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SUBOPTIMAL LINEAR-FEEDBACK QUADRATIC-COST STOCHASTIC CONTROL FOR AN OBSERVABLE LINEAR SYSTEM WITH MULTIPLICATIVE NOISE

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Abstract:
A suboptimal approach to attack stochastic control problems, when the well known Linear-Quadratic-Gaussian (LQG) algorithm cannot be used, is proposed in this paper. The stochastic system here considered is described by an observable Ito equation with linear drift and bilinear diffusion. The aim of this paper is to provide the suboptimal linear feedback (SLF) control law, with optimality criterion given by the classical quadratic cost function, for this class of nonlinear systems. The SLF control is indeed an appropriate setting that guarantees a tradeoff between easy implementation and meaningful control-goal, whereas in general the optimal control problem involves the integration of an infinite-dimensional system.

Keywords: Stochastic Systems, Stochastic Control, Ito Equations, LQG Optimal Control, Ito formula, Projection Operator

1. INTRODUCTION

As well known, the linear-quadratic Gaussian (LQG), finite-horizon, optimal control problem admits a feedback solution resulting in a linear transformation either of the state-process values (for the *complete* information case) or of the state-conditional-expectation with respect to the observations (for the *incomplete* information case). The reader is referred to Bertsekas (1976) , Fleming and Rishel (1975), and references therein, for a complete explanation of the LQG control problem. In both the complete/incomplete information cases, the optimal solution results in the *same* matrix time-function (optimal control matrix) performing the linear map from the state or from the state-expectation respectively. This result represents a particular case of the *separation theorem* (Wonham (1968)). The optimal control matrix results to be defined by means of a backward Riccati difference/differential equation for the discrete/continuous-time cases respectively, depending only on the system-parameters, and on the weights of the performance-index. When one is faced by the incomplete-information case, the state conditional mean can be obtained by optimal filtering. To this purpose, since the LQG problem concerns a linear Gaussian system, the Kalman-Bucy filter (Kalman and Bucy (1961)) can be used.

The problem of "extending", in some sense, the LQG control-scheme, to wider classes of nonlinear systems has been considered in Charalambous and Elliot (1998), Charalambous and Elliot (1997). In these papers, it is shown that, for some classes of nonlinear systems and cost-criterions, the stochastic optimal control problem in the incomplete information case can be reduced to a complete-information optimal control problem on a *finite-dimensional system*. In general, (partially observable) stochastic optimal control problems have an infinite-dimensional state space. Indeed the separation theorem allows only to be sure that the optimal controller can be always expressed as a function of the *optimal estimate* (in the mean square sense). Hence, even thought this is very useful when a finite-dimensional filter can provide the optimal state estimate (as, for instance, in the LQG case), however when a nonlinear system is involved, and an optimal-state-estimate is required, one is forced to use an *infinite-dimensional* filter (as for instance the Duncan-Mortensen-Zakai equation Zakai (1969)), and this could be prohibitive in terms of computational effort. Nevertheless in Charalambous and Elliot (1998) it is shown that, under suitable assumptions (that is when the nonlinearity entering the unobservable dynamic are gradient of potentials and satisfy a generalized version of the Riccati equation) finite-dimensional sufficient statistics are available and allows to reduce the original (incomplete information) problem to a finite-dimensional and complete-information one.

Since the general problem (concerning an *optimal controller*) remains open, it makes sense to approach the control problem in a different way in order to safe both computational requirement and meaningfulness of the control-performance criterion.

At this purpose the problem of finding a *suboptimal* controller can be taken under consideration. This consists essentialy in relaxing the requirement for the controller to be optimal among *all* the observation functions. Some suboptimal *linear* feedback (SLF) control laws are derived in Charalambous and Elliot (1998) for some kind of nonlinearity entering the system.

In this paper the solution of SLF control problem is provided for a controlled system that does not belong to the class of systems studied in Charalambous and Elliot (1998). This is given by an observable stochastic differential equation (in the Ito sense) with linear drift, bilinear diffusion and an additive control.

The paper is organized as follows. In §2 the precise setting of the SLF control problem is given. In §3 the solution of the SLF control problem is presented for the complete state information case. For readers convenience two appendices are enclosed, namely A and B, collecting some concepts and results widely used throughout the paper.

