15th Triennial World Congress, Barcelona, Spain

PERFECT REGULATION WITH CHEAP CONTROL FOR UNCERTAIN LINEAR SYSTEMS

Li Xie and **Ian R. Petersen** [1]

School of Electrical Engineering, Australian Defence Force Academy, Campbell, 2600, Australia, Phone +61 2 62688446, FAX +61 2 62688443, email: irp@ee.adfa.edu.au

Abstract: In this paper, the robust linear quadratic regulation problem with cheap control is studied for uncertain systems with norm-bounded uncertainty and integral quadratic constraint uncertainty, respectively. A Riccati equation approach is employed as a tool to investigate the limiting case in which a scalar weighting coefficient on the control input in the quadratic cost functional approaches zero. The corresponding performance limit is derived. Some results about monotonicity properties and the limiting behavior of the minimal positive definite solution to the Riccati equation are given. Using the limiting behavior of the minimal positive definite stabilizing solution to the Riccati equation, we find that perfect regulation with cheap control can be achieved if the uncertain system has a particular structure.

Keywords: Riccati equations, Cheap control, Perfect regulation, Quadratic guaranteed cost control, Norm-bounded uncertainty, Integral quadratic constraint uncertainty.

1. INTRODUCTION

One of the most important problems in control system design is concerned with the maximally achievable performance and fundamental performance limitations. For linear systems, the so-called cheap control problem has attracted much attention since the 1970's (Francis, 1979; Scherzinger and Davison, 1985). This problem consists of an optimal linear regulator problem with a quadratic cost functional which is the weighted sum of the integral squared output and the integral squared input and such that a scalar weighting coefficient on the control input in the quadratic cost functional tends to zero. If the integral squared output approaches zero as the weighting on the control input tends to zero, cheap control is said to provide perfect regulation for the control system.

Recently, the problem of performance limitations in feedback design has received a great deal of interest from many researchers. For uncertain linear systems, the results on maximally achievable performance and performance limitations are sparse to our knowledge. This paper is concerned with the problem of perfect regulation with cheap control for linear uncertain systems. The motivation for studying this problem comes from a desire to extend classical results for linear systems to linear uncertain systems. This is because many systems are inherently uncertain and a good understanding of the maximal achievable performance has a profound significance in the trade-off between system performance and control input. We hope it will be helpful to understand the relationship between the maximal achievable performance and the description of the plant to be controlled including a characterization of the uncertainty. We will employ a Riccati equation approach to quadratic guaranteed cost control to investigate the limiting case of a quadratic cost functional with cheap control for linear uncertain systems. Some results concerning the monotonicity properties and limiting behavior of the minimal positive definite solution to Riccati equations are given in Section 2. Using the limiting behavior of the minimal positive definite stabilizing solution to the Riccati equation, the performance limit for the

[1] This work was supported by the Australian Research Council

uncertain systems with norm-bounded uncertainty and integral quadratic constraint uncertainty, respectively, is derived. We find that perfect regulation with cheap control can be achieved if the uncertain system has a particular special structure.

2. NORM-BOUNDED UNCERTAINTY

The class of uncertain systems under consideration is described by the following state equation

$$\dot{x}(t) = [A + B_2\Delta(t)C_1]x(t) + [B_1 + B_2\Delta(t)D_1]u(t) \quad (1)$$
$$x(0) = x_0$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, $\Delta(t)$ is a time-varying matrix of uncertain parameters. Also $\Delta(t) \in \Xi$ is the set of all admissible uncertainties for the uncertain system (1) and is defined as follows:

$$\Xi = \{\Delta(t), \Delta'(t)\Delta(t) \leq I\}. \quad (2)$$

Associated with this system is the cost functional

$$J = \int_0^\infty \{x(t)'Rx(t) + u'(t)Gu(t)\}\mathrm{dt} \quad (3)$$

where $R, G > 0$. All matrices are real and have compatible dimensions.

Definition 1. A control law $u(t) = Kx(t)$ is said to define a quadratic guaranteed cost control with associated cost matrix $X > 0$ for the system (1) and cost functional (2) if

$$x'[R + K'GK]x + 2x'X[A + B_2\Delta(C_1 + D_1K) + B_1K]x < 0 \quad (4)$$

for all non-zero $x \in \mathbb{R}^n$ and for all $\Delta \in \Xi$.

From Theorem 5.2.1 (Petersen *et al.*, 2000), if the control law $u(t) = Kx(t)$ is a quadratic guaranteed cost control with cost matrix $X > 0$, then the corresponding value of the cost functional (3) satisfies the bound $J \leq x_0'Xx_0$ for all admissible uncertainties $\Delta(t)$.

