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ENERGY OF FRACTIONAL ORDER TRANSFER FUNCTIONS

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Abstract: The objective of the paper is to compute the impulse response energy of a fractional order transfer function having a single mode. The differentiation order *n*, defined in the sense of Riemann-Liouville, is allowed to be a strictly positive real number. A necessary and sufficient condition is established on *n*, in order for the impulse response to belong to the Lebesgue space $\mathbf{L_2}[0, \infty[$ of square integrable functions on $[0, \infty[$. *Copyright © 2002 IFAC*

Keywords: fractional order differentiation, impulse response energy, transfer function, dynamical system, fractional calculus

1. INTRODUCTION

Although as yet relatively unused in physics, the concept of differentiation to an arbitrary order (also called fractional differentiation) was defined in the 19th century by Riemann and Liouville. Their main concern was to extend differentiation by using not only integer but also non-integer (real or complex) orders. The n^{th} order derivative of fractional order is defined as (Samko, 1993):

$$\left(\mathbf{D^n}(x)\right)(t) = \frac{1}{\Gamma(m-n)}\left(\frac{d}{dt}\right)^m \left(\int_0^t \frac{x(\tau)d\tau}{(t-\tau)^{1-(m-n)}}\right)$$

with $t > 0$, $n > 0$ and $m = \lfloor n \rfloor + 1$. $\lfloor n \rfloor$ means integer part of n.

Studies on real systems such as thermal or electrochemical (Battaglia *et al.*, 2000, Cois *et al.*, 2000), reveal inherent fractional differentiation behaviour. The use of classical models (based on integer order differentiation) is thus inappropriate in representing these fractional systems. Thus, a further class (called fractional) of mathematical models has been provided since 1983 by (Oustaloup) using the concept of fractional differentiation. These models are based on either a fractional differential equation or a fractional state-space representation, namely (Matignon, 1994, Oustaloup, 1995, Cois *et al.*, 2001):

$$\begin{cases} \left(\mathbf{D}^n \mathbf{x}\right)(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,u(t) \\ y(t) = \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,u(t) \end{cases}. \qquad (1)$$

where n is a real, integer or non integer, number.

The modal decomposition of such a representation leads to express the system output as a linear combination of elements called eigenmodes governed by the following equation:

$$\mathbf{D}^n x(t) + \lambda x(t) = u(t),$$

where λ denotes a system eigenvalue. This latter equation can also be written (see (Oldham and Spanier, 1974) for example) in the Laplace domain as:

$$B_n(s) \overset{\Delta}{=} \frac{X(s)}{U(s)} = \frac{1}{s^n + \lambda}$$

One of the inherent system characteristics in control engineering is its impulse response energy. For instance, if the energy is finite, it can be concluded that a system belongs to $L_2[0, \infty[$, space of squared integrable functions.

Our concern in this paper is to compute the impulse response energy of $B_n(s)$ whatever the differentiation order n is. Up to our knowledge, the method presented herein is original since no work was found in the literature concerning the computation of impulse response energies of fractional order systems.

Based on the method proposed herein, other types of fractional order transfer functions can be treated.

2. FRACTIONAL STATE-SPACE REPRESENTATION

2.1. Definition and main properties

Definition. Fractional state-space representation is defined, as for classical state-space representation, by two equations (Matignon 1994, Oustaloup 1995):

- a state equation where the state vector is differentiated to a real, that is integer or non-integer, order. This vector has different properties compared to a classical state vector and is termed *fractional state vector*;

- an output equation, as in classical representation:

$$\begin{cases} \left(D^{(n)}\mathbf{x}\right)(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,u(t) \\ y(t) = \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,u(t) \end{cases}. \quad (2)$$

The n order derivative of the fractional state vector $\mathbf{x}(t)$ is defined by (Samko *et al.*, 1993):

$$\left(D^{n}\mathbf{x}\right)(t) \overset{\Delta}{=} \frac{1}{\Gamma(m-n)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m}\left(\int_{0}^{t}\frac{\mathbf{x}(\tau)}{(t-\tau)^{1-(m-n)}}d\tau\right), \quad (3)$$

with $t>0$, $n>0$ and $m=\lfloor n \rfloor + 1$.

