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THE JOINT MODEL AND STATE ESTIMATION PROBLEM UNDER SET-MEMBERSHIP UNCERTAINTY

I.A.Digailova, A.B.Kurzhanski

Moscow State (Lomonosov) University
Faculty of Computational Mathematics & Cybernetics
119992, Moscow, Russia
e-mail: digailova_ira@mail.ru,
kurzhans@mail.ru

Abstract: The present paper deals with combined estimation of the state and of the system model for a continuous-time system under uncertain perturbations which may arise not only in the inputs but also in the system parameters. This produces model uncertainty of the *multiplicative* type. The uncertain items are assumed either unknown but bounded by hard or soft bounds or bounded with unspecified bound.
They are obtained in recurrent form, through appropriate versions of the Principle of Optimality which produces Dynamic Programming schemes. Also discussed is the case when the perturbations are *mixed* – partly unknown but bounded and partly with unspecified bound. The result is given by a pointwise or set-valued estimator for the state space variable as well as for the transition function of the investigated system.

Keywords: dynamic programming, system model, state estimation, multiplicative uncertainty, set-membership approach, H_∞ techniques, ellipsoidal calculus, principle of optimality.

INTRODUCTION

Among the methods of estimation and identification in the absence of whatever statistical data are those of set-membership estimation introduced and discussed in Schweppe(1973), Milanese et al.(1996), Walter et al.(1997), Kurzhanski(1972, 1977). A fairly large number of papers on this approach was related to state estimation for linear systems. Ellipsoidal methods in problems of control and state estimation were treated in Boyd et al.(1994), Chernousko(1994), Kurzhanski et al.(1977, 1997). The present paper deals with an estimation problem for continuous-time systems under set-membership uncertainty when the uncertain items are not only the additive inputs but also the transition (transfer) functions of the system.

Its solution requires a combined estimation of the system model together with the state space variables. Here given is a soft or hard *measure of uncertainty* for the uncertain items which is taken to be bounded with bound either known (set-membership type) or unspecified (H_∞-type). Using some relations from tensor analysis the original system is transformed, so that the estimated parameters are the state vector and the values of the transition (transfer) functions of the transformed system.

The problem would then be within the framework of identification under unmodelled dynamics. This allows to formulate a respective Principle of Optimality and apply the techniques of Dynamic Programming, ending up with recurrent procedures.

The present approach also allows to treat systems with mixed uncertainty, where each group of uncertain items is restricted in its own way. It also allows to consider combinations of stochastic and various types of set-valued and H_∞ -type disturbances, see Digailova et al.(2000).

1. THE SYSTEM

Given is a continuous-time system

$$\dot{x} = Ux + f(t), \tag{1}$$

whose trajectories are to be estimated from observations generated due to a measurement equation

$$y(t) = G(t)x + \xi(t). \tag{2}$$

Here $x \in \mathbb{R}^n$ is the *state* , $y(t)$ - the available *measurement*, U is an unknown matrix of dimensions $n \times n$ and $\xi(t)$ - the unknown measurement noise, while $x(t_0) = \tilde{x}(t_0)$ is the initial condition. The inputs $f(t)$ and the $m \times n$ matrix coefficients $G(t)$ are assumed to be given. A specific nature of the problem is that the uncertainty U is *multiplicative* (being multiplied by the unknown vector $x(t)$). Therefore the issues under consideration are those of *nonlinear* filtering.

The problem to be studied is to estimate the vector $x(t)$ under various types of assumptions on the uncertain items $\{U, \xi(t), x(t_0)\}, s \in [t_0, \tau]$.

Due to the nature of constraints on $\{U, \xi(t), x(t_0)\}$, $s \in [t_0, \tau]$, the solution estimate may be looked for as either *pointwise or set-valued*. The basic requirement here is that the estimate should be *recurrent*. We shall therefore deal with the coming problems through a Dynamic Programming approach based on respective versions of the Principle of Optimality.