Lemma 2. (Petersen *et al.*, 2000) Suppose that there exists a constant $\varepsilon > 0$ such that the Riccati equation

$$\begin{aligned}&(A - B_1(\varepsilon G + D_1'D_1)^{-1}D_1'C_1)'X \\ &+ X(A - B_1(\varepsilon G + D_1'D_1)^{-1}D_1'C_1) + \varepsilon XB_2B_2'X \\ &- \varepsilon XB_1(\varepsilon G + D_1'D_1)^{-1}B_1'X \\ &+ \frac{1}{\varepsilon}C_1'(I - D_1(\varepsilon G + D_1'D_1)^{-1}D_1')C_1 + R = 0 \quad (5)\end{aligned}$$

has a solution $X > 0$ and consider the control law

$$u(t) = -(\varepsilon G + D_1'D_1)^{-1}(\varepsilon B_1'X + D_1'C_1)x(t) \quad (6)$$

Then given any $\delta > 0$, there exists a matrix $\tilde{X}$ such that $X < \tilde{X} < X + \delta I$ and (6) is a quadratic guaranteed cost control for the uncertain system (1) with cost matrix $\tilde{X}$, and thus $J \leq x_0'\tilde{X}x_0$. Conversely, given any quadratic guaranteed cost control with cost matrix $\tilde{X}$, there exists a constant $\tilde{\varepsilon} > 0$ such that Riccati equation (5) has a stabilizing solution $X^+ > 0$ where $X^+ < \tilde{X}$.

The following propositions give monotonicity results about the Riccati equation (5) as some matrices are varied. To prove these propositions, some results (Lancaster and Rodman, 1995; Petersen, 1988; Petersen *et al.*, 2000) about some Riccati equations will be needed. Due to space limitations, the proofs have been omitted.

Proposition 3. Let $\varepsilon = \tilde{\varepsilon} > 0$ be fixed. Suppose that for $G = \widetilde{G} > 0$, the Riccati equation (5) has a symmetric solution $\widetilde{X} > 0$. Then with any $0 < G < \widetilde{G}$, the Riccati equation (5) will have a minimal positive definite solution $X > 0$, and X is non-increasing with decreasing G. Moreover, for each $0 < G < \widetilde{G}$, X is the unique strong solution to (5).

Proposition 4. Suppose that for $\varepsilon = \tilde{\varepsilon} > 0$ and $G = \widetilde{G} > 0$, the Riccati equation (5) has a positive definite solution $X(\widetilde{G}, \tilde{\varepsilon})$. Then for any fixed $\varepsilon \in (0, \tilde{\varepsilon})$ and any $0 < G < \widetilde{G}$, the Riccati equation (5) will have a minimal positive definite stabilizing solution $X(G, \varepsilon)$ and $X(G, \varepsilon)$ is non-increasing with decreasing G. Moreover, $X(G, \varepsilon)$ is the unique stabilizing solution to the Riccati equation (5) with these values of ε and G.

Next, let $G = \eta I$ where $\eta > 0$. Suppose that there exist $\eta > 0$ and $\varepsilon = \tilde{\varepsilon} > 0$ such that the Riccati equation (5) has a positive definite solution $\tilde{X}$.

Proposition 5. Suppose that D_1 has full column rank. Then with $G = \eta I$ and for any fixed $\varepsilon \in (0, \tilde{\varepsilon})$, the unique minimal positive definite stabilizing solution to the Riccati equation (5), $X(\eta, \varepsilon)$ tends to the minimal positive definite stabilizing solution of the Riccati equation:

$$\begin{aligned}&(A - B_1(D_1'D_1)^{-1}D_1'C_1)'X + X(A - B_1(D_1'D_1)^{-1} \\ &D_1'C_1) + \varepsilon XB_2B_2'X - \varepsilon XB_1(D_1'D_1)^{-1}B_1'X \\ &+ \frac{1}{\varepsilon}C_1'(I - D_1(D_1'D_1)^{-1}D_1')C_1 + R = 0 \quad (7)\end{aligned}$$

as $\eta \to 0^+$.

