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MULTIVARIABLE CONTROL OF DISCRETE EVENT SYSTEMS IN DIOIDS

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Abstract: Timed event graphs are described in dioids by a linear state-space or transfer function model. This paper proposes a state feedback control under complete control and observation of the states. By using a model reference control approach, the closed-loop transfer function matches the control specification expressed as a given transfer function. It is shown that under certain conditions an optimal controller exists, although two sub-optimal controllers can always be obtained.

Keywords: discrete-event dynamic systems, timed Petri-nets, synchronization, control.

1. INTRODUCTION

Discrete event systems or DES are characterized by a discrete state-space where state transitions are event-driven. A timed event graph or TEG is a particular class of timed Petri nets whose places have only one transition upstream and only one downstream. They are suitable to represent DES subject to only synchronization phenomena. Most of these systems can be described by a combination of maximization or minimization and addition operations. This yields to a non linear model in conventional algebra, although a 'linear' model can be obtained in an algebraic structure named dioids (Cohen *et al.*, 1989; Baccelli *et al.*, 1992). In this context, promising control approaches have been reported as in Cohen *et al.* (1989), where an optimal trajectory control was proposed. Cofer and Garg (1996) extended the notion of supervisory control to satisfy a temporal specification based on lattice theory. Recently, de Schutter and van den Boom (2001) proposed a predictive control approach and Menguy *et al.* (2000) reported an adaptive control approach for max-plus systems. This paper proposes an approach based on model reference control very similar to that presented by Cottenceau *et al.* (2001). However, instead of using a transfer function representation to establish the closed-loop model matching equation, we use a state-space representation. This paper is organized as follows. Section 2 states some background in dioids and residuation theory and it develops linear equations used to describe a TEG in a particular dioid. Mostly, this section contains existing results in the literature whose details can be found in Baccelli *et al.* (1992) and a review of linear system theory in dioids is found in Cohen *et al.* (1999). Sections 3 and 4 present our approach, that is, under certain conditions an optimal state feedback controller is obtained although two sub-optimal controllers can always be calculated. Illustrative examples are presented in section 5 followed by concluding remarks.

2. DIOIDS AND TEG MODELING

A dioid is a set $\mathcal{D}$ endowed with two operations denoted $\oplus$ (addition) and $\otimes$ (multiplication), both associative and both having neutral elements denoted ε and e respectively, such that: $\oplus$ is commutative and idempotent $(a \oplus a = a)$, $\otimes$ is distributive with respect to $\oplus$ and ε is absorbing for $\otimes\,(a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon)$. Dioids are algebraic structures where an idempotent addition has no inverse. This idempotent addition defines a (partial) order relation (denoted $\succeq$) as $a \succeq b \Leftrightarrow a = a \oplus b$ or equivalently $b \preceq a$. The upper bound of two elements a and b in $\mathcal{D}$ is $a \oplus b$ and the bottom element of $\mathcal{D}$ is ε. If $\mathcal{D}$ is complete, then the top element of $\mathcal{D}$ is $\top$. As in conventional algebra, the symbol $\otimes$ is frequently suppressed ($ab = a \otimes b$).

Theorem 1. (Minimum solution). Given a and b in a complete dioid $\mathcal{D}$, the inequality $x \succeq ax \oplus b$ has $x = a^* b$ as the least solution, where $a^* = e \oplus a \oplus a^2 \dots$ and $a^k = a \otimes a \otimes \dots \otimes a$ (k times).This solution is also the minimum one to the corresponding equation.

Lemma 2. (Algebra of matrices). The set of square matrices of order n with entries in a dioid $\mathcal{D}$, endowed with two operations (sum and multiplication) defined respectively as

$$A \oplus B = [a_{ij} \oplus b_{ij}] \tag{1a}$$

$$A \otimes B = [\bigoplus_{k=1}^{n} (a_{ik} \otimes b_{kj})] \tag{1b}$$

is a dioid denoted $\mathcal{D}^{n \times n}$ where $A \preceq B \Leftrightarrow a_{ij} \preceq b_{ij}$.

Remark 3. A non-square matrix can be dealt by completing lines and columns with null elements ε.

Definition 4. (Power series). A formal power series a in m commutative variables x_i $(i = 1 \dots m)$ with coefficients a_k in $\mathcal{D}$, $k = (k_1, \dots, k_m) \in \mathbb{Z}^m$, is defined by

$$a = \bigoplus_{k \in \mathbb{Z}^m} a_k x_1^{k_1} \dots x_m^{k_m} \tag{2}$$

Notation: $a = (a_k)$.