2. REARRANGEMENT OF THE SYSTEM EQUATION

The original system (1) may be rearranged using the formulas of tensor analysis, particularly, the relation

$$\overline{XYZ} = (Z' \otimes X)\overline{Y} \tag{3}$$

Here $X \otimes Y$ stands for the Kronecker product of matrices X, Y (see, for example, Lancaster, 1969), and $\overline{X}$ stands for the $n \times m$-dimensional vector, constructed by stacking the n-dimensional columns $X^{(i)}$ of the $n \times m$ - matrix X, namely,

$$\overline{X} = \sum_{i=1}^{m} e^{(i)} \otimes (Xe^{(i)}),$$

where $e^{(i)}$ are the unit orths in $\mathbb{R}^m$.

For the original system, before the rearrangement, we have

$$x(t) = H(t)x^0 + \int_0^t H(t-s)f(s)ds. \tag{4}$$

Here $H(t) = \exp(Ut)$. Applying relation (3), we rearrange (4) as

$$x(t) = (x^{0\prime} \otimes I_n)\overline{H}(t) + \\ + \int_0^t (f'(t-s) \otimes I_n)\overline{H}(s)ds, \tag{5}$$

Here I_n is an $n \times n$ unit matrix. We further assume that the input $f(t)$ belongs to the class of functions that satisfy the equation

$$\dot{f} = Cf, \quad ||f(0)|| = 1,$$

where C is an $n \times n$ - dimensional diagonal matrix, so that

$$\dot{f}_i = c_{ii}f_i, \quad i = 1, \ldots, n.$$

As a result, we transform system (4) to the following form

$$x(t) = T(z^{(1)}(t) + z^{(2)}(t)), \tag{6}$$

where

$$\dot{z}^{(1)} = \mathcal{C}z^{(1)} + z^{(3)}, \dot{z}^{(2)} = \mathcal{X}^0 v(t), \\ \dot{z}^{(3)} = v(t), \tag{7}$$

$$z^{(1)}(0) = \bar{0}_n, \; z^{(2)}(0) = \mathcal{X}^0 \bar{I}_n, \; z^{(3)}(0) = \bar{I}_n.$$

Here $T = (I_n, I_n, \ldots, I_n)$ is a matrix of dimension $n \times n^2$ and $\mathcal{C} = C \otimes I_n, \;\; \mathcal{X}^0 = X^0 \otimes I_n$, where X^0 is a diagonal matrix whose elements $x_{ii} = x_i^0$.

Note that here $z^{(3)}(t) = \overline{H}(t)$ and $z^{(i)} \in \mathbb{R}^{n^2}, \; i = 1, 2, 3$.

In order to pose the state estimation problem, we now have to specify some assumptions on the uncertain items $\{U, \xi(t), x(t_0)\}, \quad t \in [t_0, \tau]$. Since $G(t)$ and the weighting matrices $K(s), N(s)$ in the forthcoming cost functionals are time-dependent, we further assume the starting time to be t_0.

3. THE UNCERTAIN ITEMS

Case 1. All the uncertain items $\{U, \xi(t), x(t_0)\}, \quad t \in [t_0, \tau]$ are *unknown but bounded* (UB) with either a *soft (integral) bound*

$$\Phi(\tau, \zeta(\cdot)) \leq 1, \tag{8}$$

$$\Phi(\tau, \zeta(\cdot)) =$$

$$\int_{t_0}^{\tau} \left(||v(t) - v^*(t)||^2_{K(t)} + ||\xi(t) - \xi^*(t)||^2_{N(t)} \right) dt +$$

$$+||z^{(1)}(t_0)||^2_{L_1} + ||z^{(2)}(t_0) - \mathcal{X}^0 \bar{I}_n||^2_{L_2}$$
$$+||z^{(3)}(t_0) - \bar{I}_n||^2_{L_3},$$

where $\zeta(\cdot) = \{z(t_0), v(\cdot)), \xi(\cdot)\}$, $z(t_0) = \{z^{(i)}(t_0),\ i = 1,2,3\}$, $||x||^2_Q = (x, Qx)$,