PROOF. For a fixed $\varepsilon \in (0, \tilde{\varepsilon})$, it follows from Proposition 4 that $\lim_{\eta \to 0^+} X(\eta, \varepsilon)$ exists and is positive semidefinite. Since D_1 has full column rank, the matrix $D_1'D_1$ is nonsingular. Also, $X(\eta, \varepsilon)$ is a continuous function of η; see Theorem 11.2.1 (Lancaster and

Rodman, 1995). Hence $\lim_{\eta\to 0^+} X(\eta,\varepsilon) = X(0,\varepsilon)$. Since the eigenvalues of a matrix are a continuous function of its elements, $X(0,\varepsilon)$ is a strong solution to the Riccati equation (7). Note that the conclusion in Proposition 4 is still valid for $G = \eta I = 0$ when $D_1'D_1$ is nonsingular. Since the strong solution to (7) is unique for $R > 0$, it follows from Proposition 4 that $X(0,\varepsilon)$ is the minimal positive definite and unique stabilizing solution to the Riccati equation (7).

From Proposition 5, we see that when D_1 has full column rank, for a given $\varepsilon \in (0,\tilde{\varepsilon})$, taking the limit $\eta \to 0^+$ does not ensure perfect regulation for the uncertain system (1). To achieve perfect regulation, we will assume that $D_1'D_1$ is singular. Hence, there exist two unitary matrices T_1 and T_2 such that we have singular value decomposition of D_1:

$$D_1 = T_1 \begin{bmatrix} \widetilde{D} & 0 \\ 0 & 0 \end{bmatrix} T_2. \quad (8)$$

Letting $T_1 = [T_{11}\ T_{12}]$, we have the following equalities

$$\begin{aligned} D_1 &= \begin{bmatrix} T_{11}\widetilde{D} & 0 \end{bmatrix} T_2 = \begin{bmatrix} \widetilde{D}_1 & 0 \end{bmatrix} T_2, \\ B_1 T_2' &= \begin{bmatrix} B_{11} & B_{12} \end{bmatrix} \end{aligned} \quad (9)$$

where $B_{11} \in \mathbb{R}^{n\times(m-m_1)}$ and $B_{12} \in \mathbb{R}^{n\times m_1}$.

The following result will characterize the limiting behavior of the minimal positive definite stabilizing solution to the Riccati equation (5) as η approaches zero under the condition that D_1 does not have full column rank.

Theorem 6. Consider Riccati equation (5) and suppose that D_1 does not have full column rank. Then there exists a unitary matrix T_2 such that D_1 and B_1 can be written as in (9). If B_{12} has full row rank, then for any given $\varepsilon \in (0,\infty)$, there exists a constant $\eta_0 > 0$ such that there is the unique positive definite stabilizing solution $X(\eta,\varepsilon)$ to Riccati equation (5) with $G = \eta I$, $\eta \in (0,\eta_0)$. Furthermore, $\lim_{\eta\to 0^+} X(\eta,\varepsilon) = 0$.

Conversely, let $X(\eta,\varepsilon)$ be the minimal positive definite solution to the Riccati equation (5) with given $\varepsilon > 0$ and $G = \eta I$ where $\eta > 0$. If $\lim_{\eta\to 0^+} X(\eta,\varepsilon) = 0$, then B_{12} has full row rank.

PROOF. First, let $D_1 \neq 0$. Hence $\widetilde{D}$ is a nonsingular square matrix and $\widetilde{D}_1$ has full column rank $m - m_1$. Substituting (9) into the Riccati equation (5), we get the following Riccati equation for the given ε:

$$\begin{gathered} (A - B_{11}(\varepsilon\eta I + \widetilde{D}'\widetilde{D})^{-1}\widetilde{D}'\hat{C}_1)'X + X(A - B_{11}(\varepsilon\eta I \\ + \widetilde{D}'\widetilde{D})^{-1}\widetilde{D}'\hat{C}_1) + \varepsilon X B_2 B_2' X - \varepsilon X B_{11}(\varepsilon\eta I + \widetilde{D}'\widetilde{D})^{-1} \\ B_{11}'X - \frac{1}{\eta} X B_{12} B_{12}' X \\ + \frac{1}{\varepsilon}\hat{C}_1'(I - \widetilde{D}(\varepsilon\eta I + \widetilde{D}'\widetilde{D})^{-1}\widetilde{D}')\hat{C}_1 + R = 0 \quad (10) \end{gathered}$$

where $\hat{C}_1 = T_{11}'C_1$. Since B_{12} has full row rank, there exists a $\eta_1 > 0$ such that $\varepsilon B_2 B_2' - \frac{1}{\eta} B_{12} B_{12}' < 0$ for any $\eta \in (0,\eta_1]$. Hence for any $\eta \in (0,\eta_1]$, Riccati equation (10) has a unique positive definite stabilizing solution $X(\eta,\varepsilon)$ and $X(\eta,\varepsilon)$ is also the maximal solution.