Lemma 5. (Algebra of power series). The set of power series endowed with two operations (sum and multiplication) defined respectively as

$$a \oplus b = (a_k \oplus b_k) \tag{3a}$$

$$a \otimes b = (\bigoplus_{i+j=k} a_i \otimes b_j) \tag{3b}$$

is a dioid denoted $\mathcal{D}[\![x_1, \dots, x_m]\!]$.

Its identity element is denoted e, corresponding to the series with coefficients $a_0 = e$ and $a_k = \varepsilon$ for $k \neq 0$. The null element is denoted ε and it corresponds to the series with all null coefficients ε. In a dioid $\mathcal{D}[\![x_1, \dots, x_m]\!]$, $a \preceq b \Leftrightarrow a_k \preceq b_k$.

Lemma 6. (Dioid $\mathcal{M}_{\text{in}}^{\text{ax}}[\![\gamma, \delta]\!]$). The set of power series in two variables (γ, δ) with (boolean) coefficients in $\{\varepsilon, e\}$ endowed with the sum and product operations defined by Equations (3a) and (3b) besides the additional simplification rules

$$\gamma^k \delta^t \oplus \gamma^l \delta^t = \gamma^{\min(k,l)} \delta^t \tag{4a}$$

$$\gamma^k \delta^t \oplus \gamma^k \delta^l = \gamma^k \delta^{\max(k,l)} \tag{4b}$$

is a dioid denoted $\mathcal{M}_{\text{in}}^{\text{ax}}[\![\gamma, \delta]\!]$.

Residuation is a lattice concept used to 'solve' equations, i.e. to invert mappings. The residual is an answer to the problem of solving $f(x) = b$ by considering the subset of so-called 'subsolutions', that is, values of x satisfying $f(x) \preceq b$ and taking the maximum element of the subset. The main result is given by Theorem 9 which uses the following definitions.

Definition 7. (Isotone and antitone mappings). A mapping f from an ordered set $\mathcal{D}$ into an ordered set $\mathcal{C}$ is isotone (antitone) if $\forall a, b \in \mathcal{D}$, $a \succeq b \Rightarrow f(a) \succeq f(b)$ $(f(a) \preceq f(b))$.

Definition 8. (Lower-semicontinuity). A mapping f from an ordered set $\mathcal{D}$ into an ordered set $\mathcal{C}$ is lower-semicontinuous if, for every (finite or infinite) subset X of $\mathcal{D}$, $f\big(\bigoplus_{x \in X} x\big) = \bigoplus_{x \in X} f(x)$.

Theorem 9. (Residuated mapping). Let f be an isotone mapping from the complete dioid $\mathcal{D}$ into the complete dioid $\mathcal{C}$. There exists a greatest solution $x_{\max} = f^\sharp(b)$ to the inequality $f(x) \preceq b$ iff $f(\varepsilon) = \varepsilon$ and f is lower-semicontinuous. The mapping f is said to be residuated and $f^\sharp$ is called its residual.

It is straightforward that $f(x) = a \otimes x$ and $g(x) = x \otimes a$ are both lower-semicontinuous mappings. Therefore, by Theorem 9 these mappings are both residuated and the following notation applies.

Definition 10. (Division). We use $f^\sharp(x) = a \backslash\!\!\circ\, x$ ('left division' by a) and $g^\sharp(x) = x /\!\!\circ\, a$ ('right division' by a) to indicate the residuals of left and right multiplication by a, respectively.

Lemma 11. (Division of matrices). Given two matrices $A \in \mathcal{D}^{m \times n}$ and $B \in \mathcal{D}^{m \times p}$, the left division of B by A is a $n \times p$ matrix given by

$$A \backslash\!\!\circ\, B = [\bigwedge_{k=1}^{m} (a_{ki} \backslash\!\!\circ\, a_{kj})] \tag{5}$$

where the symbol $\wedge$ stands for lower bound. Analogously for right division.

Lemma 12. (Division of power series). Consider two power series a and b. The left division of b by a is a power series given by

$$a \backslash\!\!\circ\, b = (\bigwedge_{i-j=k} a_j \backslash\!\!\circ\, b_i) \tag{6}$$

Analogously for right division.