The Laplace transform of this derivative (Oldham and Spanier, 1974), considering null initial conditions, is:

$$\mathcal{L}\left\{\left(D^{n}\mathbf{x}\right)(t)\right\} = s^{n}\mathbf{X}(s), \text{ where } \mathbf{X}(s)=\mathcal{L}(\mathbf{x}(t)). \quad (4)$$

Scalars $u(t)$ and $y(t)$ are system input and output respectively. $\mathbf{x}(t)$ is the fractional state vector. **A**, **B**, **C** and **D** are suitable dimension matrices.

2.2. Modal decomposition

The modal decomposition of complex-fractional systems is obtained as for classical systems. Applying a similarity transformation to representation, one can obtain a new realization, where matrix **A** becomes Jordan matrix **J**:

$$\begin{cases} \left(D_{0}^{n}\mathbf{x}_{\mathbf{J}}\right)(t) = \mathbf{J}\,\mathbf{x}_{\mathbf{J}}(t) + \mathbf{B}_{\mathbf{J}}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}_{\mathbf{J}}\mathbf{x}_{\mathbf{J}}(t) + \mathbf{E}_{\mathbf{J}}\,\mathbf{u}(t) \end{cases}. \quad (5)$$

The system eigenvalues are on the diagonal of matrix **J**. Using relation (4) and considering null initial conditions, the Laplace transform of the complex-fractional state equation is:

$$s^{n}\mathbf{x}_{\mathbf{J}}(s) = \mathbf{J}\,\mathbf{x}_{\mathbf{J}}(s) + \mathbf{B}_{\mathbf{J}}\,\mathbf{u}(s). \quad (6)$$

Then, from relation (5), system output is:

$$\mathbf{y}(t) = \mathcal{L}^{-1}\left\{\mathbf{C}_{\mathbf{J}}\left(s^{n}\mathbf{I}-\mathbf{J}\right)^{-1}\mathbf{B}_{\mathbf{J}}\right\} \otimes \mathbf{u}(t) + \mathbf{E}_{\mathbf{J}}\,\mathbf{u}(t), \quad (7)$$

where $\otimes$ denotes the convolution product. Matrix **J** is a Jordan matrix, so $\left(s^{n}\mathbf{I}-\mathbf{J}\right)^{-1}$ can be expressed as:

$$\left(s^{n}\mathbf{I}-\mathbf{J}\right)^{-1} = \begin{pmatrix} \left(s^{n}\mathbf{I}-\mathrm{J}_{d_1}(\lambda_1)\right)^{-1} & & & & \\ & \ddots & & 0 & \\ & & \left(s^{n}\mathbf{I}-\mathrm{J}_{d_l}(\lambda_l)\right)^{-1} & & \\ & 0 & & \ddots & \\ & & & & \left(s^{n}\mathbf{I}-\mathrm{J}_{d_r}(\lambda_r)\right)^{-1} \end{pmatrix} \quad (8)$$

with

$$\left(s^{n}\mathbf{I}-\mathrm{J}_{d_l}(\lambda_l)\right)^{-1} = \begin{bmatrix} \frac{1}{s^{n}-\lambda_l} & \left(\frac{1}{s^{n}-\lambda_l}\right)^{2} & \cdots & \left(\frac{1}{s^{n}-\lambda_l}\right)^{d_l} \\ & \frac{1}{s^{n}-\lambda_l} & & \vdots \\ & 0 & \ddots & \left(\frac{1}{s^{n}-\lambda_l}\right)^{2} \\ & & & \frac{1}{s^{n}-\lambda_l} \end{bmatrix} \quad (9)$$

System output $\mathbf{y}(t)$ is thus a vector composed of a linear combination of partial fractions called eigenmodes:

$\mathcal{L}^{-1}\left\{\left(\frac{1}{s^{n}-\lambda_l}\right)^{q_i}\right\} \otimes \mathbf{u}(t)$ where q_i is an integer number computed from the algebraic multiplicity of each eigenvalue λ_l.

