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ROBUST $\mathcal{H}_2$ FILTER DESIGN VIA PARAMETER-DEPENDENT LYAPUNOV FUNCTIONS [1]

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Abstract: This paper addresses the design of robust $\mathcal{H}_2$ filters for linear continuous-time systems subject to parameter uncertainty in the state-space model. The uncertain parameters are supposed to belong to a given convex bounded polyhedral domain. Two methods based on parameter-dependent Lyapunov functions are proposed for designing linear stationary asymptotically stable filters that assure asymptotic stability and a prescribed $\mathcal{H}_2$ performance, irrespective of the uncertain parameters. The proposed designs are in terms of linear matrix inequalities (LMIs).

Keywords: Robust filtering, $\mathcal{H}_2$ filtering, uncertain systems, parameter-dependent Lyapunov functions, linear matrix inequalities.

1. INTRODUCTION

Motivated by the need to accommodate significant modelling uncertainty in Kalman filtering problems, considerable attention has been devoted, over the last decade, to the so called robust $\mathcal{H}_2$ (or equivalently, minimum variance) filtering for linear systems with uncertain parameters; see, e.g. Petersen and Savkin (1999). The problem of robust $\mathcal{H}_2$ filtering consists on designing a linear stationary asymptotically stable filter that assures a prescribed bounded (optimized in a certain sense) for the worst-case asymptotic estimation error variance, irrespective of the uncertain parameters. Riccati equation approaches have been proposed for linear systems subject to norm-bounded parameter uncertainty in the state-state model (Bolzern *et al.*, 1994; Petersen and McFarlane, 1991; Shaked and de Souza, 1995; Xie and Soh, 1994), whereas systems with polytopic-type parameter uncertainty have been recently treated in de Souza and Trofino (2000) and Geromel (1999) using LMI methodologies. The aforementioned methods are based on the notion of quadratic stability and have the advantage that they are computationally simple. However, they have the drawback that stability and the guaranteed bound on the error variance are based on a parameter-independent Lyapunov function, and thus the uncertain parameters are allowed to vary arbitrarily fast, which can be quite conservative.

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Very recently, an LMI robust $\mathcal{H}_2$ filtering method has been proposed in Tuan *et al.* (2000) using an affine parameter-dependent Lyapunov function.

This paper develops new robust $\mathcal{H}_2$ filter design methods for linear continuous-time systems subject to uncertain parameters belonging to a given convex bounded polytope. The proposed design methods are based on parameter-dependent Lyapunov functions, thus reducing conservatism compared with existing design methods. In particular, the new methods include the quadratic stability filtering approach of Geromel (1999) as a particular case. The results of this paper will be demonstrated in an example that illustrates the significant improvement that can be achieved compared with the existing filter design methods.

The notation used in this paper is quite standard. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{R}^{n\times m}$ is the set of $n \times m$ real matrices, I_n is the $n \times n$ identity matrix, $\mathrm{Tr}\{\cdots\}$ denotes matrix trace, $\mathrm{diag}\{\cdots\}$ stands for a block-diagonal matrix, and the notation $S>0$ for a real matrix S, means that S is symmetric and positive definite. For a symmetric block matrix, the symbol $\star$ denotes the transpose of the symmetric blocks outside the main diagonal block.

2. PROBLEM STATEMENT

Consider the following uncertain linear system

$$\begin{aligned} \dot{x}(t) &= A(\theta)x(t) + B(\theta)w(t), \quad x(0) = x_0 \\ y(t) &= C(\theta)x(t) + D(\theta)w(t) \\ z(t) &= C_z(\theta)x(t) \end{aligned} \tag{1}$$

with $\theta := (\theta_1, \cdots, \theta_p) \in \mathbb{R}^p$, where $x(t) \in \mathbb{R}^n$ is the state, x_0 is zero-mean random variable, $w(t) \in \mathbb{R}^{n_w}$ is a zero-mean white signal (including process and measurement noises) with an identity power spectral density matrix which is uncorrelated with x_0 for all $t \geq 0$, $y(t) \in \mathbb{R}^{n_y}$ is the measurement, $z(t) \in \mathbb{R}^{n_z}$ is the signal to be estimated, $\theta_i,\ i = 1, \ldots, p$ are bounded constant uncertain parameters, and $A(\theta),\ B(\theta),\ C(\theta),\ C_z(\theta)$ and $D(\theta)$ are real matrices of appropriate dimensions that depend affinely on the parameter vector θ. It is assumed that θ belongs to a polytope $\mathcal{B}$, with ℓ known vertices.