Although various dioids can be used to represent a timed event graph or TEG, this paper assumes a description in dioid $\mathcal{M}_{\text{in}}^{\text{ax}}[\![\gamma, \delta]\!]$. In this case, it can be modeled by the following linear algebraic equations:

$$X = AX \oplus BU \tag{7a}$$

$$Y = CX \tag{7b}$$

Where U, X e Y are vectors of dimensions m, n and p corresponding to variables associated to m input transitions, n internal transitions and p output transitions, respectively. Matrices $A^{n\times n}$, $B^{n\times m}$ and $C^{p\times n}$ have entries in dioid $\mathcal{M}_{\text{in}}^{\text{ax}}[\![\gamma,\delta]\!]$ with only non-negative exponents, since tokens and bars in places introduce only non-negative integer values. By Theorem 1, Equation (7a) has the minimum solution $X = A^*BU$ for a given input U. Therefore, $Y = (CA^*B)U$ and the transfer function of the system H is then defined by

$$H = CA^*B \tag{8}$$

For example, consider a TEG as showed in Fig. 1.

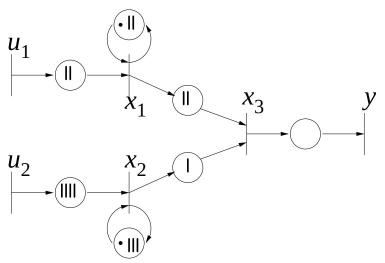

Fig. 1. Example of a TEG.

This system is described by the following state-space representation in dioid $\mathcal{M}_{\text{in}}^{\text{ax}}[\![\gamma,\delta]\!]$.

$$x_1 = (\gamma\delta^2)x_1 \oplus \delta^2 u_1 \tag{9a}$$
$$x_2 = (\gamma\delta^3)x_2 \oplus \delta^4 u_2 \tag{9b}$$
$$x_3 = \delta^2 x_1 \oplus \delta x_2 \tag{9c}$$
$$y = x_3 \tag{9d}$$

A realization (A,B,C) of this system is given by

$$A = \begin{pmatrix} (\gamma\delta^2) & \varepsilon & \varepsilon \\ \varepsilon & (\gamma\delta^3) & \varepsilon \\ \delta^2 & \delta & \varepsilon \end{pmatrix} B = \begin{pmatrix} \delta^2 & \varepsilon \\ \varepsilon & \delta^4 \\ \varepsilon & \varepsilon \end{pmatrix} C = \begin{pmatrix} \varepsilon \\ \varepsilon \\ e \end{pmatrix}' \tag{10}$$

The system transfer function is then given by

$$H = \left(\, \delta^4(\gamma\delta^2)^* \;\; \delta^5(\gamma\delta^3)^* \, \right) \tag{11}$$

3. STATE FEEDBACK CONTROL

The corresponding impulse response obtained from Equation (11) is shown by a dark gray area in Fig. 2. It represents the contribution of u_1 (on the left) and u_2 (on the right) to the output firings if an infinity number of input firings occurs at time $t = 0$ (Baccelli *et al.*, 1992). In this case, input u_1 contributes with one event each two time units and u_2 with one event each three time units. The control objective is to modify these impulse responses as shown, for instance, by a light gray area in Fig. 2. This control specification was obtained from the transfer function G_{ref} of Example 2. In a manufacturing context, this could represent a production rate reduction to satisfy a given demand, reducing intermediate stocks. This is equivalent to modify the system transfer function as below.

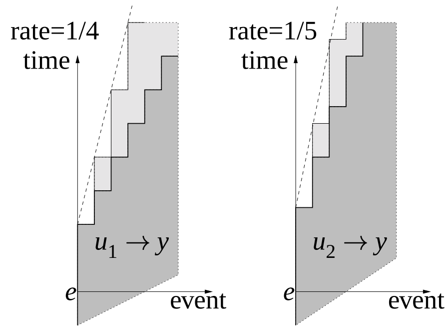

Fig. 2. Control specification as an impulse response.

Suppose all states x_i are controllable and observable. A transition is said to be controllable if its firing can be delayed or inhibited by an external input action. A transition is said to be observable if its temporal sequence of firings is accessible to the external world.

This paper proposes a feedback control framework by introducing a new control input that uses information from the states to actuate on the state transitions. This corresponds to $X = AX \oplus X_c \oplus BU$ where $X_c = FX$ is the control input and F is a controller. The state-space representation is then given by

$$X = (A \oplus F)X \oplus BU \tag{12a}$$
$$Y = CX \tag{12b}$$

Thus, the closed-loop transfer function is given by $G_{mf}(F) = C(A \oplus F)^*B$. This is depicted in Fig. 3.

