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DISCRETE-TIME ROBUST H_∞ OUTPUT FEEDBACK CONTROL OF STATE DELAYED SYSTEMS

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Abstract: The problem of H_∞ output feedback for uncertain linear discrete-time systems with state-delay and parameter uncertainties is considered. The objective is to design a linear output feedback controller such that, for the unknown state time-delay and all admissible norm-bounded parameter uncertainties, the feedback system remains robustly stable and the transfer function from the exogenous disturbances to the state-error outputs meets the prescribed H_∞ norm upper-bound constraint. The delay-independent output feedback does not depend on the uncertainties. The conditions for the existence of the robust H_∞ output feedback controller and its analytical expression is then characterized in terms of Riccati-type equations. *Copyright © 2002 IFAC*

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1. INTRODUCTION

In recent years, much work has been put into the analysis and synthesis of controllers for state-delayed systems with norm-bounded parametric uncertainties (Choi and Chung, 1997; Jeung, *et al.*, 1996; Li and De Souza, 1997; Song and Kim, 1998). This interest arises from the fact that delays and uncertainties are the two most important causes of instability. Furthermore, delays and uncertainties are typical in the process industry, which motivates the study of new stability conditions and the synthesis of high performance controllers. Most work has been directed towards the study of state feedback controller design (Fridman and Shaked, 1998; Ge, *et al.*, 1996; Yaesh, *et al.*, 1999), and state observer design (De Souza, *et al.*, 1999; Trinh and Aldeen, 1997; Wang, *et al.*, 1999), as separate issues. Very little effort has so far been put into the design and analysis of systems using output feedback with time-delays. In Yao, *et al.* (1997), both the observer and controller designs are treated as separate issues. The output feedback problem is however of the greatest practical relevance as usually the states of the system are not directly available to measurement. Also, the output feedback control case needs special attention when uncertainties are present, as for the general form of uncertainties, the separation principle does not hold and the observer design is no longer the dual of the control design. Most of the previous work involving output feedback is concerned with the continuous-time systems. Very little attention has been giving to the discrete-time case. In Wang, *et al.* (1999), a discrete-time observer for state-delayed systems with parametric uncertainties has been developed. In this context, the contribution of this paper is the development of a robust H_∞ output feedback controller which complements the observer design of Wang, *et al.* (1999), in a way to maintain robust stability of the combined system. Furthermore, uncertainty in the delayed state matrix is taken into account as an improvement over the observer design in Wang, *et al.* (1999).

More specifically, the output feedback problem addressed in this paper aims at designing observer and controller gains such that, for all admissible parameter uncertainties, the output feedback system remains robustly stable and the transfer function from the exogenous disturbances to the state error output meets a prescribed H_∞ -norm upper bound constraint, independently of the unknown time delay. The parameter uncertainties are allowed to be norm-bounded and appear in the state and the output matrices, and may be time-varying. A simple algebraic parameterized approach is exploited, which enables to derive the existence conditions for the observer and controller gains, and to characterize the set of robust H_∞ output feedback controllers in terms of several free design parameters. These free parameters, that appear in the observer and controller gains, offer additional design freedom and can be utilized to account for supplementary performance constraints.

The design formulation of the H_∞ output feedback control problem requires the solution of two Riccati matrix equalities. As shown elsewhere, see (Haurani, *et al.*, 2001), the numerical solutions to these Riccati-type equations are easiest obtained by solving two auxiliary Riccati-type inequalities.

2. PROBLEM STATEMENT

Consider the following linear uncertain discrete-time state delayed system:

$$\mathbf{x}(k+1)=(\mathbf{A}+\Delta\mathbf{A})\mathbf{x}(k)+(\mathbf{A_d}+\Delta\mathbf{A_d})\mathbf{x}(k-d)+\mathbf{Bu}(k)+\mathbf{D_1w}(k) \qquad (1)$$

with the measurement equation

$$\mathbf{y}(k)=(\mathbf{C}+\Delta\mathbf{C})\mathbf{x}(k)+\mathbf{D_2w}(k) \qquad (2)$$

where $\mathbf{x}(k)\in R^n$ is the state, $\mathbf{u}(k)\in R^q$ is the input to the plant, $\mathbf{w}(k)\in R^r$ is the square-integrable exogenous disturbance, $\mathbf{y}(k)\in R^p$ is the system output. The matrices $\mathbf{A}$, $\mathbf{A_d}$, $\mathbf{B}$, $\mathbf{D_1}$, $\mathbf{C}$ and $\mathbf{D_2}$ are assumed to be known and constant and of appropriate dimensions. The positive integer variable d denotes the unknown state delay. Here $\Delta\mathbf{A}$, $\Delta\mathbf{A_b}$ and $\Delta\mathbf{C}$ are real-valued matrix functions representing the norm-bounded parameter uncertainties and are assumed to be of the following form:

$$\begin{bmatrix}\Delta\mathbf{A}\\ \Delta\mathbf{C}\end{bmatrix}=\begin{bmatrix}\mathbf{M_1}\\ \mathbf{M_2}\end{bmatrix}\mathbf{FN_1}, \qquad \Delta\mathbf{A_d}=\mathbf{M_1FN_2} \qquad (3)$$

where $\mathbf{F}\in R^{i\times j}$, which may be time-varying, is a real uncertain matrix with Lebesgue measurable elements and which meets the requirement that $\mathbf{FF}^T\le\mathbf{I}$. The matrices $\mathbf{M_1},\mathbf{M_2}$, $\mathbf{N_1}$ and $\mathbf{N_2}$ are known, real and constant and characterize the way in which the uncertain parameters of $\mathbf{F}$ enter the nominal matrices $\mathbf{A}$, $\mathbf{A_d}$ and $\mathbf{C}$.

The following assumption is needed for the subsequent development:

Assumption 1. The matrix $\mathbf{D_2}$ or $\mathbf{M_2}$ is of full rank.

