circ \text{ given} \big\} \end{aligned} \tag{21}$$

The existing dissipative ($\lambda = 0$) result (Helton and James, 1999; James and Baras, 1996) follows as a corollary from the dynamic programming equation (21).

Corollary 5.4. Suppose that the supply rate assumption (A1) holds. Then, in the dissipative case ($\lambda = 0$), the optimal cost $W_0 := W_{\lambda=0}$ satisfies the dynamic programming equation

$$\begin{aligned} W_0(p_\circ) = \inf_K \sup_{y\in\mathscr{Y}[0,r]} \big\{ & W_0(p_r^{u,y}) : u(s) = K(y)(s), \\ & s \in [0,r], p_\circ \text{ given} \big\} \end{aligned} \tag{22}$$

for all $r \geq 0$.

6. A VARIATIONAL INEQUALITY

By definition of the information state (5), the dynamic programming equation (21) is an integral equation. The aim now is to derive an incremental form of the dynamic programming equation (21).

Recall that $W_\lambda(\cdot)$ is a functional which maps informations states (i.e. functions) to real numbers. Consequently, to meaningfully formulate a differential equation involving $W_\lambda(p)$, we require that W_λ be Frechet differentiable with respect to p, as per the dissipative case (Helton and James, 1999; James and Baras, 1996). Then, applying the chain rule,

$$\nabla_p W_\lambda(p) \left[\left. \frac{\partial p_t^{u,y}}{\partial t} \right|_{t=0} \right] = \lim_{t\downarrow 0} \left\{ \frac{W_\lambda(p_t^{u,y}) - W_\lambda(p_0)}{t} \right\}$$

where ∇_p denotes the Frechet differentiation operator. Note here that the LHS denotes the directional derivative of W_λ in the direction $\left.\frac{\partial p_t^{u,y}}{\partial t}\right|_{t=0}$. The notation $\nabla_p W_\lambda(p)[\cdot]$ does not imply multiplication.

We now state the incremental form of the dynamic programming equation (21). Unlike the dissipative ($\lambda = 0$) case, the differential equation obtained is a VI rather than a PDE.

Theorem 6.1. Suppose that the optimal almost-dissipative cost function $W_\lambda(p_\circ)$ is Frechet differentiable with respect to the information state $p_\circ$. Then, W_λ is a solution of the VI

$$0 = \max\Bigg(\langle p_\circ \rangle - W_\lambda(p_\circ), \quad (23)$$
$$-\lambda + \inf_{u\in\mathbf{R}^m} \sup_{y\in\mathbf{R}^p} \left\{ \nabla_p W_\lambda(p_\circ)\left[F(\nabla_x p_\circ, u, y)\right] \right\} \Bigg),$$

where F is the functional defined in the information state PDE (6).

7. CERTAINTY EQUIVALENCE

In the standard dissipative ($\lambda = 0$) case, a common technique for simplifying the optimal control problem is via *certainty equivalence* (Basar and Bernhard, 1995; Helton and James, 1999; James and Baras, 1996). In particular, assuming that the certainty equivalence property holds (as defined below), the measurement feedback problem can be separated into a state feedback problem and a state estimation problem. In this section, the corresponding separation is shown to occur in the almost-dissipative case under certainty equivalence.

With regard to notation, let K_{st}^λ denote a state feedback controller which renders the state feedback closed loop system (G, K_{st}^λ) almost-dissipative with supply rate r and offset λ. Additionally, let V_{st}^λ denote a corresponding storage function for the system (G, K_{st}^λ). The certainty equivalence property is then expressed as follows:

Certainty Equivalence (CE): *Given a state feedback storage function V_{st}^λ and the information state $p_t^{u,y}$, there exists a unique maximum with respect to $x \in \mathbf{R}^n$ of the function $p_t^{u,y}(x) + V_{st}^\lambda(x)$ for all measurements $y \in \mathscr{Y}[0,t]$ and all $t \geq 0$, where $u = K_{st}^\lambda(x)$. That is, the maximizer $\bar{x}_\lambda$ is unique, where*

$$\bar{x}_\lambda(p_t^{u,y}) := \operatorname*{argmax}_{x\in\mathbf{R}^n} \left\{ p_t^{u,y}(x) + V_{st}^\lambda(x) \right\}. \quad (24)$$

Next, define the *super available storage* V_λ given supply rate r and offset λ for the corresponding state feedback almost-dissipative control problem:

$$V_\lambda(x_\circ) :=$$
$$\inf_{K_{st}^\lambda} \sup_{T\geq 0} \sup_{w\in\mathscr{W}[0,T]} \Bigg\{ \int_0^T [-r(w(s), z(s)) - \lambda] ds$$
$$: z(s) = h(x(s), K_{st}^\lambda(x(s))), s\in[0,T], x(0) = x_\circ \Bigg\}. \quad (25)$$