2.3. Output analytical expression

Important results, obtained by Oustaloup and Matignon permit the analytical expression of fractional system output. Using the Mellin-Fourier inverse transformation and the residue theorem, the impulse response of an eigenmode can be expressed as:

$$h_l(t) = \sum_{k=1}^{\text{pole number}} \frac{p_k}{\lambda_l^{q_i}} Q_{q_i-1}\left(\frac{1}{n}, t\,p_k\right)\mathrm{e}^{t p_k} + \frac{1}{\pi}\int_{0}^{\infty} \frac{\mathrm{e}^{-tx}\sum_{k=0}^{q_i-1}(-1)^{k}\binom{q_i}{k}(\lambda_l)^{k}x^{n(q_i-k)}\sin[n\pi(q_i-k)]}{\left[x^{2n}-2\lambda_l x^{n}\cos(n\pi)+\lambda_l^{2}\right]^{q_i}}dx, \quad (10)$$

where the poles p_k of $H_l(s)$ are defined by:

$$\begin{cases} p_k = |p_k|\mathrm{e}^{\mathrm{j}\theta_k} \\ \theta_k \in \left]-\pi;\pi\right[\end{cases} \text{ with } \begin{cases} |p_k| = |\lambda_l|^{1/n} \\ \theta_k = \frac{\arg(\lambda_l)}{n} + \frac{2k\pi}{n} \\ -\frac{n}{2} - \frac{\arg(\lambda_l)}{2\pi} < k < \frac{n}{2} - \frac{\arg(\lambda_l)}{2\pi} \end{cases} \quad (11)$$

and where $Q_K(x,y)$ is a 2 variable polynomial defined by:

$$\begin{cases} Q_0(x,y) = x \\ \kappa\, Q_\kappa(x,y) = (xy + x - y)\, Q_{\kappa-1}(x,y) + xy \dfrac{\partial}{\partial y} Q_{\kappa-1}(x,y) \end{cases} \qquad (12)$$

Equation (10) shows the usual decomposition of fractional systems into two parts (Oustaloup, 1983):

- the *exponential mode*, resulting from the computation of residue(s) on each pole of $H_l(s)$ which generates an exponential behavior;

- the *aperiodic multimode*, the main characteristic of fractional systems, resulting from an integral along the negative real axis.

2.4. Stability condition

BIBO stability (Bounded Input-Bounded Output) is considered. A sufficient condition of this stability is given by:

$$\int_0^\infty \left| \mathbf{L}^{-1}\{H(s)\} \right| dt = K < +\infty, \qquad (13)$$

where $H(s)$ is the system transfer function. Using the structural decomposition of fractional systems (10), the stability condition is given by (Matignon, 1994):

$$\left|\arg(\lambda_l)\right| > \frac{n\pi}{2}, \text{ for } l=1,..,\dim(\mathbf{x}), \qquad (14)$$

where the λ_l are the eigenvalues of matrix **A**.

3. MAIN RESULT

The main result of the paper is announced in the following theorem:

Theorem

Let $B_n(s) = \dfrac{1}{s^n + \lambda}$ be a Laplace-domain transfer function defined for every $n \in \mathbf{R}^{+*}$ and for every $\lambda \in \mathbf{R}^{+*}$; then the Euclidean norm of $B_n(s)$ squared is:

$$\left\|B_n(j\omega)\right\|^2 = -\frac{\lambda^{\left(\frac{1}{n}-2\right)} \cot\left(\frac{\pi}{2} n\right)}{n \sin\left(\frac{\pi}{n}\right)} \quad \text{if } \frac{1}{2} < n < 2 \text{ and } n \neq 1 \qquad \textbf{(15a)}$$

$$\left\|B_1(j\omega)\right\|^2 = \frac{1}{2\lambda} \quad \text{if } n = 1 \qquad \textbf{(15b)}$$

$$\left\|B_n(j\omega)\right\|^2 = \infty \quad \text{if } 0 \le n \le \frac{1}{2} \text{ or } n \ge 2 \qquad \textbf{(15c)}$$

Proof
See appendix.

As a direct consequence, the following corollary is deduced.

Corollary

Let $b_n(t)$ be the impulse response of $B_n(s) = \dfrac{1}{s^n + \lambda}$, then $b_n(t) \in \mathbf{L_2}[0, \infty[$ if and only if $\dfrac{1}{2} < n < 2$.

Where $\mathbf{L_2}[0, \infty[$ is Lebesgue space of squared integrable functions on the interval $[0, \infty[$.