The aim of this paper is the design of a stationary asymptotically stable linear filter $\mathcal{F}$ which provides an estimate $\hat{z}$ of the signal z with a guaranteed performance in the $\mathcal{H}_2$ sense, irrespective of the uncertain parameters. The filter is assumed to be of order n and with a state-space realization

$$\mathcal{F}: \begin{aligned} \dot{x}_f(t) &= A_f x_f(t) + B_f y(t), \quad x_f(0) = 0 \\ \hat{z}(t) &= C_f x_f \end{aligned} \tag{2}$$

where the matrices $A_f \in \mathbb{R}^{n\times n}$, $B_f \in \mathbb{R}^{n\times n_y}$ and $C_f \in \mathbb{R}^{n_z\times n}$ are to be found. This paper is focused on finding a filter $\mathcal{F}$ that ensures asymptotic stability for the estimation error dynamics and minimizes a suitable upper-bound $\mu(\mathcal{F})$ for the worst-case asymptotic error variance, namely:

$$\min_{\mathcal{F}} \mu(\mathcal{F}): \ \sup_{\theta\,\in\,\mathcal{B}} \mathbf{E}[(z-\hat{z})^T(z-\hat{z})] \leq \mu(\mathcal{F}) \tag{3}$$

where $\mathbf{E}$ denotes mathematical expectation.

The dynamics of the estimation error $e = z - \hat{z}$ can be described by the following state-space model

$$\begin{aligned} \dot{x}_a(t) &= A_a(\theta)x_a(t) + B_a(\theta)w(t) \\ e(t) &= C_a(\theta)x_a(t) \end{aligned} \tag{4}$$

where

$$\begin{aligned} &x_a = \begin{bmatrix} x \\ x_f \end{bmatrix}, \quad A_a(\theta) = \begin{bmatrix} A(\theta) & 0 \\ B_f C(\theta) & A_f \end{bmatrix}, \\ &B_a(\theta) = \begin{bmatrix} B(\theta) \\ B_f D(\theta) \end{bmatrix}, \ \ C_a(\theta) = \begin{bmatrix} C_z(\theta) & -C_f \end{bmatrix}. \end{aligned} \tag{5}$$

It is well known that the above filtering problem can be solved via an optimization problem in terms of either the controllability or observability Gramians associated with the estimation error system (4); see, e.g. de Souza and Trofino (2000) and Geromel (1999). The basis for this approach is the following lemma which provides bounds for the worst-case asymptotic error variance. This lemma is a simple extension of well known results (Green and Limebeer, 1995) to the context of parameter-dependent matrices.

Lemma 1. Given the system (1) and a filter of the form (2), the following conditions hold:

(a) If there exist symmetric matrices $P(\theta)$ and $W(\theta)$ such that for all $\theta \in \mathcal{B}$

$$\begin{bmatrix} P(\theta)A_a^T(\theta) + A_a(\theta)P(\theta) & B_a(\theta) \\ B_a^T(\theta) & -I \end{bmatrix} < 0 \tag{6}$$

$$\begin{bmatrix} W(\theta) & C_a(\theta)P(\theta) \\ P(\theta)C_a^T(\theta) & P(\theta) \end{bmatrix} > 0 \tag{7}$$

then the estimation error system (4) is asymptotically stable for all $\theta \in \mathcal{B}$ and

$$\sup_{\theta\in\mathcal{B}} \mathbf{E}[e^T e] < \mu; \quad \mu = \sup_{\theta\in\mathcal{B}} \mathrm{Tr}[\,W(\theta)\,]. \tag{8}$$

(b) If there exist symmetric matrices $P(\theta)$ and $W(\theta)$ such that for all $\theta \in \mathcal{B}$

$$\begin{bmatrix} P(\theta)A_a(\theta) + A_a^T(\theta)P(\theta) & C_a^T(\theta) \\ C_a(\theta) & -I \end{bmatrix} < 0 \tag{9}$$

$$\begin{bmatrix} W(\theta) & B_a^T(\theta)P(\theta) \\ P(\theta)B_a(\theta) & P(\theta) \end{bmatrix} > 0 \tag{10}$$

then the estimation error system (4) is asymptotically stable for all $\theta \in \mathcal{B}$ and

$$\sup_{\theta\in\mathcal{B}} \mathbf{E}[e^T e] < \mu; \quad \mu = \sup_{\theta\in\mathcal{B}} \mathrm{Tr}[\,W(\theta)\,]. \tag{11}$$