Now consider the Riccati equation with the given ε:

$$A'X + XA - \frac{1}{2\eta} X B_{12} B_{12}' X + \frac{1}{\varepsilon}\hat{C}_1'\hat{C}_1 + R = 0 \quad (11)$$

For any $\eta > 0$, Riccati equation (11) has a unique positive definite stabilizing solution X_η and X_η is also the maximal solution.

We use Corollary 9.16 (Lancaster and Rodman, 1995) to compare the Riccati equations (10) and (11). Hence, we consider the matrix

$$M_1 = \begin{bmatrix} \frac{1}{\varepsilon}\hat{C}_1'(I - \widetilde{D}(\varepsilon\eta I + \widetilde{D}'\widetilde{D})^{-1}\widetilde{D}')\hat{C}_1 + R & (A - B_{11}(\varepsilon\eta I + \widetilde{D}'\widetilde{D})^{-1}\widetilde{D}'\hat{C}_1)' \\ A - B_{11}(\varepsilon\eta I + \widetilde{D}'\widetilde{D})^{-1}\widetilde{D}'\hat{C}_1 & \varepsilon B_2 B_2' - \varepsilon B_{11}(\varepsilon\eta I + \widetilde{D}'\widetilde{D})^{-1}B_{11}' - \frac{1}{\eta} B_{12} B_{12}' \end{bmatrix}$$

associated with the Riccati equation (10) and the matrix

$$M_2 = \begin{bmatrix} \frac{1}{\varepsilon}\hat{C}_1'\hat{C}_1 + R & A' \\ A & -\frac{1}{2\eta} B_{12} B_{12}' \end{bmatrix}$$

associated with the Riccati equation (11). After some straightforward manipulations, we have

$$M_2 - M_1 = \begin{bmatrix} \frac{1}{\sqrt{\varepsilon}}\hat{C}_1'\widetilde{D} \\ \sqrt{\varepsilon}B_{11} \end{bmatrix} (\varepsilon\eta I + \widetilde{D}'\widetilde{D})^{-1} \begin{bmatrix} \frac{1}{\sqrt{\varepsilon}}\widetilde{D}'\hat{C}_1 & \sqrt{\varepsilon}B_{11}' \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2\eta} B_{12} B_{12}' - \varepsilon B_2 B_2' \end{bmatrix}.$$

Using Corollary 9.16 (Lancaster and Rodman, 1995), this implies for any $\eta \in (0, \frac{\eta_1}{2}]$,

$$X_\eta \geq X(\eta,\varepsilon)$$

Since $(A, \frac{1}{\sqrt{2}}B_{12}, (\frac{1}{\varepsilon}\hat{C}_1'\hat{C}_1 + R)^{\frac{1}{2}})$ is minimal phase, from Theorem 3 (Scherzinger and Davison, 1985), $\lim_{\eta\to 0^+} X_\eta = 0$. That is, given any constant $\zeta > 0$, there exists an $\eta_2 > 0$ such that if $0 < \eta < \eta_2$ then $\|X_\eta\| < \zeta$. Hence, choosing $\eta_0 = \min(\frac{\eta_1}{2}, \eta_2)$, if $0 < \eta < \eta_0$ then $\|X(\eta,\varepsilon)\| < \zeta$. This implies $\lim_{\eta\to 0^+} X(\eta,\varepsilon) = 0$. Similar arguments can be applied to prove the first part of this theorem as $D_1 = 0$. This completes the proof of the first part of this theorem.

The second part is proved as follows. First let $D_1 \neq 0$. Hence $\widetilde{D}$ is a nonsingular square matrix. Using (9), the control law in (6) is given by

$$u(t) = [u_1(t)' \; u_2(t)']' = -T_2' K x(t)$$
$$= -T_2' \begin{bmatrix} (\varepsilon\eta I + \widetilde{D}'\widetilde{D})^{-1}(\varepsilon B_{11}' X(\eta,\varepsilon) + \widetilde{D}'\hat{C}_1) \\ \frac{1}{\eta} B_{12}' X(\eta,\varepsilon) \end{bmatrix} x(t).$$