Remark
It is worth mentioning that the discontinuity at $n = 1$ in formula (15a) is removed by formula (1b). In other words, according to (1a) and (1b):

$$\lim_{n\to 1^-} \left\|B_n(j\omega)\right\|^2 = \lim_{n\to 1^+} \left\|B_n(j\omega)\right\|^2 = \left\|B_1(j\omega)\right\|^2$$

4. PLOT AND INTERPRETATION

It can be verified that $\left\|B_n(j\omega)\right\|^2$ is minimal for $n = 1$, by solving $\dfrac{d\left\|B_n(j\omega)\right\|^2}{dn} = 0$.

$\left\|B_n(j\omega)\right\|^2$ is plotted versus n in the following figure:

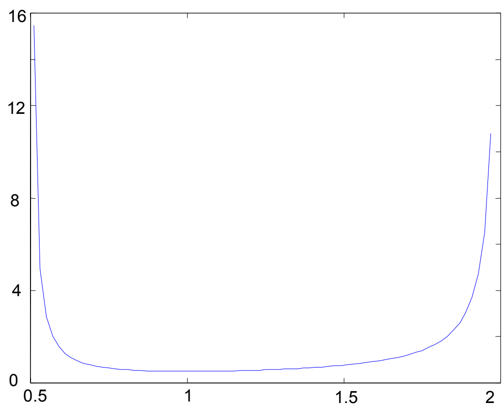
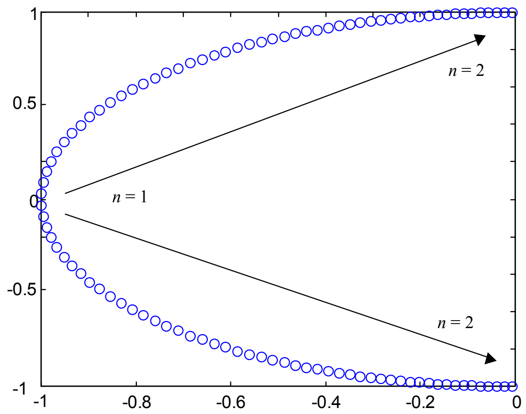


Fig 1 – Energy versus differentiation order

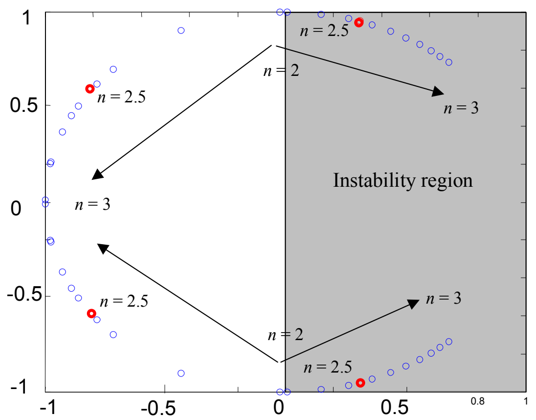
As can be seen the energy of $B_n(j\omega)$ tends to infinity as n tends to 2 which is due to the poles of $B_n(j\omega)$. Plotting pole locus versus n, it can be seen that the 2 complex conjugate poles of $B_n(s)$ tend to the $j\omega$ axis as n tends to 2. Hence, the corresponding impulse response becomes more and more oscillatory. When n reaches 2, the system becomes pseudo-stable and hence its energy infinite.

Beyond $n = 2$, the system is instable, because the number of poles of $B_n(s)$ is greater or equal to four, two of which at least are instable.

On the other hand, when the transfer function order is less than one, (Oustaloup, 1995, p.155 eq.4.83) shows that $B_n(s)$ has no pole and that the time domain impulse response of $B_n(s)$ is written as :



(a) for $1 < n \leq 2$ - Two distinct roots



(b) for $2 < n < 3$ - Four distinct roots

Fig. 2 – Pole locus versus differentiation order

$$b(t)=\frac{\lambda^2\sin(n\pi)}{\pi}\int_0^{\infty}\frac{x^n e^{-xt}\,dx}{\lambda^2+2\lambda x^n\cos(n\pi)+x^{2n}}$$

It is easy to check that

$$\lim_{t\to 0} b(t)=\infty \quad \text{as} \quad 0<n<1$$

from which follows the following remark.