In the discrete-time case, the full order linear state observer, as proposed in Wang, *et al.* (1999), is of the form:

$$\hat{\mathbf{x}}(k+1)=\mathbf{G}\hat{\mathbf{x}}(k)+\mathbf{A_d}\hat{\mathbf{x}}(k-d)+\mathbf{K_oy}(k)+\mathbf{Bu}(k) \qquad (4)$$

The controller to be designed will be assumed linear, delay-free and of the form:

$$\mathbf{u}(k)=\mathbf{K_c}\hat{\mathbf{x}}(k) \qquad (5)$$

where $\mathbf{G}$ and $\mathbf{K_o}$ are the observer gains and $\mathbf{K_c}$ is the controller gain to be determined.

Defining the state error $\mathbf{e}(k)=\mathbf{x}(k)-\hat{\mathbf{x}}(k)$, it then follows from *(1)*, *(2)*, *(4)* and *(5)* that

$$\mathbf{e}(k+1)=\mathbf{Ge}(k)+(\mathbf{A}+\Delta\mathbf{A}-\mathbf{K_o}(\mathbf{C}+\Delta\mathbf{C})-\mathbf{G})\mathbf{x}(k)+\mathbf{A_de}(k-d)+\Delta\mathbf{A_dx}(k-d)+(\mathbf{D_1}-\mathbf{K_oD_2})\mathbf{w}(k) \qquad (6)$$

Let $\mathbf{z}(k)$ then be the state-error output, which is assumed to be given by:

$$\mathbf{z}(k)=\mathbf{Le}(k) \qquad (7)$$

where $\mathbf{L}\in R^{m\times n}$ is a given constant matrix. Defining

$$\mathbf{x_f}(k)=\begin{bmatrix}\mathbf{x}(k)\\ \mathbf{e}(k)\end{bmatrix}, \qquad (8)$$

$$\mathbf{A_f}=\begin{bmatrix}\mathbf{A}+\mathbf{BK_c} & -\mathbf{BK_c}\\ \mathbf{A}-\mathbf{K_oC}-\mathbf{G} & \mathbf{G}\end{bmatrix}, \mathbf{A_{df}}=\begin{bmatrix}\mathbf{A_d} & \mathbf{0}\\ \mathbf{0} & \mathbf{A_d}\end{bmatrix} \qquad (9)$$

$$\mathbf{M_f}=\begin{bmatrix}\mathbf{M_1}\\ \mathbf{M_1}-\mathbf{K_0M_2}\end{bmatrix}, \qquad \mathbf{N_f}=[\mathbf{N_1} \quad \mathbf{0}] \qquad (10)$$

$$\mathbf{M_{df}}=\begin{bmatrix}\mathbf{M_1}\\ \mathbf{M_1}\end{bmatrix}, \qquad \mathbf{N_{df}}=[\mathbf{N_2} \quad \mathbf{0}] \qquad (11)$$

$$\mathbf{\Delta A_f} = \mathbf{M_f F N_f}\,, \qquad \mathbf{\Delta A_{df}} = \mathbf{M_{df} F N_{df}} \tag{12}$$

$$\mathbf{D_f} = \begin{bmatrix} \mathbf{D_1} \\ \mathbf{D_1} - \mathbf{K_o D_2} \end{bmatrix}, \qquad \mathbf{C_f} = \begin{bmatrix} \mathbf{0} & \mathbf{L} \end{bmatrix} \tag{13}$$

and combining *(1)*, *(2)*, *(3)*, *(5)* and *(6)*, the following augmented system is easily obtained:

$$\mathbf{x_f}(k+1) = (\mathbf{A_f} + \mathbf{\Delta A_f})\mathbf{x_f}(k) + (\mathbf{A_{df}} + \mathbf{\Delta A_{df}})\mathbf{x_f}(k-d) + \mathbf{D_f w}(k) \tag{14}$$

$$\mathbf{z}(k) = \mathbf{C_f x_f}(k) \tag{15}$$

The transfer function from the disturbance $\mathbf{w}(k)$ to the state-error output $\mathbf{z}(k)$ is thus given by:

$$\mathbf{H_{zw}}(z) = \mathbf{C_f}\left[z\mathbf{I} - (\mathbf{A_f} + \mathbf{\Delta A_f}) - (\mathbf{A_{df}} + \mathbf{\Delta A_{df}})z^{-d}\right]^{-1} \times \mathbf{D_f} \tag{16}$$

The objective of this paper is to design the parameters $\mathbf{G}$, $\mathbf{K_o}$ and $\mathbf{K_c}$, such that for all admissible parameters uncertainties $\mathbf{\Delta A}$, $\mathbf{\Delta A_d}$ and $\mathbf{\Delta C}$, the augmented system *(14)* and *(15)* is asymptotically stable and the following upper-bound constraint on the H_∞ norm of $\mathbf{H_{zw}}(z)$ is simultaneously guaranteed:

$$\left\|\mathbf{H_{zw}}(z)\right\|_\infty \le \gamma\,, \tag{17}$$

for all positive integer time-delay values $d \in R^+$, and all uncertainties *(3)*, where $\left\|\mathbf{H_{zw}}(z)\right\|_\infty = Sup_{\theta\in[0,2\pi]}\sigma_{\max}\left[\mathbf{H_{zw}}(e^{j\theta})\right]$ and $\sigma_{\max}[\mathbf{H}]$ denotes the largest singular value of $[\mathbf{H}]$, and $\gamma < 1$ is a given positive constant.

3. MAIN RESULTS

The following lemma will play a key role in designing the robust H_∞ output feedback controller for the uncertain linear discrete-time state delayed system *(1)* and *(2)*.