Directly following the proof of Theorem 2 (Scherzinger and Davison, 1985), when $\lim_{\eta\to 0^+} X(\eta,\varepsilon) = 0$, the limiting control $u_1(t) = -\widetilde{D}^{-1}\hat{C}_1 x(t)$ yields an equivalent system in the following form:

$$\dot{x}(t) = (A - B_{11}\widetilde{D}^{-1}\hat{C}_1)x(t) + B_{12}u_2(t) \qquad (12)$$

associated with the cost functional

$$J = \int_0^\infty \{x(t)'Rx(t) + \eta x(t)\hat{C}_1'(\widetilde{D}\widetilde{D}')^{-1}\hat{C}_1 x(t) + \eta u_2'(t)u_2(t)\}\mathrm{dt} \qquad (13)$$

Using Observation 5.2.2 (Petersen *et al.*, 2000) and the fact that $X(\eta,\varepsilon)$ is a positive definite solution to the Riccati equation (5), we have $J \leq x_0' X(\eta,\varepsilon)x_0$. Hence $\lim_{\eta\to 0^+} x_0' X(\eta,\varepsilon)x_0 = 0$ implies $\lim_{\eta\to 0^+} J = 0$. Now write the cost functional $J = J_1 + J_2$, where

$$J_1 = \int_0^\infty \eta x(t)\hat{C}_1'(\widetilde{D}\widetilde{D}')^{-1}\hat{C}_1 x(t)\mathrm{dt} \qquad (14)$$

and

$$J_2 = \int_0^\infty \{x(t)'Rx(t) + \eta u_2'(t)u_2(t)\}\mathrm{dt} \qquad (15)$$

From Theorem 3 (Scherzinger and Davison, 1985), the cost functional J_2 approaches zero as $\eta \to 0^+$ only if B_{12} has full row rank.

Secondly, let $D_1 = 0$, while $B_1 = B_{12}$. Riccati equation (5) can be rewritten as

$$A'X + XA + \varepsilon X B_2 B_2' X - \frac{1}{\eta} X B_1 B_1' X + \frac{1}{\varepsilon} C_1' C_1 + R = 0. \qquad (16)$$

Since Riccati equation (16) has a minimal positive definite solution $X(\eta,\varepsilon)$, the following Riccati equation has a positive definite solution $Y(\eta,\varepsilon) = X(\eta,\varepsilon)^{-1}$:

$$YA' + AY + \varepsilon B_2 B_2' - \frac{1}{\eta} B_1 B_1' + Y(\frac{1}{\varepsilon} C_1' C_1 + R)Y = 0. \qquad (17)$$

Consider the Riccati equation

$$\bar{Y}A' + A\bar{Y} - \frac{1}{\eta} B_1 B_1' + \bar{Y}(\frac{1}{\varepsilon} C_1' C_1 + R)\bar{Y} = 0. \qquad (18)$$

We consider the matrix

$$M_1 = \begin{bmatrix} -[\varepsilon B_2 B_2' - \frac{1}{\eta} B_1 B_1'] & -A \\ -A' & -[\frac{1}{\varepsilon} C_1' C_1 + R] \end{bmatrix}$$

associated with the Riccati equation (17) and the matrix

$$M_2 = \begin{bmatrix} \frac{1}{\eta} B_1 B_1' & -A \\ -A' & -[\frac{1}{\varepsilon} C_1' C_1 + R] \end{bmatrix}$$

associated with the Riccati equation (18). We have

$$M_2 - M_1 = \begin{bmatrix} \varepsilon B_2 B_2' & 0 \\ 0 & 0 \end{bmatrix} \geq 0.$$

From Corollary 9.16 (Lancaster and Rodman, 1995), Riccati equation (18) has a maximal solution $\bar{Y}(\varepsilon) \geq Y(\eta,\varepsilon) > 0$. Then, the following Riccati equation has a minimal positive definite solution $X_\eta(\varepsilon) \leq X(\eta,\varepsilon)$:

$$A'X + XA - \frac{1}{\eta} X B_1 B_1' X + \frac{1}{\varepsilon} C_1' C_1 + R = 0 \qquad (19)$$

Let $\varepsilon_1 < \varepsilon$. From Theorem 5.2.4 and its proof (Petersen *et al.*, 2000), Riccati equation (19) has a minimal positive definite stabilizing solution $X_\eta^+(\varepsilon_1) > 0$ which satisfies the inequality $X_\eta^+(\varepsilon_1) \leq \frac{\varepsilon}{\varepsilon_1} X_\eta(\varepsilon)$. Hence, $X_\eta^+(\varepsilon_1) \leq \frac{\varepsilon}{\varepsilon_1} X(\eta,\varepsilon)$ and $\lim_{\eta\to 0^+} X_\eta^+(\varepsilon_1) = 0$. It follows Theorem 3 (Scherzinger and Davison, 1985), B_1 has full row rank. This completes the proof of this theorem.