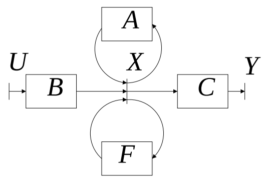

Fig. 3. State feedback control.

The controller F must be such that the closed-loop transfer function matches the control specification expressed as a given transfer function. This control approach is usually referred to as a model reference control which was also used by Cottenceau *et al.* (2001) with three particular control configurations: output feedback, state feedback (on input) and output feedback on state. The present approach is more general if the previous configurations are taken as restrictions to control and observation of the system state.

By considering a state feedback control as shown in Fig. 3, the model matching looks for the maximum controller $F_{\max}$ such that $C(A \oplus F_{\max})^*B \preceq G_{ref}$. The maximum or optimal controller are stated here in the context of residuation, i. e. delaying as maximum as possible a transition firing while providing the best matching to a given impulse response. By residuals of right and left products, this is equivalent to solving the following inequality

$$(A \oplus F)^* \preceq A_{ref} \tag{13}$$

where $A_{ref} = C \circ\!\!\!\backslash G_{ref} \circ\!\!\!/ B$.

Lemma 13. The inequality $(A \oplus F)^* \preceq A_{ref}$ has a solution iff $A^* \preceq A_{ref}$.

PROOF. ($\Rightarrow$) Suppose that there exists F_1 solution to $(A \oplus F)^* \preceq A_{ref}$. By definition, $A \preceq A \oplus F_1$ and $A^* \preceq (A \oplus F_1)^*$ due to isotony of product. Thus, $A^* \preceq A_{ref}$. ($\Leftarrow$) If $A^* \preceq A_{ref}$, then $(A \oplus \varepsilon)^* = A^* \preceq A_{ref}$ and $F = \varepsilon$ is a solution to $(A \oplus F)^* \preceq A_{ref}$.

Corollary 14. The inequality $C(A \oplus F)^* B \preceq G_{ref}$ has a solution iff $H \preceq G_{ref}$, where $H = CA^*B$ is the open-loop transfer function of the system.

PROOF. $C(A \oplus F)^* B \preceq CA_{ref}B \Leftrightarrow CA^*B \preceq CA_{ref}B$ by Lemma 13 and due to isotony of left and right products. But $CA_{ref}B = C(C \circ\!\!\!\backslash G_{ref} \circ\!\!\!/ B)B \preceq G_{ref}$ due to (A.1). Thus, $C(A \oplus F)^* B \preceq G_{ref} \Leftrightarrow CA^*B \preceq G_{ref}$.

This means that only G_{ref} delaying system events can be specified. The main result is then given by Theorem 15 whose proof is stated after intermediate Lemmas 16, 17 and 18 below.

Theorem 15. (Main theorem). If $A^* \preceq A_{ref}$, then $F_{\max}$ is the maximum solution to $(A \oplus F)^* \preceq A_{ref}$ iff $F_{\max}$ is the maximum solution to

$$F^* \preceq K \tag{14}$$

where $K = A^* \circ\!\!\!\backslash A_{ref} \circ\!\!\!/ A^*$.

Lemma 16. If F_1 is a solution to $(A \oplus F)^* \preceq A_{ref}$, then $(A \oplus F_1)^*$ is a solution too. Moreover, $F_1^* \preceq K$, where $K = A^* \circ\!\!\!\backslash A_{ref} \circ\!\!\!/ A^*$.

PROOF. By definition, $A \preceq A \oplus F_1$. Therefore,

$$\begin{aligned}
A^* &\preceq (A \oplus F_1)^* && \text{isotony of product}\\
A \oplus (A \oplus F_1)^* &= (A \oplus F_1)^* && \text{since } A \preceq A^*\\
(A \oplus (A \oplus F_1)^*)^* &= ((A \oplus F_1)^*)^* && \text{isotony of product}\\
(A \oplus (A \oplus F_1)^*)^* &= (A \oplus F_1)^* && \text{by (A.10)}\\
(A \oplus (A \oplus F_1)^*)^* &\preceq A_{ref} && F_1 \text{ is a solution}
\end{aligned}$$

Hence, $(A \oplus F_1)^*$ is a solution too. Moreover,

$$\begin{aligned}
(A \oplus F_1)^* &= (A \oplus F_1^*)^* && \text{by (A.8)}\\
(A \oplus F_1^*)^* &= (A^* F_1^*)^* A^* && \text{by (A.9)}\\
(A^* F_1^*)^* A^* &\preceq A_{ref} && F_1 \text{ is a solution}\\
(A^* F_1^*)^* &\preceq A_{ref} \circ\!\!\!/ A^* && \text{residual of right product}\\
A^* F_1^* &\preceq A_{ref} \circ\!\!\!/ A^* && \text{since } A^* F_1^* \preceq (A^* F_1^*)^*\\
F_1^* &\preceq A^* \circ\!\!\!\backslash A_{ref} \circ\!\!\!/ A^* && \text{residual of left product}
\end{aligned}$$

Lemma 17. If $F_{\max}$ is the maximum solution to $(A \oplus F)^* \preceq A_{ref}$, then $A^* \preceq F_{\max} = F_{\max}^*$.