Note that the inequalities of Lemma 1 are not convex in θ, even when $P(\theta)$ is affine in θ and the filter matrices A_f, B_f and C_f are given. Thus, the problem of finding A_f, B_f and C_f with a parameter dependent $P(\theta)$ is a very hard task. In the case where a matrix $P(\theta)$ independent of θ is used, LMI filter design methods have been developed in de Souza and Trofino (2000) and Geromel (1999). On the other hand, Tuan *et al.* (2000) presented a method based on part (a) of Lemma 1 using a matrix $P(\theta)$ that depends affinely on θ. The present paper proposes two new approaches for solving this problem using parameter-dependent Lyapunov functions.

3. ROBUST $\mathcal{H}_2$ FILTER DESIGN

This section deals with the design of robust $\mathcal{H}_2$ filters for the uncertain system (1) using parameter-dependent Lyapunov functions. Attention is focused on the following class of parameter-dependent Lyapunov functions

$$P(\theta) = M^T N^{-1}(\theta) M > 0, \ \ \forall\, \theta \in \mathcal{B} \qquad (12)$$

where $N(\theta)$ is a positive definite matrix which depends affinely on θ and M is a nonsingular constant matrix.

The first result presents a robust $\mathcal{H}_2$ filter design method based on the inequalities (6) and (7) of Lemma 1 with a matrix $P(\theta)$ as in (12).

Theorem 1. Consider the system (1) and let $\mathcal{B}$ be a polytope of admissible θ. Suppose that for some scalar $\epsilon > 0$ there exist matrices $Z \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times n}$, $F \in \mathbb{R}^{n\times n_y}$, $R \in \mathbb{R}^{n_z\times n}$, $Q \in \mathbb{R}^{n\times n}$, and symmetric matrices $\mathcal{N}_i \in \mathbb{R}^{2n\times 2n}$ and $W_i \in \mathbb{R}^{n_w\times n_w}$, $i = 1, ..., \ell$ such that the following inequalities are satisfied at all the vertices of $\mathcal{B}$

$$\begin{bmatrix} \Psi_A(\theta) + \Psi_A^T(\theta) & \star & \star \\ \Psi_M^T + \epsilon\Psi_A^T(\theta) - \mathcal{N}(\theta) & -2\epsilon\mathcal{N}(\theta) & \star \\ \Psi_B^T(\theta) & 0 & -I \end{bmatrix} < 0 \qquad (13)$$

$$\begin{bmatrix} W(\theta) & \Psi_C(\theta) \\ \Psi_C^T(\theta) & \mathcal{N}(\theta) \end{bmatrix} > 0 \qquad (14)$$

where $W(\theta)$ and $\mathcal{N}(\theta)$ are affine matrix functions of θ with values $W_1, \ldots, W_\ell$ and $\mathcal{N}_1, \ldots, \mathcal{N}_\ell$, respectively, at the vertices of $\mathcal{B}$ and

$$\Psi_A(\theta) = \begin{bmatrix} ZA(\theta) & ZA(\theta) \\ YA(\theta) + FC(\theta) + Q & YA(\theta) + FC(\theta) \end{bmatrix},$$

$$\Psi_B(\theta) = \begin{bmatrix} ZB(\theta) \\ YB(\theta) + FD(\theta) \end{bmatrix}, \qquad (15)$$

$$\Psi_C(\theta) = \begin{bmatrix} C_z(\theta) - R & C_z(\theta) \end{bmatrix}, \ \ \Psi_M = \begin{bmatrix} Z & Z \\ Y+S & Y \end{bmatrix}.$$

Then the filter $\mathcal{F}$ with transfer function matrix

$$G_{\hat{z}y}(s) = RS^{-1}(sI - QS^{-1})^{-1}F \qquad (16)$$

ensures that the estimation error system (4) is asymptotically stable for all $\theta \in \mathcal{B}$ and the asymptotic error variance satisfies

$$\mathbf{E}[e^T e] < \max_{i=1,\ldots,\ell} \mathrm{Tr}\,[W_i], \ \ \forall\, \theta \in \mathcal{B}. \qquad (17)$$

Proof. First, note that since the inequalities of (13) and (14) are affine in θ for a given $\epsilon > 0$, then they are satisfied for all $\theta \in \mathcal{B}$ if only if (13) and (14) are satisfied at all the vertices of $\mathcal{B}$.