or a *hard (instantaneous) bound*

$$\Psi(\tau, \zeta(\cdot)) \leq 1, \tag{9}$$

$$\Psi(\tau, \zeta(\cdot)) =$$
$$= \max\{\Psi_i(\tau, z(t_0), v(\cdot), \xi(\cdot)) | i = 1,2,3\},$$
$$\Psi_1(\tau, v(\cdot)) =$$
$$= \mathrm{esssup}\{||v(t) - v^*(t)||^2_{K(t)} | t \in [t_0, \tau]\},$$
$$\Psi_2(\tau, \xi(\cdot)) =$$
$$= \mathrm{esssup}\{||\xi(t) - \xi^*(t)||^2_{N(t)} | t \in [t_0, \tau]\},$$
$$\Psi_3(\tau, z(t_0)) = ||z^{(1)}(t_0)||^2_{L_1} +$$
$$+||z^{(2)}(t_0) - \mathcal{X}^0 \bar{I}_n||^2_{L_2} + ||z^{(3)}(t_0) - \bar{I}_n||^2_{L_3}.$$

The functions $v^*(t) = U^* \exp(U^* t)$, $\xi^*(t)$, and vectors $\mathcal{X}^0 \bar{I}_n$ are supposed to be known, with matrices $K(t) \geq 0, N(t) > 0$, and $L_i = \varepsilon_i I_{n^2},\ \varepsilon_i > 0$. □

Case 2. Given are *measures of uncertainty* $\Phi(\tau, z(t_0), v(\cdot), \xi(\cdot))$ or $\Psi(\tau, z(t_0), v(\cdot), \xi(\cdot))$ of Case 1, but the bounds on these measures are unspecified. □

Remark 3.1 The restrictions on the uncertain matrices U are taken in the form of constraints on the derivative of the transition ("Greene") function $H(s)$ rather than on the matrices themselves. This leads to a conservative estimate but ensures the set-valued estimates the consistency sets of forthcoming Problem 1 to be convex.

4. SOLUTION SCHEME. CASE 1 (UB)

Problem 1. Given measurement $y(t) = y^*(t),\ t \in [t_0, \tau]$, find set $\mathcal{X}(\tau)$ of states $x(\tau)$ consistent with equations (6), (7), (2) and soft constraint (8)(case 1-a) or hard constraint (9)(case 1-b).□

Case 1-a. Assuming measurement $y^*(\tau),\ t \in [t_0, \tau]$ known, let us look for the value function

$$V(\tau, z) =$$
$$= \min\{\Phi(\tau, z(t_0), v(\cdot), \xi(\cdot)) | v(\cdot), \xi(\cdot)\}, \tag{10}$$

under restrictions (6), (7) and

$$z(\tau) = z, \quad y(t) \equiv y^*(t), \quad t \in [t_0, \tau]. \tag{11}$$

Then the *information set* of states $z(\tau)$ consistent with system (6), (7), measurement $y^*(\tau),\ t \in [t_0, \tau]$ and constraint (8) is (tee Kurzhanski et al. 1997) the *level set*

$$\mathcal{Z}(\tau) = \{z : V(\tau, z) \leq 1\}. \tag{12}$$

The estimates for $x(\tau), \overline{H}(\tau)$ can then be obtained as certain projections of $\mathcal{Z}(\tau)$.