The following corollary concerning perfect regulation for the uncertain system (1) can be obtained directly from Lemma 2 and Theorem 6.

Corollary 7. Consider the uncertain system (1) and (2) with the cost functional (3) and suppose that D_1 does not have full column rank. Hence there exists a unitary matrix T_2 such that D_1 and B_1 are transformed as in (9). If B_{12} has full row rank, then there exist two constants $\varepsilon, \eta > 0$ such that there is a unique minimal positive definite stabilizing solution $X(\eta,\varepsilon)$ to the Riccati equation (5) with $G = \eta I$. Also with the corresponding control law $u(t)$ in (6), $J \to 0$ as $\eta \to 0^+$.

3. INTEGRAL QUADRATIC CONSTRAINT UNCERTAINTY

Consider the following uncertain system

$$\dot{x}(t) = Ax(t) + B_1 u(t) + B_2 \xi(t) \qquad (20)$$
$$z(t) = C_1 x(t) + D_1 u(t) \qquad (21)$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, and $z(t) \in \mathbb{R}^l$ is the uncertainty output, $\xi(t) \in$

$\mathbb{R}^r$ is the uncertain input. The cost functional for this system is defined as in (3). That is,

$$J = \int_0^\infty \{x(t)'Rx(t) + u'(t)Gu(t)\}\mathrm{d}t \qquad (22)$$

where $R, G > 0$.

The uncertainty in the above system is described by an equation of the form:

$$\xi(t) = \phi(t, x(\cdot), u(\cdot)) \qquad (23)$$

satisfying the following integral quadratic constraint

$$\int_0^\infty (\|z(t)\|^2 - \|\xi(t)\|^2)\mathrm{d}t \geq -x_0' D_s x_0 \qquad (24)$$

where $D_s > 0$ and $\|\cdot\|$ denotes the standard Euclidean norm.

We will consider a problem of optimizing of the worst case of the cost functional (22) via a linear state feedback controller of the form

$$\begin{aligned} \dot{x}_c &= A_c x_c(t) + B_c x_c(t); \quad x_c(0) = 0; \\ u(t) &= C_c x_c(t) + D_c x(t) \end{aligned} \qquad (25)$$

where A_c, B_c, C_c and D_c are given matrices. The dimension of the state vector x_c may be arbitrary. When a controller form (25) is applied to the uncertain system (20) and (21), the closed loop uncertain system is described by the state equation:

$$\begin{aligned} \dot{h} &= \hat{A}h(t) + \hat{B}_2\xi(t); z(t) = \hat{C}h(t); \\ u(t) &= \hat{K}h(t) \end{aligned} \qquad (26)$$

where

$$\begin{aligned} \hat{A} &= \begin{bmatrix} A + B_1 D_c & B_1 C_c \\ B_c & A_c \end{bmatrix}, \hat{B}_2 = \begin{bmatrix} B_2 \\ 0 \end{bmatrix}, h(t) = \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix} \\ \hat{C} &= \begin{bmatrix} C_1 + D_1 D_c & D_1 C_c \end{bmatrix}, \hat{K} = \begin{bmatrix} D_c & C_c \end{bmatrix}. \end{aligned} \qquad (27)$$

The uncertainty for this closed loop uncertain system will be described by an equation of the form $\xi(t) = \phi(t, h(\cdot))$, where the integral quadratic constraint given above is satisfied with the substitution $u(t) = \hat{K}h(t)$.

Definition 8. (Petersen *et al.*, 2000) The controller (25) is said to be a guaranteed cost controller for the uncertain system (20) and (21) with cost functional (22) and an initial condition $x(0) = x_0$ if the following conditions hold:

- (i) The matrix $\hat{A}$ defined in (27) is stable.
- (ii) There exists a constant $c_0 > 0$ such that the following conditions hold: For all admissible uncertainties, the solution to the closed loop system (26) and (24) corresponding to the initial condition $h(0) = [x_0', 0]'$ satisfies
$$[x(\cdot), u(\cdot), \xi(\cdot)] \in \mathbf{L}_2[0, \infty).$$
Also, the corresponding value of the cost functional (22) satisfies the bound $J \leq c_0$.

An uncertain system (20) and (21) with the cost functional (22) which admits a guaranteed controller (25) with an initial condition $x(0) = x_0$ is said to be guaranteed cost stabilizable with this initial condition.