It will be shown that if the inequalities of (13) and (14) hold, then the filter (16) ensures that the conditions (6) and (7) of Lemma 1 are satisfied with a matrix function $P(\theta) > 0$ as in (12) and $W(\theta)$ which depends affinely on θ.

Initially, it will be shown that the matrices S, Y and Z are nonsingular. To this end, pre- and post-multiplying (13) by $[\,\epsilon I \ \ -I \ \ 0\,]$ and its transpose, respectively, it results that

$$\Psi_M + \Psi_M^T = \begin{bmatrix} Z + Z^T & Y^T + S^T + Z \\ Y + S + Z^T & Y + Y^T \end{bmatrix} > 0 \qquad (18)$$

which implies that Z and Y are nonsingular matrices. Further, pre- and post-multiplying the above inequality by $[\,I \ -I\,]$ and its transpose, respectively, implies that $S + S^T < 0$, and thus S is a nonsingular matrix.

Inspired by Geromel (1999), define nonsingular matrices U and V such that $VUZ^T = S$ and introduce the following nonsingular matrices

$$M^T = \begin{bmatrix} Z^{-T} & \bullet \\ U & \bullet \end{bmatrix}, \quad M^{-T} = \begin{bmatrix} Y^T & \bullet \\ V^T & \bullet \end{bmatrix} \qquad (19)$$

where the elements $\bullet$ are uniquely determined from the equalities $MM^{-1} = M^{-1}M = I$. Further, define the following state-space realization for the filter (16)

$$A_f = V^{-1}QS^{-1}V, \ \ B_f = V^{-1}F, \ \ C_f = RS^{-1}V \qquad (20)$$

and let the nonsingular matrices

$$\mathcal{T} = \begin{bmatrix} Z^T & Y^T \\ 0 & V^T \end{bmatrix}, \quad N(\theta) = \mathcal{T}^{-T}\mathcal{N}(\theta)\mathcal{T}^{-1}. \qquad (21)$$

Pre- and post-multiplying the inequality (13) by N_f^T and N_f, respectively, where

$$N_f = \begin{bmatrix} I_{2n} & 0 \\ \mathcal{N}^{-1}(\theta)\Psi_A^T(\theta) & 0 \\ 0 & I_{n_w} \end{bmatrix} \qquad (22)$$

leads to

$$\begin{bmatrix} \Psi_M\mathcal{N}^{-1}(\theta)\Psi_A^T(\theta) + \Psi_A(\theta)\mathcal{N}^{-1}(\theta)\Psi_M^T & \star \\ \Psi_B^T(\theta) & -I \end{bmatrix} < 0 \qquad (23)$$

Considering the matrices in (19)-(21), and performing straightforward matrix manipulations, it can be established that (23) is equivalent to

$$\mathcal{T}_a^T\,\Phi\,\mathcal{T}_a < 0 \qquad (24)$$

where $\mathcal{T}_a = \mathrm{diag}\{\mathcal{T},\, I_{n_w}\}$ and

$$\Phi = \begin{bmatrix} M^T N^{-1}(\theta) M A_a^T(\theta) + A_a(\theta) M^T N^{-1}(\theta) M & \star \\ B_a^T(\theta) & -I \end{bmatrix}.$$

Hence, one concludes that (6) is satisfied with $P(\theta) = M^T N^{-1}(\theta) M$.

On the other hand, by Schur's complements (14) is equivalent to

$$W(\theta) - \Psi_C(\theta)\mathcal{N}^{-1}(\theta)\Psi_C^T(\theta) > 0. \qquad (25)$$

Considering the definition of $\mathcal{T}$, M and $C_a(\theta)$, it can be readily verified that $\Psi_C(\theta) = C_a(\theta) M^T \mathcal{T}$. Therefore, (25) holds if and only if

$$W(\theta) - C_a(\theta) M^T N^{-1}(\theta) M C_a^T(\theta) > 0 \qquad (26)$$

which is equivalent to the inequality (7) with $P(\theta) = M^T N^{-1}(\theta) M$.