Function $V(\tau, z) = V(\tau, z|t_0, V(t_0, \cdot))$ satisfies the *Principle of Optimality*: for $t \in [t_0, \tau]$,

$$V(\tau, z|t_0, V(t_0, \cdot)) = V(\tau, z|t, V(t, \cdot|t_0, \cdot)). \tag{13}$$

and can be sought for as a quadratic form

$$V(t, z) =$$
$$= (z - z^*(t), \mathcal{P}(t)(z - z^*(t))) + k^2(t). \tag{14}$$

Using standard techniques of Dynamic Programming, we then come to the relations

$$\dot{z}^* = \mathcal{A} z^* + \mathcal{B} v^*(t) +$$
$$+\mathcal{P}^{-1} \mathcal{T}_1' G'(t) N(t) (y^*(t) -$$
$$-G(t) \mathcal{T}_1 z^* - \xi^*(t)), \tag{15}$$

with

$$z^{*\prime}(t_0) = \{z^{(1)\prime}(t_0), z^{(2)\prime}(t_0), z^{(3)\prime}(t_0)\}' =$$
$$= \{(\bar{0}_n)', (\mathcal{X}^0 \bar{I}_n)', (\bar{I}_n)'\}, \tag{16}$$

$$\dot{\mathcal{P}} = -\mathcal{P}\mathcal{A}' - \mathcal{A}\mathcal{P} - \mathcal{P}\mathcal{B}K^{-1}(t)\mathcal{B}'\mathcal{P} +$$
$$+\mathcal{T}_1' G'(t) N(t) G(t) \mathcal{T}_1, \quad \mathcal{P}(t_0) = \mathcal{L}, \tag{17}$$

and

$$\dot{k}^2 = \|y^*(t) - G(t)\mathcal{T}_1 z^* - \xi^*(t)\|^2_{N(t)}, \tag{18}$$

with $k^2(t_0) = 0$.

Here $\mathcal{T}_1 = (T, T, R)$ is a matrix of dimension $n \times 3n^2$ and $R = (0_n, \dots, 0_n)$ is a zero-valued matrix of dimension $n \times n^2$,

$$\mathcal{A} = \begin{pmatrix} \mathcal{C} & 0_{n^2} & I_{n^2} \\ 0_{n^2} & 0_{n^2} & 0_{n^2} \\ 0_{n^2} & 0_{n^2} & 0_{n^2} \end{pmatrix}, \mathcal{B} = \begin{pmatrix} 0_{n^2} \\ \mathcal{X}^0 \\ I_{n^2} \end{pmatrix},$$

$$\mathcal{L} = \begin{pmatrix} \varepsilon_1 I_{n^2} & 0_{n^2} & 0_{n^2} \\ 0_{n^2} & \varepsilon_2 I_{n^2} & 0_{n^2} \\ 0_{n^2} & 0_{n^2} & \varepsilon_3 I_{n^2} \end{pmatrix}.$$

Theorem 5.1 (i)The solution to Problem 1-a is given by the vector $x^*(\tau) = \mathcal{T}_1 z^*(\tau)$.

(ii) The set $\mathcal{X}(\tau)$ of states $x(\tau)$ of system (1) consistent with measurement $y^*(t),\ t \in [t_0, \tau]$, may be found as $\mathcal{X}(\tau) = \mathcal{T}_1 \mathcal{Z}(\tau),\ \mathcal{Z}(\tau) = \{z : V(\tau, z) \leq 1\}$. It is an ellipsoid [1]

$$\mathcal{X}(\tau) =$$

[1] Denote $\mathcal{E}(q, Q) = \{x : (x - q, Q^{-1}(x - q) \leq 1\}$.

$$= \mathcal{E}(\mathcal{T}_1 z^*(\tau), (1-k^2(\tau))(\mathcal{T}_1\mathcal{P}(\tau)\mathcal{T}_1')^{-1}), \quad (19)$$

so that any possible state $x(\tau) \in x^*(\tau) + \mathcal{E}_x[\tau]$, where $\mathcal{E}_x[\tau] = \mathcal{E}(T\bar{0}_n, (1-k^2(\tau))(\mathcal{T}_1\mathcal{P}(\tau)\mathcal{T}_1')^{-1})$ is the *estimation error.*

(iii) The estimate $\overline{H}^*(t)$ for the transition function $\overline{H}(t)$ is given by the variable

$$\overline{H}^*(t) = \mathcal{T}_2 z^*(t), t \in [t_0, \tau], \quad (20)$$

and the respective error

$$\mathcal{E}_H[t] = \mathcal{E}(T\bar{0}_n, (1-k^2(t))(\mathcal{T}_2\mathcal{P}(t)\mathcal{T}_2)^{-1}).$$

Here $\mathcal{T}_2 = (R, R, T)$ is a matrix of dimension $n \times 3n^2$.