Let Ξ be the set of all admissible uncertainties for the uncertain system (20), (21) and (24). Let Θ denote the set of all linear time-invariant dynamic state feedback guaranteed cost controllers of the form (25) for the uncertain system with the given initial condition. Consider the Riccati equation (5) and define a set Γ as follows:

$$\begin{aligned} \Gamma := \{\varepsilon > 0 \; \text{ such that Riccati equation (5) has a} \\ \text{positive definite solution} X_\varepsilon\}. \end{aligned} \qquad (28)$$

Here Γ is an open interval in $\mathbb{R}$. That is, $\Gamma = (0, \varepsilon)$. For any $\varepsilon \in \Gamma$, the Riccati equation (5) has a positive definite solution X_ε. It should be noted that although we only consider the case of unstructured uncertainty with a single integral quadratic constraint, the main result of this section can be extended to the multi-block uncertainty case (Milliken *et al.*, 1999).

Lemma 9. (Petersen *et al.*, 2000) Consider the uncertain system (20), (21) with the cost functional (22) and Integral Quadratic Constraint (24). Suppose that $B_2 \neq 0$ and the set Γ is not empty. Then, for any initial condition $x_0 \neq 0$, the uncertain systems (20) and (21) will be guaranteed cost stabilizable and

$$J^* \stackrel{\triangle}{=} \inf_{u(\cdot) \in \Theta} \sup_{\xi(\cdot) \in \Xi} J = \inf_{\varepsilon \in \Gamma} \left[x_0' X_\varepsilon x_0 + \frac{1}{\varepsilon} x_0' D_s x_0\right]$$

where X_ε is the minimal positive definite solution to Riccati equation (5).

Next, let $G = \eta I$ where $\eta > 0$. Using the results of Section 2, we will give the main result of this section concerning perfect regulation with cheap control for the uncertain system (20), (21).

Proposition 10. J^* is monotonically non-increasing and converges to some J_0 as $\eta \to 0$.

PROOF. Since $x_0' X_\varepsilon x_0 + \frac{1}{\varepsilon} x_0' D_s x_0 > 0$ has zero as a lower bound, J^* is finite. Let η_1 and η_2 satisfy $0 < \eta_2 < \eta_1$. For any $\varepsilon \in \Gamma(\eta_1)$, from Proposition 3, Riccati equation (5) with ε and η_2 has a positive definite solution. Hence $\varepsilon \in \Gamma(\eta_2)$. Furthermore $\Gamma(\eta_1) \subseteq \Gamma(\eta_2)$ and $0 \leq J^*(\eta_2) \leq J^*(\eta_1)$, which implies that $J^*(\eta)$ is monotonically non-increasing and converges to some J_0 as $\eta \to 0$.

Theorem 11. Consider the uncertain system (20), (21) with the cost functional (22) and Integral Quadratic

Constraint (24). Suppose that $B_2 \neq 0$, the initial state x_0 is known and $x_0 \neq 0$, and D_1 does not have full column rank. Hence there exists a unitary matrix T_2 such that D_1 and B_1 are transformed as in (9). Then, if B_{12} has full row rank,

$$\lim_{\eta \to 0^+} \inf_{u(\cdot)\in\Theta} \sup_{\xi(\cdot)\in\Xi} J = 0.$$

PROOF. Given any constant $\zeta > 0$, then there exists an $\varepsilon > 0$ such that $\frac{1}{\varepsilon} x_0' D_s x_0 < \frac{\zeta}{2}$. From Theorem 6, for this ε, there exists an $\eta_1 > 0$ such that the Riccati equation (5) with this ε and $G = \eta I$ where $0 < \eta < \eta_1$, has a minimal positive definite stabilizing solution $X(\eta, \varepsilon)$. This implies that this ε is an interior point of the set Γ for the Riccati equation (5) with this ε and $G = \eta I$ where $0 < \eta < \eta_1$. Furthermore, since $\lim_{\eta \to 0^+} x_0' X(\eta, \varepsilon) x_0 = 0$, we can choose a suitable η_2 where $\eta_1 > \eta_2 > 0$ such that $x_0' X(\eta, \varepsilon) x_0 < \frac{\zeta}{2}$ whenever $0 < \eta \leq \eta_2$. Letting $\eta = \eta_2$, we have $x_0' X_\varepsilon x_0 + \frac{1}{\varepsilon} x_0' D_s x_0 < \zeta$. That is, $J^*(\eta_2) < x_0' X_\varepsilon x_0 + \frac{1}{\varepsilon} x_0' D_s x_0 < \zeta$, where $X_\varepsilon = X(\eta_2, \varepsilon)$. From Proposition 10, since $J^*(\eta)$ is monotonically non-increasing with decreasing η, we have for $0 < \eta < \eta_2$, $J^*(\eta) < \zeta$. Note that since $\zeta > 0$ is arbitrary, it follows that for any given $\zeta > 0$, there exists an $\eta_0 > 0$ such that $J^*(\eta) < \zeta$ whenever $0 < \eta < \eta_0$. This implies $\lim_{\eta \to 0^+} J^*(\eta) = 0$.