Finally, by Lemma 1 one concludes that for all $\theta \in \mathcal{B}$, the estimation error system (4) is asymptotically stable and

$$\mathbf{E}(e^T e) < \sup_{\theta\in\mathcal{B}} \mathrm{Tr}\,[W(\theta)] < \max_{i=1,\dots,\ell} \mathrm{Tr}\,[\,W_i\,] \qquad (27)$$

$\nabla\nabla\nabla$

Theorem 1 provides an LMI method for the design of a robust $\mathcal{H}_2$ filter for the uncertain system (1) using an affine parameter-dependent Lyapunov function. Observe that the upper-bound (17) on the asymptotic estimation error variance is also dependent on the uncertain parameters.

Remark 1. Note that, as for a given $\epsilon > 0$ the upper-bound on the asymptotic estimation error variance is an affine function of the LMI unknown matrices, the filter that minimizes this upper-bound, for a given $\epsilon > 0$, can be determined via the following convex optimization problem:

minimize μ
subject to (13) and (14), for θ at the vertices of $\mathcal{B}$, and $\mu - \mathrm{Tr}\,[W_i] \geq 0$, $i = 1,\dots,\ell$.

Moreover, $\mathbf{E}[e^T e] < \mu$. Observe that μ depends on the parameter ϵ and thus one should find the $\epsilon > 0$ which minimizes the upper-bound μ.

It should be remarked that (13) is not jointly convex in the LMI unknown matrices and ϵ. Further, (13) fails to be satisfied for too large values of ϵ. A gridding procedure seems to be the best way to find the $\epsilon > 0$ that minimizes μ. $\square$

It follows from the proof of Theorem 1 that this theorem specializes to a filter design method based on quadratic stability by constraining $\mathcal{N}(\theta)$ to be independent of θ, i.e. by setting $\mathcal{N}_i = \mathcal{N}$, $i = 1,\dots,\ell$. Further, as it will be shown in the next lemma, the conditions of Theorem 1 with the above constraints are necessary for (6) and (7) to hold with a parameter-independent matrix $P(\theta)$.

Lemma 2. Consider the system (1) and suppose that there exists an asymptotically stable filter with state-space realization (A_f, B_f, C_f) such that the estimation error system (4) satisfies (6) and (7) with $P(\theta) = Q_0$ and $W(\theta)$ which is affine in θ with values $W_1,\cdots,W_\ell$ at the vertices of $\mathcal{B}$. Then the conditions of Theorem 1 hold with the same matrix $W(\theta)$ and a sufficiently small $\epsilon > 0$.

Proof. Introduce the following partitions of Q_0 and Q_0^{-1}, where all the blocks are $n\times n$ matrices

$$Q_0 = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix}, \quad Q_0^{-1} = \begin{bmatrix} \Xi_1 & \Xi_2 \\ \Xi_2^T & \Xi_3 \end{bmatrix}. \qquad (28)$$

Further, without loss of generality, the matrices Q_2 and Ξ_2 can be assumed to be nonsingular. In addition, define the matrices

$$\begin{aligned} &Z = Q_1^{-1}, \quad Y = \Xi_1, \quad F = \Xi_2 B_f, \\ &S = \Xi_2 Q_2^T Q_1^{-1}, \quad Q = \Xi_2 A_f Q_2^T Q_1^{-1}; \end{aligned} \qquad (29)$$

$$R = C_f Q_2^T Q_1^{-1}, \quad \mathcal{T} = \begin{bmatrix} Q_1^{-1} & \Xi_1 \\ 0 & \Xi_2^T \end{bmatrix}, \quad \mathcal{N} = \mathcal{T}^T Q_0 \mathcal{T}$$

As θ belongs to a bounded polytope, it follows from (6) and (7) that there exists a sufficiently small scalar $\alpha > 0$ such that for all $\theta \in \mathcal{B}$

$$\begin{bmatrix} A_a(\theta)Q_0 + Q_0 A_a^T(\theta) + \frac{\alpha}{2} A_a(\theta) Q_0 A_a^T(\theta) & \star \\ B_a^T(\theta) & -I \end{bmatrix} < 0,$$

$$\begin{bmatrix} W(\theta) & C_a(\theta)Q_0 \\ Q_0 C_a^T(\theta) & Q_0 \end{bmatrix} > 0. \qquad (30)$$