Therefore any possible realization $\overline{H}(t) \in \mathcal{H}(t)$, where $\mathcal{H}(\cdot)$ is an ellipsoidal tube generated by sets $\mathcal{H}(t) = \mathcal{E}(\overline{H}^*(t), (1-k^2(t))(\mathcal{T}_2\mathcal{P}(t)\mathcal{T}_2)^{-1})$, $t \in [t_0, \tau]$. □

Case 1-b. This case is treated through a substitution of constraints (9) by one single constraint

$$h^{-1}(\tau)\Phi(\tau, \zeta(\cdot)|\alpha, \beta(t), \gamma(t),\ t \in [t_0, \tau]) \leq 1,$$

where

$$\Phi(\tau, \zeta(\cdot)|\alpha, \beta(t), \gamma(t),\ t \in [t_0, \tau]) =$$
$$\int_{t_0}^{\tau} \Big(\beta(s)||v(s) - v^*(s)||^2_{K(s)} +$$
$$+\gamma(s)||\xi(s) - \xi^*(s)||^2_{N(s)}\Big)ds +$$
$$+\alpha\Big(||z^{(1)}(t_0)||^2_{L_1} + ||z^{(2)}(t_0) - \mathcal{X}^0\bar{I}_n||^2_{L_2}$$
$$+||z^{(3)}(t_0) - \overline{I}_n||^2_{L_3}\Big),$$

and $\alpha \geq 0, \beta(s) \geq 0, \gamma(s) \geq 0,\ s \in [t_0, \tau]$,

$$h(\tau) = \alpha + \int_{t_0}^{\tau} (\beta(s) + \gamma(s))ds. \quad (21)$$

Denote the triplet

$$\omega[\tau] = \{\alpha, \beta(t), \gamma(t) \mid t \in [t_0, \tau]\}$$

and the class of triplets that satisfy (21) as $\Omega[\tau]$.

Then (see Kurzhanski et al. 1997, Part IV) the value function

$$\mathcal{V}(\tau, z) = \min\{\Psi(\tau, \zeta(\cdot))|\zeta(\cdot)\}, \quad (22)$$

under restrictions (6), (7), (11) also satisfies the Principle of Optimality (13). It may be found as

$$\mathcal{V}(\tau, z) = \max\{\mathcal{V}(\tau, z|\omega[\tau])|\omega[\tau] \in \Omega[\tau]\}, (23)$$

where

$$\mathcal{V}(\tau, z|\omega[t]) = h^{-1}(\tau)V(\tau, z|\omega[t]),$$
$$V(\tau, z|\omega[t]) =$$
$$\min\{\Phi(\tau, \zeta(\cdot)|\omega[\tau])|z(t_0), v(\cdot), \xi(\cdot)\}$$

under conditions (6), (7), (11). The function $V(t, z|\omega[t]),\quad t \in [t_o, \tau]$ may be again sought for as a quadratic form that this time depends on $\omega[t]$:

$$V(t, z|\omega[t]) =$$
$$= (z - z^*_\omega(t), \mathcal{P}_\omega(t)(z - z^*_\omega(t)) + k^2_\omega(t).$$

The parameters of this form are described by equations similar to (15)-(18) but with the next substitutions:

$$K(s) \Rightarrow \beta(s)K(s),\ N(s) \Rightarrow \gamma(s)N(s), \quad (24)$$

$$\mathcal{L} \Rightarrow \alpha\mathcal{L}.$$

Following Kurzhanski et al., 1997, Part IV, we come to the assertion.