Lemma 1. *If there exist a positive-definite matrix* $\mathbf{P}$ *and positive scalars* $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ *and* $\varepsilon_3 > 0$ *such that the following inequalities,*

$$\mathbf{P}^{-1} - \gamma^{-2}\mathbf{C_f}^T\mathbf{C_f} - \varepsilon_1\mathbf{A_{df}}^T\mathbf{A_{df}} - \varepsilon_2\mathbf{N_f}^T\mathbf{N_f} - \varepsilon_3\mathbf{N_{df}}^T\mathbf{N_{df}} > 0 \tag{18}$$

$$\mathbf{A_f}\left(\mathbf{P}^{-1} - \gamma^{-2}\mathbf{C_f}^T\mathbf{C_f} - \varepsilon_1\mathbf{A_{df}}^T\mathbf{A_{df}} - \varepsilon_2\mathbf{N_f}^T\mathbf{N_f} - \varepsilon_3\mathbf{N_{df}}^T\mathbf{N_{df}}\right)^{-1} \times\mathbf{A_f}^T - \mathbf{P} + \mathbf{D_f D_f}^T + \varepsilon_1^{-1}\mathbf{I} + \varepsilon_2^{-1}\mathbf{M_f M_f}^T + \varepsilon_3^{-1}\mathbf{M_{df} M_{df}}^T < 0 \tag{19}$$

hold, then the augmented system (14) and (15) is asymptotically stable and meets the specified H_∞ *norm upper-bound constraint* $\left\|\mathbf{H_{zw}}(z)\right\|_\infty \le \gamma$, *independent of the positive integer state time-delay* d.

Proof. It is easy to see that the system *(14)*, *(15)* is asymptotically stable if and only if the following auxiliary system is asymptotically stable:

$$\mathbf{y_f}(k+1) = (\mathbf{A_f} + \mathbf{\Delta A_f})^T\mathbf{y_f}(k) + \mathbf{A_{df}}^T\mathbf{y_f}(k-d) + \mathbf{C_f}^T\mathbf{w_1}(k) \tag{20}$$

$$\mathbf{z_1}(k) = \mathbf{D_f}^T\mathbf{y_f}(k) \tag{21}$$

where the state $\mathbf{y_f}(k) \in R^{2n}$, the disturbance input $\mathbf{w_1}(k) \in R^m$, the system output $\mathbf{z_1}(k) \in R^r$, and the transfer functions of the systems *(14)*,*(15)* and *(20)* ,*(21)* have the same H_∞-norm values. Then, based on the auxiliary system *(20)*,*(21)*, the proof of this lemma is completely similar to that of Theorem 2 in Song and Kim (1998) and is thus omitted. QED

For the sake of simplicity, the following definitions are introduced prior to stating the main results of the paper :

$$\mathbf{\Phi_1} \triangleq \left(\mathbf{P_1}^{-1} - \varepsilon_1\mathbf{A_d}^T\mathbf{A_d} - \varepsilon_2\mathbf{N_1}^T\mathbf{N_1} - \varepsilon_3\mathbf{N_2}^T\mathbf{N_2}\right)^{-1} \tag{22}$$

$$\mathbf{\Phi_2} \triangleq \left(\mathbf{P_2}^{-1} - \gamma^{-2}\mathbf{L}^T\mathbf{L} - \varepsilon_1\mathbf{A_d}^T\mathbf{A_d}\right)^{-1} \tag{23}$$

$$\mathbf{\Gamma_1} \triangleq -\mathbf{A\Phi_2}(\mathbf{\Phi_1} + \mathbf{\Phi_2})^{-1} + \left(\mathbf{A}\left(\mathbf{\Phi_1}^{-1} + \mathbf{\Phi_2}^{-1}\right)^{-1}\mathbf{A}^T + \mathbf{D_1 D_1}^T + \left(\varepsilon_2^{-1} + \varepsilon_3^{-1}\right)\mathbf{M_1 M_1}^T\right) \times(\mathbf{E_c U_c})^{-T}(\mathbf{\Phi_1} + \mathbf{\Phi_2})^{-1/2} \tag{24}$$

where $\mathbf{E_c} \in R^{n\times n}$ is an invertible matrix and $\mathbf{U_c} \in R^{n\times n}$ is an arbitrary chosen orthogonal matrix $\left(\mathbf{U_c U_c}^T = \mathbf{I}\right)$.

$$\mathbf{\bar{A}} \triangleq \mathbf{A} + \mathbf{\Gamma_1} \tag{25}$$

$$\mathbf{\Gamma_2} \triangleq -\mathbf{C\Phi_2}(\mathbf{\Phi_1} + \mathbf{\Phi_2})^{-1} + \left(\mathbf{C}\left(\mathbf{\Phi_1}^{-1} + \mathbf{\Phi_2}^{-1}\right)^{-1}\mathbf{A}^T + \mathbf{D_2 D_1}^T + \varepsilon_2^{-1}\mathbf{M_2 M_1}^T\right) \times(\mathbf{E_c U_c})^{-T}(\mathbf{\Phi_1} + \mathbf{\Phi_2})^{-1/2} \tag{26}$$

$$\mathbf{\bar{C}} \triangleq \mathbf{C} + \mathbf{\Gamma_2} \tag{27}$$

$$\mathbf{\Theta_1} \triangleq \varepsilon_1\mathbf{A_d}^T\mathbf{A_d} + \varepsilon_2\mathbf{N_1}^T\mathbf{N_1} + \varepsilon_3\mathbf{N_2}^T\mathbf{N_2}\,, \qquad \mathbf{\Theta_2} \triangleq \gamma^{-2}\mathbf{L}^T\mathbf{L} + \varepsilon_1\mathbf{A_d}^T\mathbf{A_d} \tag{28}$$

$$\mathbf{R_c} \triangleq \mathbf{\Phi_1} + \mathbf{\Phi_2}\,, \qquad \mathbf{S_c} \triangleq -\mathbf{\Phi_1 A}^T \tag{29}$$

$$\mathbf{R_o} \triangleq \mathbf{\Gamma_2 \Phi_1 \Gamma_2}^T + \mathbf{\bar{C}\Phi_2\bar{C}}^T + \mathbf{D_2 D_2}^T + \varepsilon_2^{-1}\mathbf{M_2 M_2}^T \tag{30}$$