2. STATEMENT OF THE SLF CONTROL PROBLEM

Let introduce the basic notations and symbols that will be used throughout the paper. $(\Omega, \mathcal{F}, P)$ will denote the basic probability triple. $\mathbf{E}\{\cdot\}$ denotes the expectation operator. $L^2(\mathcal{E})$, with $\mathcal{E}$ linear space, denotes the Hilbert space of all the $\mathcal{E}$-valued square-integrable random variables defined on $(\Omega, \mathcal{F}, P)$. Let $\mathcal{I}$ be a linear space endowed with some inner product, and $\xi, \eta \in \mathcal{I}$. The notation $\big(\xi\ ,\ \eta\big)$ will be used to denote the inner product between ξ and η. For any matrix M, the notation $M_{i,j}$ will be used to denote its (i,j)-entry. For the identity matrix in $\mathbb{R}^n$ it will be used the symbol I_n. Let I be a real interval and $\xi : I \to L^2(\mathbb{R}^i)$ an $\mathbb{R}^i$-valued stochastic process; denote with $\mathcal{F}_t^{\xi}$ the σ-algebra generated by $\{\xi_s;\ s \in I, s \le t\}$. For a vector-valued process $\{\xi_t\}$, the notation ξ_t^j shall indicate the j-th entry. If $\{\xi_t\}$ and $\{\eta_t\}$ are two second-order scalar martingales, the notation $\{\langle \xi, \eta \rangle_t\}$ will be used to indicate the mutual quadratic variation process. The notation $\langle \xi \rangle_t$ will be also used in place of $\langle \xi, \xi \rangle_t$, and will be said the quadratic variation of ξ. The reader is referred to Liptser and Shiryayev (1978) for the definition of quadratic variation of a martingale. Anyway, this paper is concerned only with martingales given by Ito integrals in the form $\int_0^t \zeta_\tau dW_\tau$ where ζ is an Ito-integrable process and W is the Wiener process. For such a martingale the quadratic variation process has a simple expression, and is given by the path-integral $\int_0^t \zeta_\tau^2 d\tau$. Similarly, given another process ζ' the mutual quadratic variation between $\int_0^t \zeta_\tau dW_\tau$ and $\int_0^t \zeta'_\tau dW_\tau$ is given by the path-integral $\int_0^t \zeta_\tau \zeta'_\tau d\tau$. When ξ and η are vector-valued, the same notation $\langle \xi, \eta \rangle$ will denote the matrix whose (i,j) entry is given by $\langle \xi^i, \eta^j \rangle$. Moreover, it will be used the symbol $\langle M \rangle_t^{(2)}$ to denote:

$$\langle M \rangle_t^{(2)} \doteq st \langle M \rangle_t \tag{2.1}$$

where $st(\cdot)$ denotes the stack operator (see Appendix A, formula (A.1)).

Let $\mathcal{S} \subset L^2(\mathcal{E})$ be a linear space and $X \in L^2(\mathcal{E})$; then the symbol $\mathbf{\Pi}\left\{X / \mathcal{S}\right\}$ will denote the orthogonal projection of X onto $\mathcal{S}$. Anytime the underlying space is understood it will be used the notation $\widehat{X}$ to denote the orthogonal projection. As well known, the projection $\widehat{X}$ represents the best (in the sense of the error-variance) estimate of X using estimators $\alpha \in \mathcal{S}$, and it is characterized by the following property:

$$\mathbf{E}\big\{(X - \widehat{X}, \alpha)\big\} = 0, \quad \forall \alpha \in \mathcal{S}. \tag{2.2}$$

Consider the following stochastic system:

$$dX_t = A(t)X_t dt + H(t)u_t dt + B(X_t, dW_t), \tag{2.3}$$
$$dY_t = C(t)X_t dt + dW_t, \tag{2.4}$$

with initial values: $X_{t_i} = X_i$, $Y_{t_i} = 0$, where $B : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the bilinear map:

$$B(X_t, dW_t) = \sum_{k=1}^{m} \lambda_k X_t dW_t^k, \tag{2.5}$$

where $\lambda_k \in \mathbb{R}$, $k = 1, ..., m$, and W^k denote the k-th entry of the standard m-dimensional Brownian motion, and where $t \in I$, $I = [t_i, t_f] \subset \mathbb{R}$,

$A : I \to \mathbb{R}^{n\times n}$, $H : I \to \mathbb{R}^{n\times p}$, $C : I \to \mathbb{R}^{m\times n}$, are continuous matrix functions. The σ-algebra generated by W will be denoted with $\mathcal{F}_t$. The initial point X_i is a deterministic vector in $\mathbb{R}^n$. As shown later, the hypotesis non-randomness for X_i is without loss of generality. The control function $u : (\Omega \times I) \to \mathbb{R}^p$ is assumed to be adapted to the non-decreasing family $\{\mathcal{F}_t^Y\}_{t\in I}$. The symbol $\mathcal{L}_t^i(Y)$ wiil be used to denote the set of $\mathbb{R}^i$-valued linear transformations of $\{Y_s;\ s \in I, s \le t\}$. One has that $\mathcal{L}_t^i(Y)$ is a closed linear subspace of $L^2(\mathbb{R}^i)$. Finally, let $\widehat{X}_t = \mathbf{\Pi}\left\{X_t/\mathcal{L}_t^n(Y)\right\}$, the *linear-optimal estimate* of the state X solution of (2.3). It is now possible to give a precise statement of the SLF control problem in the general case (incomplete information):