PROOF. Suppose that $F_{\max}$ is the maximum solution. By definition, $F_{\max} \preceq A \oplus F_{\max}$. Then,

$$\begin{aligned}
F_{\max}^* &\preceq (A \oplus F_{\max})^* && \text{isotony of product}\\
F_{\max} &\preceq (A \oplus F_{\max})^* && \text{since } F_{\max} \preceq F_{\max}^*
\end{aligned}$$

Nevertheless, due to Lemma 13 $(A \oplus F_{\max})^*$ is also a solution which contradicts the fact that $F_{\max}$ is maximum. Hence,

$$\begin{aligned}
F_{\max} &= (A \oplus F_{\max})^* && \\
F_{\max}^* &= ((A \oplus F_{\max})^*)^* && \text{isotony of product}\\
F_{\max}^* &= (A \oplus F_{\max})^* && \text{by (A.10)}\\
F_{\max}^* &= F_{\max} &&
\end{aligned}$$

Moreover, $A \preceq A \oplus F_{\max}$ by definition. Consequently,

$$\begin{aligned}
A^* &\preceq (A \oplus F_{\max})^* && \text{isotony of product}\\
A^* &\preceq F_{\max} && \text{since } F_{\max} = (A \oplus F_{\max})^*
\end{aligned}$$

Lemma 18. If $A^* \preceq A_{ref}$, then $A^* \preceq K \preceq A_{ref}$, where $K = A^* \circ\!\!\!\backslash A_{ref} \circ\!\!\!/ A^*$.

PROOF. If $A^* \preceq A_{ref}$, then

$$\begin{aligned}
A^* \circ\!\!\!\backslash A^* &\preceq A^* \circ\!\!\!\backslash A_{ref} && f(x) = a \circ\!\!\!\backslash x \text{ is isotone}\\
A^* &\preceq A^* \circ\!\!\!\backslash A_{ref} && \text{by (A.3)}\\
A^* \circ\!\!\!/ A^* &\preceq A^* \circ\!\!\!\backslash A_{ref} \circ\!\!\!/ A^* && f(x) = x \circ\!\!\!/ a \text{ is isotone}\\
A^* &\preceq A^* \circ\!\!\!\backslash A_{ref} \circ\!\!\!/ A^* && \text{by (A.3)}
\end{aligned}$$

Since $A^* \succeq e$,

$$\begin{aligned}
A^* \circ\!\!\!\backslash A_{ref} &\preceq e \circ\!\!\!\backslash A_{ref} && f(x) = x \circ\!\!\!\backslash a \text{ is antitone}\\
e \circ\!\!\!\backslash A_{ref} &= A_{ref} && \text{by (A.4)}\\
A^* \circ\!\!\!\backslash A_{ref} \circ\!\!\!/ A^* &\preceq A_{ref} \circ\!\!\!/ A^* && f(x) = x \circ\!\!\!/ a \text{ is isotone}\\
A_{ref} \circ\!\!\!/ A^* &\preceq A_{ref} \circ\!\!\!/ e && f(x) = a \circ\!\!\!/ x \text{ is antitone}\\
A^* \circ\!\!\!\backslash A_{ref} \circ\!\!\!/ A^* &\preceq A_{ref} && \text{by (A.4)}
\end{aligned}$$

PROOF. [Main theorem] ($\Rightarrow$) If $F_{\max}$ is the maximum solution to $(A \oplus F)^* \preceq A_{ref}$, then $A^* \preceq F_{\max}$ by Lemma 17. Consequently,

$$\begin{aligned}
F_{\max} &= A \oplus F_{\max} && \text{since } A \preceq A^*\\
F_{\max}^* &= (A \oplus F_{\max})^* && \text{isotony of product}
\end{aligned}$$