$$\mathbf{S_o} \triangleq \mathbf{\Gamma_2 \Phi_1 \Gamma_1}^T + \mathbf{\bar{C}\Phi_2\bar{A}}^T + \mathbf{D_2 D_1}^T + \varepsilon_2^{-1}\mathbf{M_2 M_1}^T \tag{31}$$

$$\mathbf{\Omega_1} \triangleq \mathbf{AP_1A}^T + \mathbf{AP_1\Theta_1}^{1/2}\left(\mathbf{I} - \mathbf{\Theta_1}^{1/2}\mathbf{P_1\Theta_1}^{1/2}\right)^{-1}\mathbf{\Theta_1}^{1/2}\mathbf{P_1A}^T - \mathbf{P_1} + \mathbf{D_1 D_1}^T + \left(\varepsilon_2^{-1} + \varepsilon_3^{-1}\right)\mathbf{M_1 M_1}^T - \mathbf{S_c}^T\mathbf{R_c}^{-1}\mathbf{S_c} + \varepsilon_1^{-1}\mathbf{I} \tag{32}$$

$$\mathbf{\Omega_2} \triangleq \mathbf{\bar{A}P_2\bar{A}}^T + \mathbf{\bar{A}P_2\Theta_2}^{1/2}\left(\mathbf{I} - \mathbf{\Theta_1}^{1/2}\mathbf{P_2\Theta_2}^{1/2}\right)^{-1}\mathbf{\Theta_2}^{1/2}\mathbf{P_2\bar{A}}^T$$

$$-\mathbf{P_2}+\mathbf{\Gamma_1\Phi_1\Gamma_1}^T+\mathbf{D_1D_1}^T+\left(\varepsilon_2^{-1}+\varepsilon_3^{-1}\right)\mathbf{M_1M_1}^T$$
$$-\mathbf{S_o}^T\mathbf{R_o}^{-1}\mathbf{S_o}+\varepsilon_1^{-1}\mathbf{I} \qquad (33)$$

The following theorem provides the theoretical basis for achieving the desired design goal.

Theorem 1. *Let $\delta_1>0$ and $\delta_2>0$ be sufficiently small numbers, and let the matrices $\mathbf{\Phi_1}$, $\bar{\mathbf{A}}$, $\mathbf{\Theta_1}$, $\mathbf{\Theta_2}$, $\mathbf{\Gamma_1}$, $\mathbf{S_c}$, $\mathbf{R_c}$, $\mathbf{S_o}$ and $\mathbf{R_o}$ be defined as in (22)-(31). Suppose there exist positive scalars ε_1, ε_2 and ε_3, an invertible matrix $\mathbf{E_c}\in R^{n\times n}$, and a matrix $\mathbf{E_o}\in R^{n\times p}$ such that the following Riccati-type matrix equations*

$$\mathbf{AP_1A}^T-\mathbf{P_1}+\mathbf{AP_1\Theta_1}^{1/2}\left(\mathbf{I}-\mathbf{\Theta_1}^{1/2}\mathbf{P_1\Theta_1}^{1/2}\right)^{-1}\mathbf{\Theta_1}^{1/2}\mathbf{P_1A}^T$$
$$+\mathbf{D_1D_1}^T+\left(\varepsilon_2^{-1}+\varepsilon_3^{-1}\right)\mathbf{M_1M_1}^T-\mathbf{S_c}^T\mathbf{R_c}^{-1}\mathbf{S_c}$$
$$+\mathbf{E_cE_c}^T+\left(\varepsilon_1^{-1}+\delta_1\right)\mathbf{I}=\mathbf{0} \qquad (34)$$

$$\bar{\mathbf{A}}\mathbf{P_2}\bar{\mathbf{A}}^T-\mathbf{P_2}+\bar{\mathbf{A}}\mathbf{P_2\Theta_2}^{1/2}\left(\mathbf{I}-\mathbf{\Theta_1}^{1/2}\mathbf{P_2\Theta_2}^{1/2}\right)^{-1}\mathbf{\Theta_2}^{1/2}\mathbf{P_2}\bar{\mathbf{A}}^T$$
$$+\mathbf{\Gamma_1\Phi_1\Gamma_1}^T+\mathbf{D_1D_1}^T+\left(\varepsilon_2^{-1}+\varepsilon_3^{-1}\right)\mathbf{M_1M_1}^T-\mathbf{S_o}^T\mathbf{R_o}^{-1}\mathbf{S_o}$$
$$+\mathbf{E_oE_o}^T+\left(\varepsilon_1^{-1}+\delta_2\right)\mathbf{I}=\mathbf{0} \qquad (35)$$

along side with the corresponding matrix inequality constraints

$$\mathbf{P_1}^{-1}-\varepsilon_1\mathbf{A_d}^T\mathbf{A_d}-\varepsilon_2\mathbf{N_1}^T\mathbf{N_1}-\varepsilon_3\mathbf{N_2}^T\mathbf{N_2}>0 \qquad (36)$$
$$\mathbf{P_2}^{-1}-\gamma^{-2}\mathbf{L}^T\mathbf{L}-\varepsilon_1\mathbf{A_d}^T\mathbf{A_d}>0 \qquad (37)$$

have symmetric positive-definite solutions $\mathbf{P_1}$ and $\mathbf{P_2}$ respectively.
Under these conditions, if $\mathbf{G}$, $\mathbf{K_o}$ and $\mathbf{K_c}$ are gain matrices which for some chosen orthogonal matrices $\mathbf{U_c}\in R^{n\times n}$ ($\mathbf{U_cU_c}^T=\mathbf{I}$) and $\mathbf{U_o}\in R^{p\times p}$ ($\mathbf{U_oU_o}^T=\mathbf{I}$), satisfy:

$$\mathbf{BK_c}=\mathbf{S_c}^T\mathbf{R_c}^{-1}+\mathbf{E_cU_cR_c}^{-1/2} \qquad (38)$$
$$\mathbf{K_o}=\mathbf{S_o}^T\mathbf{R_o}^{-1}+\mathbf{E_oU_oR_o}^{-1/2}, \qquad (39)$$
$$\mathbf{G}=\bar{\mathbf{A}}-\mathbf{K_o}\bar{\mathbf{C}} \qquad (40)$$

then the resulting output feedback system using $\mathbf{G}$, $\mathbf{K_o}$ and $\mathbf{K_c}$ will be such that, for all admissible parameter uncertainties $\Delta\mathbf{A}$, $\Delta\mathbf{A_d}$ and $\Delta\mathbf{C}$, and for all positive integer time-delay values d,

(1) *the augmented state-delayed system (14) and (15) is asymptotically stable.*

(2) $\left\|\mathbf{H_{zw}}(z)\right\|_\infty\le\gamma$.