$$\min_{u\in\mathcal{L}^p(Y)} J(u), \tag{2.6}$$

$$J(u) = \frac{1}{2}\mathbf{E}\left\{\int_{t_i}^{t_f}\Big[\Big(X_t, Q(t)X_t\Big) + \Big(u_t, R(t)u_t\Big)\Big]dt + \Big(X_{t_f}, FX_{t_f}\Big)\right\}, \tag{2.7}$$

where $\forall t$, $Q(t) = Q(t)^T \ge 0$, $R(t) = R(t)^T > 0$, and $F = F^T \ge 0$, under the differential constraints represented by system (2.3), (2.4).

When the state process solution of eq. (2.3) is available the controlled system reduces to a single equation, namely the only state equation (2.3). Since the state process is available, we are concerned with a different family of controls than $\mathcal{L}^p(Y)$, previously considered for the general case. As a matter of fact, we will consider as admissible control functions the linear maps of X_t. This kind of space will be denoted with $\mathcal{L}_t^p(X)$. Hence, the SLF control problem for the *complete information case* can be stated as follows:

$$\min_{u\in\mathcal{L}^p(X)} J(u), \tag{2.8}$$

under the differential contraint represented by eq. (2.3) alone, where J is given by (2.7).

3. SOLUTION OF THE SLF CONTROL PROBLEM IN THE COMPLETE INFORMATION CASE

Theorem 3.1. *The solution of the problem* (2.8), *under the differential contraint* (2.3), *is the following:*

$$u_t^o = L^o(t)X_t, \tag{3.1}$$
$$L^o(t) = -R(t)^{-1}H(t)^TG(t), \tag{3.2}$$
$$\dot G(t) = -A(t)^TG(t) - G(t)A(t) - Q(t) + G(t)^TH(t)R(t)^{-1}H(t)^TG(t), \tag{3.3}$$
$$G(t_f) = F. \tag{3.4}$$

Proof. In the following time dependencies are omitted, for short, provided that this does not cause confusion. Since u has the form $u = LX$ with $L : [t_i, t_f] \to \mathbb{R}^{p\times n}$ continuous, the index J can be rewritten as:

$$J(L) = \frac{1}{2}\mathbf{E}\left\{\int_{t_i}^{t_f}\Big(X_\tau\ ,\ (Q + L^TRL)X_\tau\Big)d\tau + \Big(X_{t_f}\ ,\ FX_{t_f}\Big)\right\}.$$

Let $V(t)$ be the (unique) solution of the following (backward) equation

$$\dot V = -(A + HL)^TV - V(A + HL) - Q - L^TRL, \tag{3.5}$$

where the final condition is $V(t_f) = F$. Note that $V = V^T$. Let the function $\xi(s, \alpha)$, $s \in [t_i, t_f]$, be defined as: $\alpha \in \mathbb{R}^n$, as

$$\xi(t, X_t) = \Big(X_t\ ,\ V(t)X_t\Big). \tag{3.6}$$

Since

$$\int_{t_i}^{t_f} d\xi = \Big(X_{t_f}\ ,\ FX_{t_f}\Big) - \Big(X_{t_i}\ ,\ V(t_i)X_{t_i}\Big),$$

one has

$$J(L) = \frac{1}{2}\mathbf{E}\left\{\int_{t_i}^{t_f}\Big(X_\tau\ ,\ (Q + L^TRL)X_\tau\Big)d\tau + \int_{t_i}^{t_f} d\xi + \Big(X_{t_i}\ ,\ V(t_i)X_{t_i}\Big)\right\}. \tag{3.7}$$