Then, applying results in (Soravia, 1996), V_λ is the unique viscosity solution of the VI

$$0 = \max\Bigg(-V_\lambda(x_\circ),$$
$$-\lambda + \inf_{u\in\mathbf{R}^m} \sup_{w\in\mathbf{R}^s} \{ \nabla_x V_\lambda(x_\circ)\cdot f(x_\circ, u, w)$$
$$-r(w, h(x_\circ, u)) \} \Bigg). \quad (26)$$

Using the super available storage V_λ, define the functional

$$\hat{W}_\lambda(p_\circ) := \langle p_\circ + V_\lambda \rangle. \quad (27)$$

As was shown in (James and Baras, 1996), given a function $h: \mathbf{R}^n \to \mathbf{R}$, functionals of the form of (27) are Frechet differentiable with respect to p in the direction h. Furthermore, this directional derivative is given by the evaluation map provided that the CE property holds. That is,

$$\nabla_p \hat{W}_\lambda(p_\circ)[h] = h(\bar{x}_\lambda(p_\circ)), \quad (28)$$

where $\bar{x}_\lambda(p_\circ)$ is the maximizer (24) for the information state $p_\circ$. Using this fact yields the following simple result.

Lemma 7.1. Suppose that the CE assumption holds and that $p_\circ$ and V_λ are differentiable at $\bar{x}_\lambda(p_\circ)$. Then,

$$\inf_{u\in\mathbf{R}^m} \sup_{y\in\mathbf{R}^p} \left\{ \nabla_p \hat{W}_\lambda(p_\circ)\left[F(\nabla_x p_\circ, u, y)\right] \right\}$$
$$= H(\bar{x}_\lambda(p_\circ), \nabla_x V_\lambda(\bar{x}_\lambda(p_\circ))) \quad (29)$$

where F is the operator defined by (6) and H is the Hamiltonian

$$H(x, \rho) =$$
$$\inf_{u\in\mathbf{R}^m} \sup_{w\in\mathbf{R}^s} \{ \rho\cdot f(x,u,w) - r(w, h(x,u))) \}. \quad (30)$$

Finally, this means that the functional (27) is an explicit solution of the VI (23) provided that the CE property holds.

Theorem 7.2. Suppose that the CE property holds and that $p_\circ$ and V_λ are differentiable at $\bar{x}_\lambda(p_\circ)$. Then, the functional $\hat{W}_\lambda$ given by (27) is a solution of the VI (23).

8. REFERENCES

Basar, T. and P. Bernhard (1995). *H^∞-Optimal Control and Related Minimax Design Problem s: A Dynamic Game Approach*. Birkhäuser. Boston.

Dower, P.M. (2002). A variational inequality for measurement feedback almost-dissipative control. *Preprint, La Trobe University*.

Dower, P.M. and M.R. James (1998). Dissipativity and nonlinear systems with finite power gain. *Int. J. Robust and Nonlinear Control* **8**, 699–724.

Helton, J.W. and M.R. James (1999). *Extending H^∞ Control to Nonlinear Systems*. SIAM Frontiers in Applied Mathematics.

Hill, D.J. and P.J. Moylan (1976). The stability of nonlinear dissipative systems. *IEEE Trans. Autom. Contr.* **21-10**, 708–711.
Hill, D.J. and P.J. Moylan (1980). Dissipative dynamical systems: Basic input-output and state properties. *Journal of The Franklin Institute, Pergammon Press* **309-5**, 327–357.
Isidori, A. and A. Astolfi (1992). Disturbance attenuation and H^∞ control via measurement feedback in nonlinear systems. *IEEE Trans. Autom. Contr.* **37**, 1283–1293.
James, M.R. and J.S. Baras (1996). Partially observed differential games, infinite-dimensional hamilton-jacobi-isaacs equations, and nonlinear H^∞ control. *SIAM J. Control and Optimization* **34(4)**, 1342–1364.
Soravia, P. (1996). H^∞ control of nonlinear systems: differential games and viscosity solutions. *SIAM J. Contr. and Opt.* **34(3)**, 1071–1097.
van der Schaft, A.J. (1992). L_2-gain analysis of nonlinear systems and nonlinear state feedback H^∞ control. *IEEE Trans. Autom. Contr.* **AC-37(6)**, 770–784.
Willems, J.C. (1972). Dissipative dynamical systems, part i: General theory. *Arch. Rat. Mech. Anal.* **45**, 321–351.
Z.P. Jiang, A.R. Teel, L. Praly (1994). Small-gain theorem for iss systems and applications. *J. Math. Contr. Sys. Sign.* **7(2)**, 95–120.