Proof. By virtue of Lemma 1, the validity of *(18)* and *(19)* needs to be shown. To this end, defining,

$$\mathbf{P}\triangleq\begin{bmatrix}\mathbf{P_1} & \mathbf{0}\\ \mathbf{0} & \mathbf{P_2}\end{bmatrix}>0 \qquad (41)$$

and considering the definitions *(9)-(13)* and *(22)-(31)*, it is easy to see that inequality *(18)* follows from inequalities *(36)* and *(37)*. Also, for simplicity of notation, define the left-hand side of *(19)* by $\mathbf{\Sigma}$, where

$$\mathbf{\Sigma}\triangleq\begin{bmatrix}\mathbf{\Sigma_{11}} & \mathbf{\Sigma_{12}}\\ \mathbf{\Sigma_{12}^T} & \mathbf{\Sigma_{22}}\end{bmatrix} \qquad (42)$$

Substituting *(41)* yields:

$$\mathbf{\Sigma_{11}}=\left(\mathbf{A}+\mathbf{BK_c}\right)\mathbf{\Phi_1}\left(\mathbf{A}+\mathbf{BK_c}\right)^T+\mathbf{BK_c\Phi_2}\left(\mathbf{BK_c}\right)^T$$
$$-\mathbf{P_1}+\mathbf{D_1D_1}^T+\varepsilon_1^{-1}\mathbf{I}+\left(\varepsilon_2^{-1}+\varepsilon_3^{-1}\right)\mathbf{M_1M_1}^T \qquad (43)$$

$$\mathbf{\Sigma_{12}}=\left(\mathbf{A}+\mathbf{BK_c}\right)\mathbf{\Phi_1}\left(\mathbf{A}-\mathbf{K_oC}-\mathbf{G}\right)^T-\mathbf{BK_c\Phi_2G}^T$$
$$+\mathbf{D_1}\left(\mathbf{D_1}-\mathbf{K_oD_2}\right)^T+\varepsilon_2^{-1}\mathbf{M_1}\left(\mathbf{M_1}-\mathbf{K_oM_2}\right)^T$$
$$+\varepsilon_3^{-1}\mathbf{M_1M_1}^T \qquad (44)$$

$$\mathbf{\Sigma_{22}}=\left(\mathbf{A}-\mathbf{K_oC}-\mathbf{G}\right)\mathbf{\Phi_1}\left(\mathbf{A}-\mathbf{K_oC}-\mathbf{G}\right)^T+\mathbf{G\Phi_2G}^T$$
$$-\mathbf{P_2}+\left(\mathbf{D_1}-\mathbf{K_oD_2}\right)\left(\mathbf{D_1}-\mathbf{K_oD_2}\right)^T$$
$$+\varepsilon_2^{-1}\left(\mathbf{M_1}-\mathbf{K_oM_2}\right)\left(\mathbf{M_1}-\mathbf{K_oM_2}\right)^T$$
$$+\varepsilon_3^{-1}\mathbf{M_1M_1}^T+\varepsilon_1^{-1}\mathbf{I} \qquad (45)$$

It follows from the matrix inversion Lemma,

$$\left(\mathbf{A_{11}}-\mathbf{A_{12}A_{22}^{-1}A_{21}}\right)^{-1}=$$
$$\mathbf{A_{11}^{-1}}+\mathbf{A_{11}^{-1}A_{12}}\left(\mathbf{A_{22}}-\mathbf{A_{21}A_{11}^{-1}A_{12}}\right)^{-1}\mathbf{A_{21}A_{11}^{-1}}$$

and the definitions of $\mathbf{\Theta_1}$ and $\mathbf{\Theta_2}$ given in *(28)*, that

$$\mathbf{\Phi_1}=\mathbf{P_1}+\mathbf{P_1\Theta_1}^{1/2}\left(\mathbf{I}-\mathbf{\Theta_1}^{1/2}\mathbf{P_1\Theta_1}^{1/2}\right)^{-1}\mathbf{\Theta_1}^{1/2}\mathbf{P_1} \qquad (46)$$

$$\mathbf{\Phi_2}=\mathbf{P_2}+\mathbf{P_2\Theta_2}^{1/2}\left(\mathbf{I}-\mathbf{\Theta_2}^{1/2}\mathbf{P_2\Theta_2}^{1/2}\right)^{-1}\mathbf{\Theta_2}^{1/2}\mathbf{P_2} \qquad (47)$$