With the matrices F, $\mathcal{N}$, Q, R, S, Y and Z as above and $\epsilon = \alpha$, it can be established that the left-hand side of (13) and (14), denoted by Φ_1 and Φ_2, respectively, become

$$\Phi_1 = \bar{\mathcal{T}}^T \begin{bmatrix} Q_0 A_a^T(\theta) + A_a(\theta) Q_0 & \star & \star \\ \alpha Q_0 A_a^T(\theta) & -2\alpha Q_0 & \star \\ B_a^T(\theta) & 0 & -I \end{bmatrix} \bar{\mathcal{T}} \qquad (31)$$

$$\Phi_2 = \hat{\mathcal{T}}^T \begin{bmatrix} W(\theta) & C_a(\theta)Q_0 \\ Q_0 C_a^T(\theta) & Q_0 \end{bmatrix} \hat{\mathcal{T}} \qquad (32)$$

where $\bar{\mathcal{T}} = \mathrm{diag}\{\mathcal{T}, \mathcal{T}, I_{n_w}\}$, $\hat{\mathcal{T}} = \mathrm{diag}\{I_{n_z}, \mathcal{T}\}$.

Finally, using Schur's complements and considering (30), it can be readily established that $\Phi_1 < 0$ and $\Phi_2 > 0$, which concludes the proof. $\nabla\nabla\nabla$

In the light of Lemma 2, the $\mathcal{H}_2$ filtering method of Theorem 1 with $\mathcal{N}(\theta)$ independent of θ and

$\epsilon > 0$ sufficiently small is equivalent to an LMI based quadratic stability approach, such as that of Geromel (1999). Thus, it turns out that the method of Theorem 1 is at most as conservative as the $\mathcal{H}_2$ filter design approach of Geromel (1999).

The next theorem provides a "dual" filter design approach of that of Theorem 1. This approach is based on the inequalities (9) and (10) of Lemma 1 with a matrix $P(\theta)$ as in (12).

Theorem 2. Consider the system (1) and let $\mathcal{B}$ be a polytope of admissible θ. Suppose that for some scalar $\epsilon > 0$ there exist matrices $Z \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times n}$, $F \in \mathbb{R}^{n\times n_y}$, $R \in \mathbb{R}^{n_z\times n}$, $Q \in \mathbb{R}^{n\times n}$, and symmetric matrices $\mathcal{N}_i \in \mathbb{R}^{2n\times 2n}$ and $W_i \in \mathbb{R}^{n_w\times n_w}$, $i = 1, ..., \ell$ such that the following inequalities are satisfied at all the vertices of $\mathcal{B}$

$$\begin{bmatrix} \Psi_A^T(\theta) + \Psi_A(\theta) & \star & \star \\ \Psi_M^T + \epsilon\Psi_A(\theta) - \mathcal{N}(\theta) & -2\epsilon\mathcal{N}(\theta) & \star \\ \Psi_C(\theta) & 0 & -I \end{bmatrix} < 0 \quad (33)$$

$$\begin{bmatrix} W(\theta) & \Psi_B^T(\theta) \\ \Psi_B(\theta) & \mathcal{N}(\theta) \end{bmatrix} > 0 \quad (34)$$

where $W(\theta)$ and $\mathcal{N}(\theta)$ are affine matrices functions of θ, with values $W_1, \ldots, W_l$ and $\mathcal{N}_1, \ldots, \mathcal{N}_l$, respectively, at the vertices of $\mathcal{B}$ and $\Psi_A(\theta)$, $\Psi_B(\theta)$, $\Psi_C(\theta)$ and Ψ_M are as defined in (15). Then the filter $\mathcal{F}$ with transfer function matrix

$$G_{\hat{z}y}(s) = RS^{-1}(sI - QS^{-1})^{-1}F \quad (35)$$

ensures that the estimation error system (4) is asymptotically stable for all $\theta \in \mathcal{B}$ and the asymptotic error variance satisfies

$$\mathbf{E}[e^T e] < \max_{i=1,\ldots,\ell} \mathrm{Tr}\,[W_i], \quad \forall\, \theta \in \mathcal{B}. \quad (36)$$

Proof. It is similar to the proof of Theorem 1, except that now

$$M^T = \begin{bmatrix} Y^T & \bullet \\ V^T & \bullet \end{bmatrix}, \quad M^{-T} = \begin{bmatrix} Z^{-T} & \bullet \\ U & \bullet \end{bmatrix} \quad (37)$$

and the matrix $M^{-T}\mathcal{T}$ is used *in lieu* of $\mathcal{T}$.

$\nabla\nabla\nabla$

Note that remarks similar to those related to Theorem 1 also apply to Theorem 2.