Therefore, $F_{\max}^* = (A \oplus F_{\max})^* \preceq A_{ref}$. In other words, $F_{\max}$ can be made as great as $F_{\max}^* \preceq A_{ref}$. Hence, $F_{\max}$ is the maximum solution to $F^* \preceq A_{ref}$. However, $F_{\max}^* \preceq K$ by Lemma 16 and $K \preceq A_{ref}$ by Lemma 18. Thus, $F_{\max}$ can be made as great as $F_{\max}^* \preceq K$, i.e. $F_{\max}$ is the maximum solution to $F^* \preceq K$. ($\Leftarrow$) If $F_{\max}$ is the maximum solution to $F^* \preceq K$, then it is enough to prove that $F_{\max}$ is a solution to $(A \oplus F)^* \preceq A_{ref}$, since every solution F_1 to $(A \oplus F)^* \preceq A_{ref}$ is such that $F_1 \preceq F_{\max}$, by Lemma 16. Since $A^* \preceq K$ by Lemma 18, $F = A^*$ is a solution to $F^* \preceq K$ and $A^* \preceq F_{\max}$. Thus,

$$\begin{aligned}
A \oplus F_{\max} &= F_{\max} && \text{since } A \preceq A^* \preceq F_{\max}\\
(A \oplus F_{\max})^* &= F_{\max}^* && \text{isotony of product}\\
(A \oplus F_{\max})^* &\preceq K && \text{since } F_{\max}^* \preceq K\\
(A \oplus F_{\max})^* &\preceq A_{ref} && \text{by Lemma 18}
\end{aligned}$$

Corollary 19. $F_{\max} = K$ is a solution to $(A \oplus F)^* \preceq A_{ref}$ iff $K^* = K$, where $K = A^* \circ\backslash A_{ref} \circ/ A^*$. Moreover, it is the maximum one.

PROOF. $(\Rightarrow)$ If $F_{\max} = K$ is a solution to $(A \oplus F)^* \preceq A_{ref}$, then $K^* \preceq K$ by Lemma 13. Thus, $K^* = K$ since $K \preceq K^* \preceq K$. In this case, $F_{\max} = K$ is the maximum solution to $F^* \preceq K$ since $F^*_{\max} = K^* = K$ and every solution F_1 to $F^* \preceq K$ is such that $F_1 \preceq F_1^* \preceq K = F_{\max}$. $(\Leftarrow)$ If $K^* = K$, then $F_{\max} = K$ is a solution to $F^* \preceq K$ and thus it is the maximum one.

Corollary 19 provides an optimal state feedback controller for a system and specification satisfying $K^* = K$. Nevertheless, two sub-optimal controllers do exist.

4. SUB-OPTIMAL CONTROLLERS

The main result is two sub-optimal controllers given by Lemma 21 below. Moreover, if $K^* = K$, these controllers can be easily checked to be equal to the maximum controller K, by using (A.5). Therefore, they provide an alternative solution to the model matching when conditions imposed by Corollary 19 or even by Cottenceau *et al.* (2001) are not satisfied.

Lemma 20. Given $F_a = K \circ\backslash K$ and $F_b = K \circ/ K$, if $A^* \preceq A_{ref}$, then $A^* \preceq F_a = F_a^* \preceq K$ and $A^* \preceq F_b = F_b^* \preceq K$, where $K = A^* \circ\backslash A_{ref} \circ/ A^*$.

PROOF.

$$
\begin{array}{rl}
KA^* = (A^* \circ\backslash A_{ref} \circ/ A^*)A^* = K & \text{by (A.6)} \\
F_a = K \circ\backslash K = K \circ\backslash (KA^*) \succeq A^* & \text{by (A.2)} \\
e \preceq K & \text{by Lemma 18} \\
e(K \circ\backslash K) \preceq K(K \circ\backslash K) & \text{isotony of product} \\
F_a \preceq K & \text{by (A.7)} \\
F_a = F_a^* \preceq K & \text{by (A.5)}
\end{array}
$$

Analogously,

$$
\begin{array}{rl}
A^*K = A^*(A^* \circ\backslash A_{ref} \circ/ A^*) = K & \text{by (A.6)} \\
F_b = K \circ/ K = (A^*K) \circ/ K \succeq A^* & \text{by (A.2)} \\
(K \circ/ K)e \preceq (K \circ/ K)K & \text{isotony of product} \\
F_b \preceq K & \text{by (A.7)} \\
F_b = F_b^* \preceq K & \text{by (A.5)}
\end{array}
$$

Lemma 21. (Sub-optimal controllers). If $A^* \preceq A_{ref}$, then $F_a = K \circ\backslash K$ and $F_b = K \circ/ K$ are solutions to $(A \oplus F)^* \preceq A_{ref}$, where $K = A^* \circ\backslash A_{ref} \circ/ A^*$.