Re-writing $\mathbf{\Sigma_{11}}$ as,

$$\mathbf{\Sigma_{11}}=\mathbf{BK_c}\left(\mathbf{\Phi_1}+\mathbf{\Phi_2}\right)\left(\mathbf{BK_c}\right)^T+\mathbf{BK_c}\left(\mathbf{\Phi_1A}^T\right)$$
$$+\mathbf{A\Phi_1}\left(\mathbf{BK_c}\right)^T+\mathbf{A\Phi_1A}^T-\mathbf{P_1}+\mathbf{D_1D_1}^T$$
$$+\varepsilon_1^{-1}\mathbf{I}+\left(\varepsilon_2^{-1}+\varepsilon_3^{-1}\right)\mathbf{M_1M_1}^T$$

and using the definitions of $\mathbf{R_c}$ and $\mathbf{S_c}$ in *(29)*, while noting that $\mathbf{R_c}$ is invertible because $\mathbf{\Phi_1}$ and $\mathbf{\Phi_2}$ are positive-definite (due to *(36)* and *(37)*),

$$\mathbf{\Sigma_{11}}=\left(\mathbf{BK_cR_c}^{1/2}-\mathbf{S_c}^T\mathbf{R_c}^{-1/2}\right)\left(\mathbf{BK_cR_c}^{1/2}-\mathbf{S_c}^T\mathbf{R_c}^{-1/2}\right)^T$$
$$-\mathbf{S_c}^T\mathbf{R_c}^{-1}\mathbf{S_c}+\mathbf{A\Phi_1A}^T-\mathbf{P_1}+\mathbf{D_1D_1}^T$$
$$+\varepsilon_1^{-1}\mathbf{I}+\left(\varepsilon_2^{-1}+\varepsilon_3^{-1}\right)\mathbf{M_1M_1}^T$$

Using the definition of $\mathbf{BK_c}$ in *(38)*,

$$\mathbf{\Sigma_{11}}=\mathbf{A\Phi_1A}^T-\mathbf{P_1}+\mathbf{D_1D_1}^T+\varepsilon_1^{-1}\mathbf{I}+\left(\varepsilon_2^{-1}+\varepsilon_3^{-1}\right)\mathbf{M_1M_1}^T$$
$$-\mathbf{S_c}^T\mathbf{R_c}^{-1}\mathbf{S_c}+\mathbf{E_cE_c}^T$$

so that by *(46)*,

$$\mathbf{\Sigma_{11}}=\mathbf{AP_1A}^T+\mathbf{AP_1\Theta_1}^{1/2}\left(\mathbf{I}-\mathbf{\Theta_1}^{1/2}\mathbf{P_1\Theta_1}^{1/2}\right)^{-1}\mathbf{\Theta_1}^{1/2}\mathbf{P_1A}^T$$
$$-\mathbf{P_1}+\mathbf{D_1D_1}^T+\varepsilon_1^{-1}\mathbf{I}+\left(\varepsilon_2^{-1}+\varepsilon_3^{-1}\right)\mathbf{M_1M_1}^T$$
$$-\mathbf{S_c}^T\mathbf{R_c}^{-1}\mathbf{S_c}+\mathbf{E_cE_c}^T$$

From *(34)*, $\mathbf{\Sigma_{11}}=-\delta_1\mathbf{I}<0$.

Similarly, $\mathbf{\Sigma_{22}}$ of *(45)* can be re-written as,

$$\begin{aligned}\mathbf{\Sigma_{22}} &= \left(\mathbf{A}-\mathbf{K_oC}-\bar{\mathbf{A}}+\mathbf{K_o}\bar{\mathbf{C}}\right)\mathbf{\Phi_1}\left(\mathbf{A}-\mathbf{K_oC}-\bar{\mathbf{A}}+\mathbf{K_o}\bar{\mathbf{C}}\right)^T\\ &+\left(\bar{\mathbf{A}}-\mathbf{K_o}\bar{\mathbf{C}}\right)\mathbf{\Phi_2}\left(\bar{\mathbf{A}}-\mathbf{K_o}\bar{\mathbf{C}}\right)^T\\ &+\left(\mathbf{D_1}-\mathbf{K_oD_2}\right)\left(\mathbf{D_1}-\mathbf{K_oD_2}\right)^T\\ &-\mathbf{P_2}+\varepsilon_1^{-1}\mathbf{I}+\varepsilon_2^{-1}\left(\mathbf{M_1}-\mathbf{K_oM_2}\right)\left(\mathbf{M_1}-\mathbf{K_oM_2}\right)^T\\ &+\varepsilon_3^{-1}\mathbf{M_1M_1^T}\end{aligned}$$

where $\mathbf{G}$ has been replaced by its expression *(40)*.
Grouping the terms with respect to $\mathbf{K_o}$,

$$\begin{aligned}\mathbf{\Sigma_{22}} &= \mathbf{K_o}\left[\mathbf{\Gamma_2\Phi_1\Gamma_2^T}+\bar{\mathbf{C}}\mathbf{\Phi_2}\bar{\mathbf{C}}^T+\mathbf{D_2D_2^T}+\varepsilon_2^{-1}\mathbf{M_2M_2^T}\right]\mathbf{K_o^T}\\ &-\mathbf{K_o}\left[\mathbf{\Gamma_2\Phi_1\Gamma_1^T}+\bar{\mathbf{C}}\mathbf{\Phi_2}\bar{\mathbf{A}}^T+\mathbf{D_2D_1^T}+\varepsilon_2^{-1}\mathbf{M_2M_1^T}\right]\\ &-\left[\mathbf{\Gamma_2\Phi_1\Gamma_1^T}+\bar{\mathbf{C}}\mathbf{\Phi_2}\bar{\mathbf{A}}^T+\mathbf{D_2D_1^T}+\varepsilon_2^{-1}\mathbf{M_2M_1^T}\right]^T\mathbf{K_o^T}\\ &+\mathbf{\Gamma_1\Phi_1\Gamma_1^T}+\bar{\mathbf{A}}\mathbf{\Phi_2}\bar{\mathbf{A}}^T-\mathbf{P_2}+\mathbf{D_1D_1^T}\\ &+\left(\varepsilon_2^{-1}+\varepsilon_3^{-1}\right)\mathbf{M_1M_1^T}+\varepsilon_1^{-1}\mathbf{I}\end{aligned}$$