Theorem 5.2 (i)The set $\mathcal{X}(t)$ of states $x(t)$ consistent with measurement $y^*(s),\ s \in [t_0, t]$, is the projection

$$\mathcal{X}(t) = \Pi_x\mathcal{Z}(t) = \cup\{\mathcal{T}_1 z | z \in \mathcal{Z}(t)\}, \quad (25)$$

of the set $\mathcal{Z}(t) = \{z : \mathcal{V}(t, z) \leq 1\}$, where $\mathcal{Z}(t)$ is an intersection of ellipsoids

$$\mathcal{Z}(t) =$$
$$\bigcap_{\omega[t]\in\Omega[t]} \{\mathcal{E}(z^*_\omega(t), (1 - k^2_\omega(t))\mathcal{P}^{-1}_\omega(t))\ \}.$$

(ii) The following inclusion is true: $(\omega[t] \in \Omega[t])$

$$\mathcal{X}(t) \subseteq \mathcal{X}_+(t) = \quad (26)$$

$$= \bigcap_{\omega[t]\in\Omega[t]} \{\mathcal{E}(\mathcal{T}_1 z^*_\omega(t), (1 - k^2_\omega(t))(\mathcal{T}_1\mathcal{P}_\omega(t)\mathcal{T}_1')^{-1})\}.$$

(iii) The pointwise estimate $x^*(t)$ of the vector $x(t)$ is the "Chebyshev center" [2] of $\mathcal{X}(t)$, defined through the relation:

$$\max\{||x - x^*(t)||\ |x \in \mathcal{X}(t)\} =$$
$$\min_p \max_x\{||x - p||\ |x, p \in \mathcal{X}(t)\},$$

so that the estimation error set is $\mathcal{E}(t) = \mathcal{X}(t) - x^*(t)$.

(iv) The transition function $H(s)$ is estimated by the variable $\overline{H}^*(s)$ with $\overline{H}(s) \in \mathcal{H}(s),\ s \in [t_0, t]$, where $\mathcal{H}(\cdot)$ is the tube generated by sets

$$\mathcal{H}(s) = \Pi_H\mathcal{Z}(s) = \cup\{\mathcal{T}_2(s)z\ | z \in \mathcal{Z}(s)\}.$$

(v) The following inclusion is true

$$\mathcal{H}(t) =$$
$$\bigcap_{\omega[t]\in\Omega[t]} \{\mathcal{E}(\mathcal{T}_2 z^*_\omega(t), (1 - k^2_\omega(t))(\mathcal{T}_2\mathcal{P}_\omega(t)\mathcal{T}_2')^{-1})\ \}.$$

The pointwise estimate $\overline{H}^*(t)$ is the "Chebyshev center" of set $\mathcal{H}(t)$ and the error set is $\mathcal{H}(t) - \overline{H}^*(t)$. □

[2] The Chebyshev center of a compact set $\mathcal{X} \subseteq \mathbb{R}^n$ is the center of the smallest ball that includes $\mathcal{X}$.

5. SOLUTION SCHEME. CASE 2 (UUB)

Let us now suppose that the bound on $\Phi(\tau,\zeta(\cdot))$, $\Psi(\tau,\zeta(\cdot))$ is not specified as in (8) or(9). Starting with the "soft" functional $\Phi(\tau,\zeta(\cdot))$ we will consider the following problem.

Problem 2-a. Find the smallest number σ_0^2 among the numbers σ^2 that satisfy the inequality

$$\min_h \max_{\zeta(\cdot)}\{||x(\tau)-h||_Q^2-\sigma^2\Phi(\tau,\zeta(\cdot))\}\le 0$$

under restrictions $y(t)=y^*(t),\ t\in[t_0,\tau], x(\tau)=\mathcal{T}_1 z(\tau)$, due to equations (6), (7), (2). □

The last problem is to determine the smallest number σ^2 that allows

$$\min_h \max_z\{||\mathcal{T}_1 z-h||_Q^2\}\le\sigma^2 V(\tau,z), \quad (27)$$

under restriction $V(\tau,z)\le r^2$, whatever is the number r^2. Here $Q>0$ is given.