The next result is similar to that of Lemma 2; the proof follows along the same lines as for Lemma 2.

Lemma 3. Consider the system (1) and suppose that there exists an asymptotically stable filter with state-space realization (A_f, B_f, C_f) such that the estimation error system (4) satisfies (9) and (10) with $P(\theta) = Q_0$ and $W(\theta)$ which is affine in θ with values $W_1, \cdots, W_\ell$ at the vertices of $\mathcal{B}$. Then the conditions of Theorem 2 hold with the same matrix $W(\theta)$ and a sufficiently small $\epsilon > 0$.

It should be remarked that the filtering methods of Theorems 1 and the 2 are not equivalent and, in general, they provide different filters and upper-bounds for the asymptotic variance of the estimation error. In applications, both designs should be tested and the one that provides the best result, in terms of guaranteed $\mathcal{H}_2$ performance, should be adopted. An example in the next section illustrates this fact.

4. EXAMPLE

Consider the following example which has been studied in Geromel (1999) and Tuan *et al.* (2000)

$$\dot{x}(t) = \begin{bmatrix} 0 & -1 + 0.3\alpha \\ 1 & -0.5 \end{bmatrix} x(t) + \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} w(t)$$

$$y(t) = \begin{bmatrix} -100 + 10\beta & 100 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1 \end{bmatrix} w(t) \quad (38)$$

$$z(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

where α and β are bounded uncertain parameters.

Similarly to Tuan *et al.* (2000), the following two cases will be considered

(a) $|\alpha| \leq \gamma$ and $|\beta| \leq 1$;

(b) $|\alpha| \leq \gamma$ and $\beta = \alpha$.

We shall address the design of a robust $\mathcal{H}_2$ filter for the above system for different values of γ. Note that as the system (38) is asymptotically stable whenever $\alpha < 10/3$, it turns out that γ should be smaller than 10/3.

Optimum filters have been designed for the system (38) using Theorems 1 and 2. The optimum ϵ for each γ, in the sense of minimizing the upper-bound μ on the asymptotic error variance, has been obtained via a gridding procedure. Fig. 1 illustrates the behavior of μ obtained from Theorem 1 in function of ϵ for the case (b) with $\gamma = 1$. In this situation, the optimum ϵ is 0.9 and the corresponding minimum μ is 2.1856. Since, for this example, the results obtained from Theorem 1 are better than those from Theorem 2, only the former results will be presented and will be referred to as approach (A).

For comparison purpose, the following robust $\mathcal{H}_2$ filter design methods have also been implemented: (i) the approach of Tuan *et al.* (2000), which is based on an affine parameter-dependent Lyapunov function - referred to as (B); (ii) the quadratic stability based approach of Geromel (1999) - referred to as (Q). Fig. 2 displays the optimized upper-bound μ on the asymptotic estimation error variance in function of γ obtained from the three approaches as above for the case (a), whereas Fig. 3 refers to the case (b).

As expected, it can be readily concluded from Figs. 2 and 3 that the results for the approach

(A) are much superior than those for the quadratic stability method. On the other hand, the approach (A) overall gives better performance than the approach (B) for the case (a), whereas for the case (b) the approach (A) provides smaller upper-bounds μ than the approach (B) for all values of γ. Further, it should be remarked that the approach (A) can handle larger parameter uncertainty. Indeed, it turns out that the approach (Q) is restricted to $\gamma < 1.6$, which is the admissible uncertainty range for quadratic stability of the system (38), whereas the approach (B) provides solutions only for $\gamma < 3.05$. In contrast, the approach (A) can solve the problem for $\gamma < 10/3$, i.e. as long as the system (38) is asymptotically stable. Note that for $\gamma \geq 2$ the approach (A) provides significant performance improvement as compared to the approach (B).

It should be observed that the asymptotic error variance for a Kalman filter designed for the nominal system of (38) with $\alpha = \beta = 0$ and applied to that system with $\alpha = \beta = 1$ is 31.12, whereas for $\alpha = \beta = 3$ it becomes 10037.

5. CONCLUSIONS

This paper investigated the design of robust $\mathcal{H}_2$ filters for linear continuous time-invariant systems with uncertain convex bounded parameters in the matrices of the system state-space model. Two LMI filter design methods based on parameter-dependent Lyapunov functions have been proposed that guarantee asymptotic stability and an optimized upper-bound for the asymptotic variance of the estimation error in spite of significant parameter uncertainty. The proposed methods have been applied to an example proposed in the literature and exhibited superior performance as compared to the existing techniques.