From *(30)* and *(31)*,

$$\begin{aligned}\mathbf{\Sigma_{22}} &= \mathbf{K_oR_oK_o^T}-\mathbf{K_oS_o}-\mathbf{S_o^TK_o^T}+\mathbf{\Gamma_1\Phi_1\Gamma_1^T}+\bar{\mathbf{A}}\mathbf{\Phi_2}\bar{\mathbf{A}}^T\\ &-\mathbf{P_2}+\mathbf{D_1D_1^T}+\left(\varepsilon_2^{-1}+\varepsilon_3^{-1}\right)\mathbf{M_1M_1^T}+\varepsilon_1^{-1}\mathbf{I}\end{aligned}$$

Assumption 1 implies that the matrix $\mathbf{R_o}$ defined in *(30)* is positive-definite and hence invertible, thus

$$\begin{aligned}\mathbf{\Sigma_{22}} &= \left(\mathbf{K_oR_o^{1/2}}-\mathbf{S_o^TR_o^{-1/2}}\right)\left(\mathbf{K_oR_o^{1/2}}-\mathbf{S_o^TR_o^{-1/2}}\right)^T\\ &-\mathbf{S_o^TR_o^{-1}S_o}+\mathbf{\Gamma_1\Phi_1\Gamma_1^T}+\bar{\mathbf{A}}\mathbf{\Phi_2}\bar{\mathbf{A}}^T-\mathbf{P_2}+\mathbf{D_1D_1^T}\\ &+\left(\varepsilon_2^{-1}+\varepsilon_3^{-1}\right)\mathbf{M_1M_1^T}+\varepsilon_1^{-1}\mathbf{I}\end{aligned}$$

From *(39)* for $\mathbf{K_o}$,

$$\begin{aligned}\mathbf{\Sigma_{22}} &= \bar{\mathbf{A}}\mathbf{\Phi_2}\bar{\mathbf{A}}^T-\mathbf{P_2}+\mathbf{\Gamma_1\Phi_1\Gamma_1^T}+\mathbf{D_1D_1^T}\\ &+\left(\varepsilon_2^{-1}+\varepsilon_3^{-1}\right)\mathbf{M_1M_1^T}-\mathbf{S_o^TR^{-1}S_o}+\mathbf{E_oE_o^T}+\varepsilon_1^{-1}\mathbf{I}\end{aligned}$$

and by *(47)*,

$$\begin{aligned}\mathbf{\Sigma_{22}} &= \bar{\mathbf{A}}\mathbf{P_2}\bar{\mathbf{A}}^T+\bar{\mathbf{A}}\mathbf{P_2\Theta_2^{1/2}}\left(\mathbf{I}-\mathbf{\Theta_1^{1/2}P_2\Theta_2^{1/2}}\right)^{-1}\mathbf{\Theta_2^{1/2}P_2}\bar{\mathbf{A}}^T\\ &-\mathbf{P_2}+\mathbf{\Gamma_1\Phi_1\Gamma_1^T}+\mathbf{D_1D_1^T}+\left(\varepsilon_2^{-1}+\varepsilon_3^{-1}\right)\mathbf{M_1M_1^T}\\ &-\mathbf{S_o^TR^{-1}S_o}+\mathbf{E_oE_o^T}+\varepsilon_1^{-1}\mathbf{I}\end{aligned}$$

From *(35)*, $\mathbf{\Sigma_{22}}=-\delta_2\mathbf{I}<0$.
Finally,

$$\begin{aligned}\mathbf{\Sigma_{12}} &= \left(\mathbf{A}+\mathbf{BK_c}\right)\mathbf{\Phi_1}\left(\mathbf{A}-\mathbf{K_oC}-\mathbf{G}\right)^T-\mathbf{BK_c\Phi_2G^T}\\ &+\mathbf{D_1}\left(\mathbf{D_1}-\mathbf{K_oD_2}\right)^T+\varepsilon_2^{-1}\mathbf{M_1}\left(\mathbf{M_1}-\mathbf{K_oM_2}\right)^T\\ &+\varepsilon_3^{-1}\mathbf{M_1M_1^T}\end{aligned}$$

Grouping the terms with respect to $\mathbf{G}^T$,

$$\begin{aligned}\mathbf{\Sigma_{12}} &= -\left(\mathbf{A\Phi_1}+\mathbf{BK_c\Phi_1}+\mathbf{BK_c\Phi_2}\right)\mathbf{G}^T\\ &+\mathbf{A\Phi_1}\left(\mathbf{A}-\mathbf{K_oC}\right)^T+\mathbf{BK_c\Phi_1}\left(\mathbf{A}-\mathbf{K_oC}\right)^T\\ &+\mathbf{D_1}\left(\mathbf{D_1}-\mathbf{K_oD_2}\right)^T+\varepsilon_2^{-1}\mathbf{M_1}\left(\mathbf{M_1}-\mathbf{K_oM_2}\right)^T\\ &+\varepsilon_3^{-1}\mathbf{M_1M_1^T}\end{aligned}$$

Replacing $\mathbf{BK_c}$ by it expression *(38)* and grouping the terms with respect to $\mathbf{K_o}$,

$$\begin{aligned}\mathbf{\Sigma_{12}} &= -\mathbf{E_cU_c}\left(\mathbf{\Phi_1}+\mathbf{\Phi_2}\right)^{1/2}\mathbf{G}^T\\ &+\Big[\mathbf{A\Phi_1}\left(\mathbf{\Phi_1}+\mathbf{\Phi_2}\right)^{-1}\mathbf{\Phi_1C^T}-\mathbf{A\Phi_1C^T}\\ &-\mathbf{E_cU_c}\left(\mathbf{\Phi_1}+\mathbf{\Phi_2}\right)^{-1/2}\mathbf{\Phi_1C^T}-\mathbf{D_1D_2^T}-\varepsilon_2^{-1}\mathbf{M_1M_2^T}\Big]\mathbf{K_o^T}\\ &+\mathbf{A\Phi_1A^T}-\mathbf{A\Phi_1}\left(\mathbf{\Phi_1}+\mathbf{\Phi_2}\right)^{-1}\mathbf{\Phi_1A^T}\\ &+\mathbf{E_cU_c}\left(\mathbf{\Phi_1}+\mathbf{\Phi_2}\right)^{-1/2}\mathbf{\Phi_1A^T}+\mathbf{D_1D_1^T}\\ &+\left(\varepsilon_2^{-1}+\varepsilon_3^{-1}\right)\mathbf{M_1M_1^T}\end{aligned}$$