Remark

$b_n(t) \in \mathbf{L_1}[0, \infty[$ if $1 \leq n < 2$, where $\mathbf{L_1}[0, \infty[$ is Lebesgue space of modulus integrable functions.

Hence, it is because of the behaviour of $b_n(t)$ next to the origin that $\|B_n(j\omega)\|^2 = \|b_n(t)\|^2 = \infty$ when ($n < 1/2$).

As shown in fig.3 through three examples, the surface of impulse responses squared (the computed energy) is expected to be minimal when $n = 1$. For lower values, the surface is augmented due to the behavior near zero. For bigger values, the system becomes oscillatory and hence the surface greater.

5. CONCLUSION AND PERSPECTIVES

In this correspondence, the impulse response energy of the fractional order transfer function $B_n(s) = (s-\lambda)^{-1}$ was computed for all values of n. Moreover, it was shown that $B_n(s)$ belongs to $\mathbf{L_2}[0, \infty[$ if and only if $1/2 < n < 2$. Actual work goes towards the generalization of this result to any fractional transfer function.

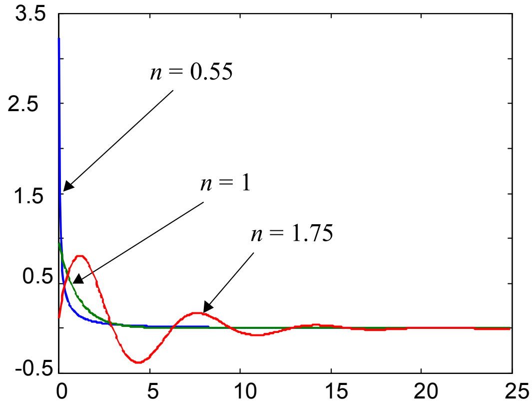


Fig. 3 – Impulse responses for three values of *n* with λ = 1

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7. APPENDIX - PROOF OF THE THEOREM

Energy of $B_n(s)$ can be written either in the time or frequency domain by applying Parseval's theorem which is valid only for stable systems. The stability of $B_n(s)$, depending on the number of poles, was widely studied in (Oustaloup, 1995). The conclusions are reported below in terms of the poles of the transfer function $B_n(s)$:

- ✓ If $0 < n < 1$ ➔ no pole,
- ✓ if $n = 1$ ➔ 1 pole,
- ✓ if $1 < n < 2$ ➔ 2 complex conjugate poles,
- ✓ if $n = 2$ ➔ 2 complex conjugate poles on the imaginary axis. The system is pseudo-stable and hence its energy infinite.
- ✓ if $n > 2$ ➔ at least 4 poles two of which at least have positive real parts. i.e. the system is unstable.

$B_n(s)$ is a multivalued complex function **if** $\boldsymbol{n \neq 1}$, hence it is important, at this stage to cut the complex plane **C** on]- ∞, 0]. The arguments of s are allowed to vary in the interval:]-π, π[.

Consequently, the norm of $B_n(s)$ will be computed only for $0 < n < 2$ and $n \neq 1$.

The energy of $b_n(t)$ is defined in the time domain as:

$$\left\| b_n(t) \right\|^2 = \int_{-\infty}^{\infty} b_n(t)\,\overline{b_n(t)}\,dt$$

where $\overline{b_n(t)}$ is the complex conjugate of $b_n(t)$.

Applying Parseval's theorem, for stable systems, yields the following:

$$\left\| B_n(j\omega) \right\|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{(j\omega)^n + \lambda} \left(\overline{\frac{1}{(j\omega)^n + \lambda}} \right) d\omega$$

or due to the symmetry in the frequencies:

$$\left\| B_n(j\omega) \right\|^2 = \frac{1}{\pi} \int_0^{\infty} \frac{1}{\left(\omega^n + \lambda e^{-j\frac{\pi}{2}n} \right)\left(\omega^n + \lambda e^{j\frac{\pi}{2}n} \right)} d\omega$$

Letting $\omega^n = x$, and $d\omega = \frac{1}{n} x^{\frac{1}{n}-1} dx$, yields the following integral to solve:

$$\left\| B_n(j\omega) \right\|^2 = \frac{1}{n\pi} \int_0^{\infty} \frac{x^{\frac{1}{n}-1}}{x^2 + 2\lambda \cos\left(\pi \frac{n}{2} \right) x + \lambda^2} dx \qquad \textbf{(16)}$$

The remaining part of the appendix is devoted to presenting the method used for solving (16).