Appendix A. PROOF OF THEOREM 5.3

Proof: Lemma 5.2 provides inequality in (21) in one direction. To prove the opposite direction, denote the RHS of (21) by $R_r(p_\circ)$. The aim is then to show that $R_r(p_\circ) \geq W_\lambda(p_\circ)$. Fix $r \geq 0$. Choose $K_{r,p_\circ}$ to be a δ-optimal controller for $R_r(p_\circ)$. Note that the sup over $\mathscr{Y}[0, r\wedge T]$ is identical to that over $\mathscr{Y}[0,T]$. Then, for all $y \in \mathscr{Y}[0,T]$, $T \geq 0$,

$$\begin{aligned} &R_r(p_\circ)+\delta \\ &\geq [W_\lambda(p_r^{u,y}) - \lambda r]\chi_{r<T} + [\langle p_T^{u,y}\rangle - \lambda T]\chi_{r\geq T} \\ &\ : u(s) = K(y)(s),\ s\in[0,T],\ p_\circ \text{ given.} \end{aligned} \quad \text{(A.1)}$$

Choose K_{r,p_r} δ-optimal in $W_\lambda(p_r^{u,y})$. If $T > r$,

$$\begin{aligned} &W_\lambda(p_r^{u,y})+\delta \\ &\geq J_{(p_r^{u,y}),\lambda}(K_{r,p_r}) \\ &\geq \sup_{T\geq r}\ \sup_{\hat{y}\in\mathscr{Y}[r,T]} \{\langle p_T^{\hat{u},\hat{y}}\rangle - \lambda(T-r) \\ &: \hat{u}(s) = K_{r,p_r}(\hat{y})(s),\ s\in[r,T],\ p_r^{\hat{u},\hat{y}} = p_r^{u,y}, \\ &\ \ u(s) = K_{r,p_\circ}(y)(s), s\in[0,r],\ p_\circ \text{ given}\} \\ &\geq \langle p_T^{\hat{u},\hat{y}}\rangle - \lambda(T-r) \\ &: \hat{u}(s) = K_{r,p_r}(\hat{y})(s),\ s\in[r,T],\ p_r^{\hat{u},\hat{y}} = p_r^{u,y}, \\ &\ \ u(s) = K_{r,p_\circ}(y)(s),\ s\in[0,r],\ p_\circ \text{ given,} \end{aligned} \quad \text{(A.2)}$$

for all $\hat{y} \in \mathscr{Y}[r,T]$. Define the augmented output and controller as

$$\begin{aligned} y_1(s) &= \begin{cases} y(s) & s\in[0,r], \\ \hat{y}(s) & s\in[r,T], \end{cases} \\ K_1(y_1)(s) &= \begin{cases} K_{r,p_\circ}(y)(s), s\in[0,r], \\ K_{r,p_r}(\hat{y})(s), s\in[r,T]. \end{cases} \end{aligned} \quad \text{(A.3)}$$

Combining (A.1), (A.2) and (A.3),

$$\begin{aligned} &R_r(p_\circ)+\delta \\ &\geq [\langle p_T^{u_1,y_1}\rangle - \lambda(T-r) - \lambda r - \delta]\chi_{r<T} \\ &\quad +[\langle p_T^{u_1,y_1}\rangle - \lambda T]\chi_{r\geq T} \\ &: u_1(s) = K_1(y_1)(s),\ s\in[0,\hat{T}],\ p_\circ \text{ given.} \\ &\geq [\langle p_T^{u_1,y_1}\rangle - \lambda T]\chi_{r<T} + [\langle p_T^{u_1,y_1}\rangle - \lambda T]\chi_{r\geq T} - \delta \\ &: u_1(s) = K_1(y_1)(s),\ s\in[0,T],\ p_\circ \text{ given} \\ &= \langle p_T^{u_1,y_1}\rangle - \lambda T - \delta \\ &: u_1(s) = K_1(y_1)(s),\ s\in[0,T],\ p_\circ \text{ given,} \end{aligned}$$

for all $y_1 \in \mathscr{Y}[0,T]$, $T \geq 0$. Therefore, taking the supremum over $y_1 \in \mathscr{Y}[0,T]$,

$$\begin{aligned} &R_r(p_\circ)+2\delta \\ &\geq \sup_{y_1\in\mathscr{Y}[0,T]} \{\langle p_T^{u_1,y_1}\rangle - \lambda T \\ &\quad : u_1(s) = K_1(y_1)(s),\ s\in[0,T],\ p_\circ \text{ given}\} \\ &= J_{p_\circ}(K_1;T) - \lambda T, \end{aligned}$$

for any $T \geq 0$. Taking the supremum over $T \geq 0$ and noting that K_1 is suboptimal yields that $R_r(p_\circ)+2\delta \geq W_\lambda(p_\circ)$. Sending $\delta \downarrow 0$ completes the proof. ■