Lemma 5.1. For $V(\tau,z)=$

$$=(z-z^*(\tau),\mathcal{P}(\tau)(z-z^*(\tau))+k^2(\tau)$$

relation (27) is equivalent to the following

$$||\mathcal{T}_1 z-\mathcal{T}_1 z^*(\tau)||_Q^2\le\sigma^2 V(\tau,z). \quad (28)$$

□

Our aim is therefore to find the smallest σ^2 that satisfies (28), whatever be the vector $z\in\mathbb{R}^{3n^2}$ or, in other words, whatever be the bound r^2 in the inequality $V(\tau,z)\le r^2$. (Note that vector $z^*(\tau)$ *does not depend* upon the bound r^2).

Suppose r^2 is known. Then we could find the smallest number $\sigma^2(r)$ that ensures the inclusion

$$\mathcal{E}(\mathcal{T}_1 z-\mathcal{T}_1 z^*(\tau)),(r^2-k^2(\tau))(\mathcal{T}_1'\mathcal{P}^{-1}(\tau)\mathcal{T}_1)^{-1}$$
$$\subseteq\mathcal{E}(x-x^*(\tau)),\sigma^2r^2Q^{-1}),$$

for any $z\in\mathbb{R}^{3n^2}$, provided $x=\mathcal{T}_1 z,\ x^*=\mathcal{T}_1 z^*$. This yields

$$\sigma_0^2(r)=\frac{(r^2-k^2(\tau))}{r^2}\lambda_0^2(\tau), \quad (29)$$

$$\lambda_0^2(\tau)=\max\Big\{\frac{(l,(\mathcal{T}_1'\mathcal{P}^{-1}(\tau)\mathcal{T}_1)^{-1}l)}{(l,Q^{-1}l)}|l\in\mathbb{R}^n\Big\}.$$

In order that σ_0^2 would satisfy (28) *for any* r^2, we have to take

$$\sigma_0^2=\lim_{r\to\infty}\frac{(r^2-k^2(\tau))}{r^2}\lambda_0^2(\tau)=\lambda_0^2(\tau).$$

Summarizing the results, we have the following.

Theorem 6.1. The solution to Problem 2-a is given by vector $x^*(\tau)=\mathcal{T}_1 z^*(\tau)$ (the same as in Problem 1-a). The estimation error is determined through the relation

$$||x(\tau)-x^*(\tau)||_Q^2\le\lambda_0^2(\tau)V(\tau,z(\tau)),$$

where $x(\tau)=\mathcal{T}_1 z(\tau), x^*(\tau)=\mathcal{T}_1 z^*(\tau)$.

A similar result is true for estimating $\overline{H}(\tau)$ with $\mathcal{T}_1$ substituted for $\mathcal{T}_2$.□

Problem 2-b. Find the smallest number σ_0^2 among the numbers σ^2 that satisfy the inequality

$$\min_h \max_{\zeta(\cdot)}\{||x(\tau)-h||_Q^2-\sigma^2\Psi(\tau,\zeta(\cdot))\}\le 0$$

under restrictions $y(s)=y^*(s),\ t\in[t_0,\tau], x(\tau)=\mathcal{T}_1 z(\tau)$ due to equations (6), (7), (2).□

The inequality in Problem 2-b may be substituted by the next one.

$$\min_h \max_z\{||\mathcal{T}_1 z-h||_Q^2-\sigma^2\mathcal{V}(\tau,z)\}\le 0, \quad (30)$$

where $h\in\mathbb{R}^n,\ z\in\mathbb{R}^{3n^2}$. However, since $\Psi(\tau,\zeta(\cdot))$ and therefore $\mathcal{V}(\tau,z)$ are nonquadratic, the further solution will be more complicated than in case 2-a.