PROOF. It is similar for F_a and F_b as below.

$$
\begin{array}{rl}
A \oplus F_a = F_a & \text{by Lemma 20} \\
(A \oplus F_a)^* = F_a^* & \text{isotony of product} \\
(A \oplus F_a)^* \preceq K & \text{by Lemma 20} \\
(A \oplus F_a)^* \preceq A_{ref} & \text{by Lemma 18}
\end{array}
$$

5. EXAMPLES

Example 1. Given the example of Fig. 1, we wish to find the maximum controller $F_{\max}$ such that the closed-loop transfer function be equal to the open-loop transfer function of the system. In this case, G_{ref} should be set to $H = \begin{pmatrix} \delta^4(\gamma\delta^2)^* & \delta^5(\gamma\delta^3)^* \end{pmatrix}$.

According to Corollary 14 this problem has a solution since $H \preceq G_{ref}$. Moreover, every solution F satisfies $G_{mf}(F) = H$ due to Corollary 14. By calculating $A_{ref} = C \circ\backslash G_{ref} \circ/ B$ gives

$$
A_{ref} = \begin{pmatrix} \top & \top & \top \\ \top & \top & \top \\ \delta^2(\gamma\delta^2)^* & \delta(\gamma\delta^3)^* & \top \end{pmatrix} \tag{15}
$$

Hence, $K = A^* \circ\backslash A_{ref} \circ/ A^*$ is given by

$$
K = \begin{pmatrix} (\gamma\delta^2)^* & \delta^{-1}(\gamma\delta^3)^* & \delta^{-2}(\gamma\delta^2)^* \\ \varepsilon & (\gamma\delta^3)^* & \varepsilon \\ \delta^2(\gamma\delta^2)^* & \delta(\gamma\delta^3)^* & (\gamma\delta^2)^* \end{pmatrix} \tag{16}
$$

In this case, it can be shown that $K^* = K$ and by Corollary 19 the maximum controller that keeps unchanged the transfer function of the system is equal to K. According to (Cottenceau *et al.*, 2001), we use the maximum realizable controller $F_{\max}$ that is equal to K without terms with negative exponents, i. e.

$$
F_{\max} = \begin{pmatrix} (\gamma\delta^2)^* & \gamma\delta^2(\gamma\delta^3)^* & \gamma(\gamma\delta^2)^* \\ \varepsilon & (\gamma\delta^3)^* & \varepsilon \\ \delta^2(\gamma\delta^2)^* & \delta(\gamma\delta^3)^* & (\gamma\delta^2)^* \end{pmatrix} \tag{17}
$$

The Fig. 4 shows the controlled system where dotted places and transitions corresponds to the controller.

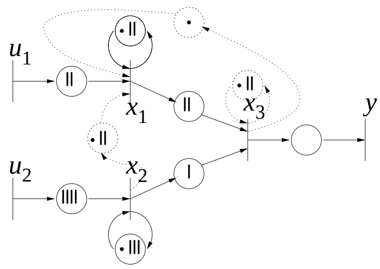

Fig. 4. Optimal state feedback controller.

In a manufacturing context, since tasks are subject to only synchronization, some of them can reach a maximum delay without change the overall system performance. In Fig. 4, internal transitions are delayed without change the non-controlled system behavior.

Example 2. $G_{ref} = \begin{pmatrix} \delta^4(\gamma\delta^4)^* & \delta^5(\gamma\delta^5)^* \end{pmatrix}$.

The corresponding impulse response is shown by a light gray area in Fig. 2. Thus, $A_{ref} = C \circ\backslash G_{ref} \circ/ B$ is

$$
A_{ref} = \begin{pmatrix} \top & \top & \top \\ \top & \top & \top \\ \delta^2(\gamma\delta^4)^* & \delta(\gamma\delta^5)^* & \top \end{pmatrix} \tag{18}
$$

Thus, $K = A^* \backslash A_{ref} / A^*$ is given by

$$K = \begin{pmatrix} (\gamma\delta^4)^* & \delta^{-1}(\gamma\delta^5)^* & \delta^{-2}(\gamma\delta^4)^* \\ \delta(\gamma\delta^4)^* & (\gamma\delta^5)^* & \delta^{-1}(\gamma\delta^4)^* \\ \delta^2(\gamma\delta^4)^* & \delta(\gamma\delta^5)^* & (\gamma\delta^4)^* \end{pmatrix} \quad (19)$$