Example 12. Consider the uncertain system described by the state equation

$$\dot{x}(t) = \begin{bmatrix} 1+\Delta(t) & -2+\Delta(t) \\ 16 & 8 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} u(t)$$

where x(0)=[1 0]' and $\Delta(t)$ is a scalar subject to the bound $\Delta'(t)\Delta(t) \leq I$. Associated with this system is the cost functional

$$J = \int_0^\infty \{x_1^2(t) + x_2^2(t) + \eta(u_1^2(t) + u_2^2(t))\} \mathrm{d}t.$$

The corresponding matrices for the above uncertain system are given below

$$A = \begin{bmatrix} 1 & -2 \\ 16 & 8 \end{bmatrix}, B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C_1 = \begin{bmatrix} 1 & 1 \end{bmatrix},$$
$$D_1 = \begin{bmatrix} 0 & 0 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, G = \begin{bmatrix} \eta & 0 \\ 0 & \eta \end{bmatrix}.$$

Let $\varepsilon = 0.33$. A plot of cost functional and cost bound $x_0' X(\eta, 0.33) x_0$ versus η and time-invariant $\Delta \in [-1,\ 1]$ is shown in Figure 1. Since B_1 has full row rank, the maximal achievable performance for this uncertain system is zero. The cost bound $x_0' X(\eta, 0.33) x_0$ is 0.1684×10^{-4} as $\eta = 10^{-10}$.

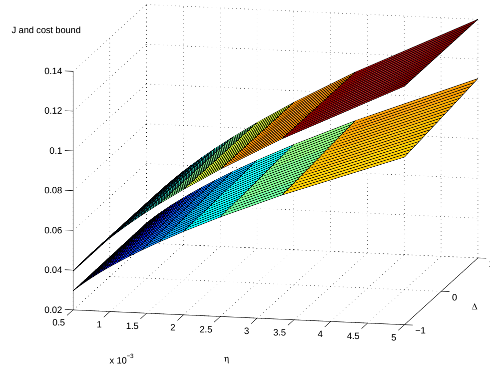

Fig. 1. Cost functional and cost bound versus η and time-invariant Δ.

4. CONCLUSION

We have derived the cheap control performance limit for the uncertain systems with both norm-bounded uncertainty and integral quadratic constraint uncertainty. Also the condition under which perfect regulation can be achieved has been given. This has been done by investigating the monotonicity properties and limiting behavior of the minimal positive definite solutions to the corresponding Riccati equations. It is noted that the uncertain systems under consideration require a special structure to achieve perfect regulation. The reason for this is that the quadratic guaranteed cost control approach adopted in this paper requires that the weighting matrix on the state in the cost functional must be a positive definite; i.e., $R > 0$. Hence, conditions for perfect regulation are not given in terms of minimal phase and right-invertible assumptions but rather in terms of certain rank assumptions.

5. REFERENCES

Francis, B.A. (1979). The optimal linear-quadratic time-invariant regulator with cheap control. *IEEE, Trans. Automat. Control* **24**, 616–621.

Lancaster, P. and L. Rodman (1995). *Algebraic Riccati Equations*. Oxford University Press, Oxford.

Milliken, P., C. Marsh and B. Brunt (1999). Minimax controller design for a class of uncertain linear systems. *Automatica* **35**, 583–590.

Petersen, I.R. (1988). Some new results on algebraic Riccati equations arising in linear quadratic equations differential games and the stabilization of uncertain linear systems. *Systems & Control Letters* **10**, 341–348.

Petersen, I.R., V.A. Ugrinovskii and A.V. Savkin (2000). *Robust Control Design Using H^∞ Methods*. Springer-Verlag, London.

Scherzinger, B.M. and E.J. Davison (1985). The optimal LQ regulator with cheap control for not strictly proper systems. *Optimal Control Applications & Methods* **6**, 291–303.