6. REFERENCES

Bolzern, P., P. Colaneri and G. de Nicolao (1994). Optimal robust state estimation for uncertain linear systems. In *Proc. 1st IFAC Symp. Robust Control Design*, Rio de Janeiro, Brazil, 152–161.

de Souza, C.E. and A. Trofino (2000). An LMI Approach to the Design of Robust $\mathcal{H}_2$ Filters. In *Advances in Linear Matrix Inequality Methods in Control* (L. El Ghaoui and S.-I. Niculescu, Eds.), 175–185. SIAM.

Geromel, J.C. (1999). Optimal linear filtering under parameter uncertainty. *IEEE Trans. Signal Processing* **47**, 168–175.

Green, M. and D.J.N. Limebeer (1995). *Linear Robust Control.* Prentice-Hall.

Petersen, I.R. and A.V. Savkin (1999). *Robust Kalman Filtering for Signals and Systems with Large Uncertainties.* Birkhuser, Boston.

Petersen, I.R. and D.C. McFarlane (1991). Robust estimation for uncertain systems. In *Proc. 30th IEEE Conf. Decision and Control*, Brighton, UK, 2630–2631.

Shaked, U. and C.E. de Souza (1995). Robust minimum variance filtering. *IEEE Trans. Signal Processing* **43**, 2474–2483.

Tuan, H.D., P. Apkarian and T.Q. Nguyen (2000). Robust and reduced-order filtering: New LMI-based characterizations and methods. In *Proc. 2000 American Control Conf.*, Chicago, 1327–1331.

Xie, L. and Y.C. Soh (1994). Robust Kalman filtering for uncertain systems. *System & Control Letts.* **22**, 123–129.

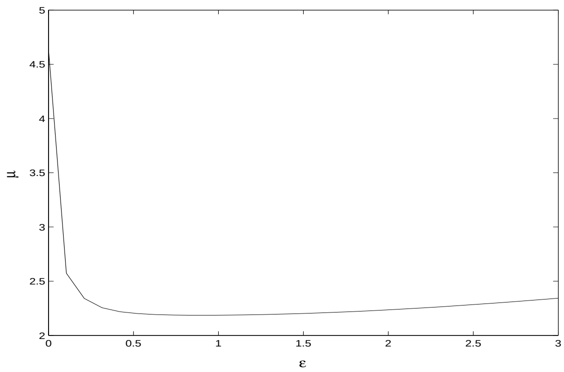

Fig. 1. μ versus ϵ for $|\alpha| \leq 1$ and $\beta = \alpha$.

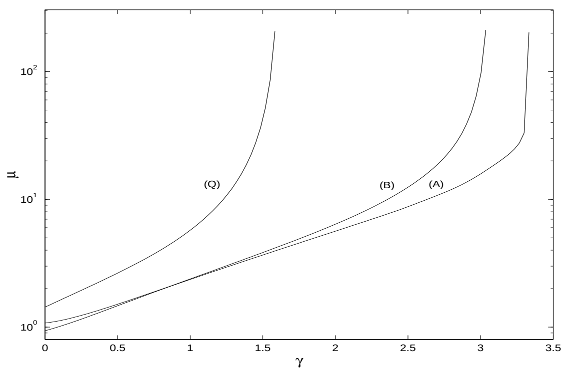

Fig. 2. Upper-bound μ for $|\alpha| \leq \gamma$ and $|\beta| \leq 1$.

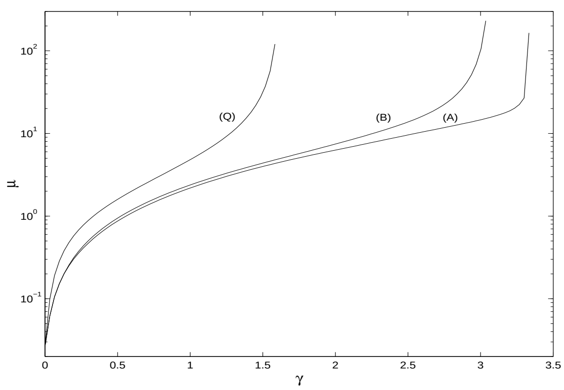

Fig. 3. Upper-bound μ for $|\alpha| \leq \gamma$ and $\beta = \alpha$.