By analogy to (16) define a complex valued function $G(z)$ as:

$$G(z) = \frac{z^{\frac{1}{n}-1}}{\left(z - \lambda e^{j\left(\pi\frac{n}{2} + \pi \right)} \right)\left(z - \overline{\lambda e^{j\left(\pi\frac{n}{2} + \pi \right)}} \right)}$$

Set:

$$I = \int_0^{\infty} G(z)\,dz \qquad \textbf{(17)}$$

The integral (16) is now equivalent to $\left\| B_n(j\omega) \right\|^2 = \frac{1}{n\pi} I$. For sake of simplicity a new plane cut is defined in the complex **C** plane:

- ✓ it excludes from **C** the axis [0, +∞[,
- ✓ arguments of complex numbers are allowed to vary in $\left]0, 2\pi\right[$. Hence, the second pole of $G(z)$ reads $\lambda e^{j\left(\pi - \pi\frac{n}{2} \right)}$.

To compute I, it is preferred to evaluate the integral of $G(z)$ along the contour Γ of fig.3 and the residues inside.

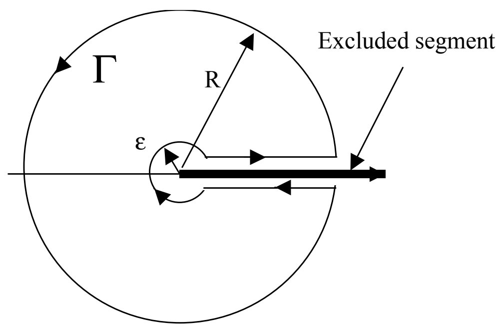

Fig. 4 – Integration contour

Applying residue theorem, the integral along Γ reads:

$$I_\Gamma = \int_\Gamma G(z)\,dz = 2\pi j \;\mathrm{Res}(G(z), poles)$$

A.1 Residues inside the closed contour

$$I_\Gamma = 2\pi j \left(Res\left(G(z), z = \lambda e^{j\left(\pi + \frac{\pi}{2}n \right)} \right) + Res\left(G(z), z = \lambda e^{j\left(\pi - \frac{\pi}{2}n \right)} \right) \right)$$

Evaluating the last expression yields:

$$I_\Gamma = \pi \frac{\lambda^{\left(\frac{1}{n} - 2 \right)}}{\sin\left(\frac{\pi}{2} n \right)} \left[e^{j\left(\pi - \frac{\pi}{2}n \right)\left(\frac{1}{n} - 1 \right)} - e^{j\left(\pi + \frac{\pi}{2}n \right)\left(\frac{1}{n} - 1 \right)} \right]$$

A.2 Integral along the small circle γ_ε

Set: $z \in \gamma_\varepsilon \Leftrightarrow z = \varepsilon e^{j\theta}$, θ varies from 2π to 0, $dz = \varepsilon j e^{j\theta} d\theta$

$$I_\varepsilon = \int_{\gamma\varepsilon} G(z)dz$$

$$I_\varepsilon = \int_{2\pi}^{0} \frac{\varepsilon^{\left(\frac{1}{n}-1\right)} e^{j\left(\frac{1}{n}-1\right)\theta} \varepsilon j e^{j\theta}}{\left(\varepsilon e^{j\theta} - \lambda e^{j\left(\pi+\frac{\pi}{2}n\right)}\right)\left(\varepsilon e^{j\theta} - \lambda e^{j\left(\pi-\frac{\pi}{2}n\right)}\right)} d\theta$$

If $\varepsilon \rightarrow 0$:

$$I_\varepsilon \approx \frac{n\varepsilon^{\frac{1}{n}}}{\lambda^2}\left[1 - e^{j\frac{2\pi}{n}}\right]$$

Consequently, $\lim_{\varepsilon \to 0} I_\varepsilon = 0$ if $n > 0$ which condition is always satisfied.