In this case, it can be shown that $K^* \neq K$. Nevertheless, two sub-optimal controllers are given by $F_a = K \backslash K$ and $F_b = K / K$ due to Lemma 21. By considering only the realizable part, it yields

$$F_a = \begin{pmatrix} (\gamma\delta^4)^* & \gamma\delta^4(\gamma\delta^5)^* & \gamma\delta^2(\gamma\delta^4)^* \\ \varepsilon & (\gamma\delta^5)^* & \varepsilon \\ \delta^2(\gamma\delta^4)^* & \delta(\gamma\delta^5)^* & (\gamma\delta^4)^* \end{pmatrix} \quad (20)$$

$$F_b = \begin{pmatrix} (\gamma\delta^4)^* & \gamma\delta^3(\gamma\delta^4)^* & \gamma\delta^2(\gamma\delta^4)^* \\ \delta(\gamma\delta^4)^* & (\gamma\delta^4)^* & \gamma\delta^3(\gamma\delta^4)^* \\ \delta^2(\gamma\delta^4)^* & \delta(\gamma\delta^4)^* & (\gamma\delta^4)^* \end{pmatrix} \quad (21)$$

Figs. 5 e 6 show the controlled system with controllers F_a and F_b, respectively in dotted line.

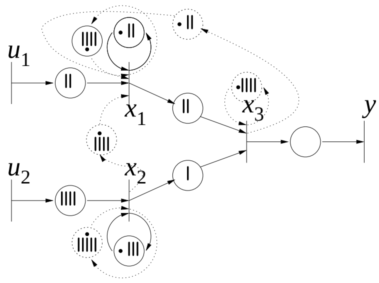

Fig. 5. Sub-optimal controller F_a.

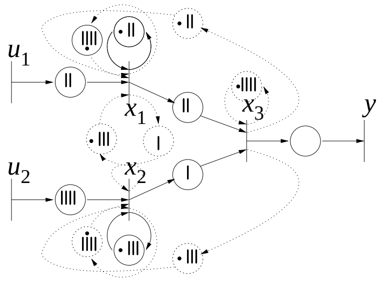

Fig. 6. Sub-optimal controller F_b.

The corresponding closed-loop transfer functions are: $G_{mf}(F_a) = \left(\delta^4(\gamma\delta^4)^* \;\; \delta^5(\gamma\delta^5)^* \right)$ and $G_{mf}(F_b) = \left(\delta^4(\gamma\delta^4)^* \;\; \delta^5(\gamma\delta^4)^* \right)$. Note that F_a and F_b are both solutions to the model matching Equation (13), although only F_a exactly matches the specification G_{ref}. Nevertheless, the controlled system in Fig. 6 is a strongly connected graph which has important stability properties (Baccelli *et al.*, 1992).

6. CONCLUDING REMARKS

This paper proposes a state feedback control under complete control and observation of the states. A model reference control is accomplished by solving a state-space model matching equation using residuation theory. This approach is more general than previous control configurations considered in literature if they are taken as restrictions to control and observation of the system state. Moreover, when certain conditions to obtain an optimal controller are not satisfied, two sub-optimal controllers provides an interesting alternative solution. Future work should consider partial access to states where usual feedback control configurations can be dealt in a uniform way.

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Appendix A. FORMULAE

In the following, a, b and x are elements of a dioid (Baccelli *et al.*, 1992).

$$a(a \backslash x) \preceq x \qquad (x / a)a \preceq x \quad (A.1)$$

$$a \backslash (ax) \succeq x \qquad (xa) / a \succeq x \quad (A.2)$$

$$a^*x = a^* \backslash (a^*x) \qquad xa^* = (xa^*) / a^* \quad (A.3)$$

$$e \backslash a = a \qquad a / e = a \quad (A.4)$$

$$(a \backslash a)^* = a \backslash a \qquad (a / a)^* = a / a \quad (A.5)$$

$$a^* \backslash x = a^*(a^* \backslash x) \qquad x / a^* = (x / a^*)a^* \quad (A.6)$$

$$a(a \backslash (ax)) = ax \qquad ((xa) / a)a = xa \quad (A.7)$$

$$(a \oplus b)^* = (a \oplus b^*)^* \quad (A.8)$$

$$b^*(ab^*)^* = (b^*a)^*b^* \quad (A.9)$$

$$(a^*)^* = a^* \quad (A.10)$$