Let $z^*(\tau,r)$ be *the Chebyshev center* of the set $\mathcal{Z}(\tau,r)=\{z:\mathcal{V}(\tau,z)\le r^2\}$, so that

$$\min_w \max_z\{||z-w||\ \ |w\in\mathbb{R}^{3n^2},z\in\mathcal{Z}(\tau,r)\}=$$
$$=\max_z\{||z-z^*(\tau,r)||\ \ |z\in\mathcal{Z}(\tau,r)\}.$$

Then

$$\mathcal{Z}(\tau,r)=z^*(\tau,r)+\mathcal{Z}_0(\tau,r),$$

where $\mathcal{Z}_0(\tau,r)$ is *the error set* which, together with $z^*(\tau,r)$, depends on r.

Suppose the number r is given. Consider set

$$\mathcal{X}(\tau,r)=\mathcal{T}_1\mathcal{Z}(\tau,r)=\mathcal{T}_1 z^*(\tau,r)+\mathcal{T}_1\mathcal{Z}_0(\tau,r)$$

Then the *Chebyshev radius* r_c of this set will be

$$r_c(r)=$$
$$=\frac{1}{2}\max\{\rho(l|\mathcal{X}(\tau,r))+\rho(-l|\mathcal{X}(\tau,r))|(l,l)\le 1\},$$

where $\rho(l|\mathcal{X})=\sup\{(l,x)|x\in\mathcal{X}\}$ is the support function of set $\mathcal{X}$. We may now find the smallest $\sigma^2(r)$ for which the ball $\mathcal{E}(x^*(\tau),r_c^2(r)I_n)$ satisfies the inclusion

$$\mathcal{E}(x^*(\tau,r),r_c^2(r)I_n)\subseteq\mathcal{E}(x^*(\tau,r),\sigma^2r^2Q).$$

This gives

$$\sigma^2(r)=\frac{r_c^2(r)}{r^2}\lambda_Q,\quad \lambda_Q=\max\Big\{\frac{(l,l)}{(l,Ql)}\Big\}.$$

The final value of σ^2, which does not depend on r, is $\sigma^2=\sigma_0^2$

$$\sigma_0^2=\lambda_Q\lim_{r\to\infty}\Big\{\frac{r_c^2(r)}{r^2}|r>0\Big\} \quad (31)$$

This is due to the fact that the ratio $r_c^2(r)/r^2$ is bounded for $r>0$, which can be proved by constructing a lower majorant of $\mathcal{V}(\tau, z)$ as a nondegenerate quadratic form $V(\tau, z) = (z - z^*(\tau), \mathcal{P}(\tau)(z - z^*(\tau)))$ with $\mathcal{P}(\tau), z^*(\tau)$ independent of $\omega(\cdot)$.

Theorem 6.2. The solution $\sigma_0^2(\tau)$ to problem 2-b is given by relation (31). The pointwise estimate $x^*(\tau)$ for problem 2-b exists if the variety $\{z^*(\tau, r)\}$ is bounded for $r > 0.\square$.

A similar scheme is true for the transition function $\overline{H}(\tau)$ with $\mathcal{T}_1$ substituted for $\mathcal{T}_2$.

Approximate solutions to Problem 2-b can be reached by nondegenerate quadratic approximations of the value function $\mathcal{V}(\tau, z)$.

6. CONCLUSION

1. This report indicates an estimation technique for systems with uncertainty both in the model and in the system inputs. The restrictions, in the form of soft or hard bounds, are imposed on the unknown transition matrix functions $H(\cdot)$ of the system rather than on the system coefficients. The proposed estimators simultaneously produce both an estimate of the state $x(t)$ and an estimate of the unknown transition function.

2. The unified approach of this report is based on Dynamic Programming techniques which ensure recurrence in the estimation process.

3. The approach allows to consider mixed uncertainty when the acting disturbances may be unknown but bounded by hard or soft bounds, with bounds partly given and partly unspecified. It also allows to treat systems that also include stochastic disturbances.

4. The results are formulated through pointwise estimates or through set-valued estimators. Effective calculations are available by applying ellipsoidal techniques.

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