A.3 Integral along the big circle γ_R

$$I_R = \int_{\gamma_R} G(z)dz$$

$z \in \gamma_R \Leftrightarrow z = Re^{j\theta}$, θ varies from 0 to 2π, $dz = Rje^{j\theta}d\theta$

$$I_R = \int_{0}^{2\pi} \frac{R^{\left(\frac{1}{n}-1\right)} e^{j\left(\frac{1}{n}-1\right)\theta} Rje^{j\theta}}{\left(Re^{j\theta} - \lambda e^{j\left(\pi+\frac{\pi}{2}n\right)}\right)\left(Re^{j\theta} - \lambda e^{j\left(\pi-\frac{\pi}{2}n\right)}\right)} d\theta$$

If $R \rightarrow \infty$:

$$I_R \approx \frac{1}{\left(\frac{1}{n}-2\right)R^{2-\frac{1}{n}}}\left(e^{j2\pi\left(\frac{1}{n}-2\right)} - 1\right).$$

Hence,

$$\lim_{R\to\infty} I_R = 0 \text{ if } n > \frac{1}{2}$$

and $\lim_{R\to\infty} |I_R| = \infty$ if $\frac{1}{2} \geq n > 0$

A.4 Integral along positive and negative segments

Let γ_+ be the segment $z = x$, $\varepsilon < x < R$, $dz = dx$.

$$I_+ = \int_{\varepsilon}^{R} G(z)dz$$

Note that:

$\lim_{\substack{\varepsilon\to 0\\ R\to\infty}} I_+ = I$ where I is the desired integral of eq. (17).

Let γ_- be the segment $z = xe^{2\pi j}$, $R > x > \varepsilon$, $dz = dx$.

$$I_- = \int_{R}^{\varepsilon} G(z)dz$$

$$I_- = -e^{2\pi j\left(\frac{1}{n}-1\right)} \int_{\varepsilon}^{R} \frac{x^{\left(\frac{1}{n}-1\right)}}{\left(x - \lambda e^{j\left(\pi+\frac{\pi}{2}n\right)}\right)\left(x - \lambda e^{j\left(\pi-\frac{\pi}{2}n\right)}\right)} dx$$

$\lim_{\substack{\varepsilon\to 0\\ R\to\infty}} I_- = -e^{2\pi j\left(\frac{1}{n}-1\right)} I$, I being the integral of (17).

A.5 Deducing *I*

$$I_\Gamma = I_R + I_\varepsilon + I_+ + I_- = 2\pi j \sum Res(G(z), pôles)$$

Replacing each term by its computed value yields:

$$I = \pi \frac{\lambda^{\left(\frac{1}{n}-2\right)}}{\sin\left(\frac{\pi}{2}n\right)} \left(\frac{e^{j\left(\pi-\frac{\pi}{2}n\right)\left(\frac{1}{n}-1\right)} - e^{j\left(\pi+\frac{\pi}{2}n\right)\left(\frac{1}{n}-1\right)}}{1 - e^{2\pi j\left(\frac{1}{n}-1\right)}}\right)$$

Or after some tedious simplifications:

$$I = -\pi\lambda^{\left(\frac{1}{n}-2\right)} \frac{\cot\left(\frac{\pi}{2}n\right)}{\sin\left(\frac{\pi}{n}\right)}$$

Back to the norm: $\|B_n(j\omega)\|^2 = \frac{1}{n\pi} I$

$$\|B_n(j\omega)\| = -\lambda^{\left(\frac{1}{n}-2\right)} \frac{\cot\left(\frac{\pi}{2}n\right)}{n \sin\left(\frac{\pi}{n}\right)}$$

when ½ < n < 2 and $n \neq 1$

$\|B_n(j\omega)\| = \infty$ for $0 < n \leq ½$.

And due to the instability of $B_n(s)$ when $n \geq 2$:

$\|B_n(j\omega)\| = \infty$ for $n \geq 2$.

A.6 Special case *n* = 1

In this case the integral to solve is much easier:

$$\|B_n(j\omega)\|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{(\omega + \lambda j)(\omega - \lambda j)}$$

The solution is obtained by evaluating residues inside a closed contour without any plane cut. The result is straightforward:

$$\|B_n(j\omega)\|^2 = \frac{1}{2\lambda}$$