Noting that $\left(\mathbf{\Phi_1}+\mathbf{\Phi_2}\right)^{-1}=\mathbf{\Phi_2^{-1}}\left(\mathbf{\Phi_1^{-1}}+\mathbf{\Phi_2^{-1}}\right)^{-1}\mathbf{\Phi_1^{-1}}$ $=\mathbf{\Phi_1^{-1}}-\mathbf{\Phi_1^{-1}}\left(\mathbf{\Phi_1^{-1}}+\mathbf{\Phi_2^{-1}}\right)^{-1}\mathbf{\Phi_1^{-1}}$, the following is obtained:

$$\begin{aligned}\mathbf{\Sigma_{12}} &= -\mathbf{E_cU_c}\left(\mathbf{\Phi_1}+\mathbf{\Phi_2}\right)^{1/2}\mathbf{G}^T\\ &-\Big[\mathbf{A}\left(\mathbf{\Phi_1^{-1}}+\mathbf{\Phi_2^{-1}}\right)^{-1}\mathbf{C}^T+\mathbf{E_cU_c}\left(\mathbf{\Phi_1}+\mathbf{\Phi_2}\right)^{-1/2}\mathbf{\Phi_1C^T}\\ &+\mathbf{D_1D_2^T}+\varepsilon_2^{-1}\mathbf{M_1M_2^T}\Big]\mathbf{K_o^T}\\ &+\mathbf{A}\left(\mathbf{\Phi_1^{-1}}+\mathbf{\Phi_2^{-1}}\right)^{-1}\mathbf{A}^T+\mathbf{E_cU_c}\left(\mathbf{\Phi_1}+\mathbf{\Phi_2}\right)^{-1/2}\mathbf{\Phi_1A^T}\\ &+\mathbf{D_1D_1^T}+\left(\varepsilon_2^{-1}+\varepsilon_3^{-1}\right)\mathbf{M_1M_1^T}\end{aligned}$$

Replacing $\mathbf{G}$, $\bar{\mathbf{A}}$ and $\bar{\mathbf{C}}$ by their expressions in *(40)*, *(25)* and *(27)* respectively and subsequently $\mathbf{\Gamma_1}$ and $\mathbf{\Gamma_2}$ by their expressions in *(24)* and *(26)* respectively, it then implies that $\mathbf{\Sigma_{12}}=\mathbf{0}$ and so,

$$\mathbf{\Sigma}=\begin{bmatrix}\mathbf{\Sigma_{11}} & \mathbf{0}\\ \mathbf{0} & \mathbf{\Sigma_{22}}\end{bmatrix}<0, \qquad (48)$$

as $\mathbf{\Sigma_{11}}<0$ and $\mathbf{\Sigma_{22}}<0$ as shown above. By virtue of Lemma 1, the output feedback system *(14)* and *(15)* is robustly asymptotically stable and $\left\|\mathbf{H_{zw}}(z)\right\|_\infty\le\gamma$ for all values of positive integer time-delay d, and all uncertainties *(3)*. QED

Remark 1. It is important to point out that, as opposed to the observer design case procedure of Wang, *et al.* (1999), the output feedback design procedure proposed in this paper applies to systems which can be open loop unstable. For the observer design alone, i.e. when $\mathbf{K_c}=\mathbf{0}$, the Riccati equation *(34)* is solvable in terms of a positive-definite matrix $\mathbf{P_1}$ only if $\mathbf{A}$ is stable, see (Wang, *et al.*, 1999) to confirm this restrictive assumption. However, in the output feedback case, when $\mathbf{K_c}$ is given by *(38)*, equation *(34)* reads as,

$$\mathbf{\Sigma_{11}}+\delta_1\mathbf{I}=\mathbf{0} \qquad (49)$$

with $\mathbf{\Sigma_{11}}$ as in *(43)* in which $\mathbf{K_c}$ is no longer a zero matrix. Again, for the existence of solutions to *(34)*, only the stability of $\mathbf{A}+\mathbf{BK_c}$ is needed, which does not necessitate a priori stability of $\mathbf{A}$. Also, the invertibility of $\mathbf{A}$ required in Wang, *et al.* (1999) is not necessary in this paper.

Remark 2. the numerical solutions to *(34)* and *(35)* are easiest obtained by solving two auxiliary Riccati-type inequalities as explained in Haurani, *et al.* (2001).

Remark 3. **T**he presented robust H_∞ output feedback control design procedure still offers much additional design freedom. This freedom is reflected by the arbitrary choice of the free gains parameters $\mathbf{E_c}$ $\left(\mathbf{E_c} \in R^{n\times n}\right)$ and $\mathbf{E_o}$ $\left(\mathbf{E_o} \in R^{n\times p}\right)$, and the orthogonal matrices $\mathbf{U_c} \in R^{n\times n}$ and $\mathbf{U_o} \in R^{p\times p}$. Introducing additional performance constraints into the problem formulation *(1)* and *(2)* of Theorem 1, which would exploit this design freedom is currently under investigation.

4. CONCLUSION

This paper presents, in what is believed to be the first approach to robust, delay-independent, discrete-time H_∞ output feedback control design procedure. Specifically, the conditions for solvability of the robust H_∞ output feedback control problem is characterized in terms of the existence of solutions of two algebraic Riccati inequalities. The analytical expressions for the resulting observer and controller gains are given.

Ongoing work is concerned with the incorporation of further performance constraints into the output feedback problem which can be accommodated by exploiting the additional freedom in the choice of design parameters.

Future research will aim at the extension of the above presented results to the more general case in which a penalty on the control input is incorporated.

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