By the Ito formula it results:

$$d\xi(t, X_t) = \frac{\partial\xi}{\partial s}(t, X_t)dt + \sum_{i=1}^n \frac{\partial\xi}{\partial\alpha_i}(t, X_t)dX_t^i + \frac{1}{2}\sum_{i,j}\frac{\partial^2\xi}{\partial\alpha_i\partial\alpha_j}(t, X_t)d\langle M^i, M^j\rangle_t, \tag{3.8}$$

where M_t denotes the bilinear diffusion term:

$$M_t = \sum_{k=1}^m \int_{t_i}^t \lambda_k X_\tau dW_\tau^k. \tag{3.9}$$

Now, from the definition of the process ξ given in (3.6), one has:

$$\frac{\partial\xi}{\partial s}(t, X_t) = \Big(X_t\ ,\ \dot V(t)X_t\Big), \tag{3.10}$$

$$\sum_{i=1}^{n} \frac{\partial \xi}{\partial \alpha_i}(t, X_t) dX_t^i$$
$$= \Big(dX_t \ , \ V(t) X_t \Big) + \Big(X_t \ , \ V(t) dX_t \Big), \tag{3.11}$$

$$\frac{\partial^2 \xi}{\partial \alpha_i \partial \alpha_j}(t, X_t) = 2V_{i,j}(t). \tag{3.12}$$

Taking into account that $\langle W^i, W^j \rangle_t = 0$, $i \neq j$, it results:

$$\langle M \rangle_t^{(2)} = \sum_{k=1}^{m} \int_{t_i}^{t} \lambda_k^2 X_\tau^{[2]} d\tau = \int_{t_i}^{t} \Lambda X_\tau^{[2]} d\tau, \tag{3.13}$$

where $\Lambda \doteq \sum_{k=1}^{m} \lambda_k^2$. From (3.13) and taking into account of (3.12) one has:

$$\frac{1}{2} \sum_{i,j} \frac{\partial^2 \xi}{\partial \alpha_i \partial \alpha_j}(t, X_t) d\langle M^i, M^j \rangle_t$$
$$= \int_{t_i}^{t} \left[st(V(\tau)) \right]^T \Lambda X_\tau^{[2]} d\tau.$$

Substituting this and (3.10), (3.11), in (3.8), taking into account of the system equation (2.3), and recalling that u has the form $u = LX$, it results:

$$d\xi(t, X)$$
$$= \Big(X \ , \ \big(\dot{V} + (A + HL)^T V + V(A + HL) \big) X \Big) dt$$
$$+ 2 \Big(X \ , \ VB(X, dW) \Big) + \left[st(V(t)) \right]^T \Lambda X_t^{[2]} dt.$$

Since $\mathbf{E} \int (X \ , \ VB(X, dW)) = 0$, taking into account of (3.5), the substitution of the above equation in (3.7) results in

$$J(L) = \frac{1}{2} \mathbf{E} \Bigg\{ \int_{t_i}^{t_f} \left[st(V(t)) \right]^T \Lambda \mathbf{E} \left\{ X_t^{[2]} \right\} dt$$
$$+ \Big(X_{t_i} \ , \ V(t_i) X_{t_i} \Big) \Bigg\}. \tag{3.14}$$

Now, a version of the Ito formula in the Kronecker formalism (see Appendix B) will be used in order to write a closed form expression for the process $X_t^{[2]}$ (apply Theorem B.2 and then use formulas (B.1), (B.2)), with the help of formulas (A.2) and (B.3) after some manipulations one has:

$$d{X_t}^{[2]} = U_n^2 \left[(A(t) + H(t)L(t)) \otimes I_n \right] X_t^{[2]} dt$$
$$+ \Lambda X_t^{[2]} dt + \sum_{k=1}^{m} U_n^2 \lambda_k {X_t}^{[2]} dW_t^k, \tag{3.15}$$

with U_n^2 as in Lemma B.1.

Denoting with $\Phi_L(t, \tau)$ the transition matrix associated to $U_n^2 \left[(A(t) + H(t)L(t)) \otimes I_n \right]$, one has:

$${X_t}^{[2]} = \exp\{\Lambda I_{n^2}(t - t_i)\} \Phi_L(t, t_i) {X_i}^{[2]}$$
$$+ \sum_{k=1}^{m} \int_{t_i}^{t} \exp\{\Lambda I_{n^2}(t - \tau)\} \Phi_L(t, \tau) U_n^2 \lambda_k {X_\tau}^{[2]} dW_\tau^k,$$

from which, taking the expectations:

$$\mathbf{E}({X_t}^{[2]}) = \exp\{\Lambda I_{n^2}(t - t_i)\} \Phi_L(t, t_i) {X_i}^{[2]}.$$

Substituting the above expression in (3.14), and denoting $\tilde{\Phi}(t - t_i) \doteq \Lambda \exp\{\Lambda I_{n^2}(t - t_i)\}$ one has:

$$J(L) = \frac{1}{2} \mathbf{E} \Bigg\{ \int_{t_i}^{t_f} \left[st(V(t)) \right]^T \tilde{\Phi}(t - t_i) \Phi_L(t, t_i) X_i^{[2]} dt$$
$$+ (X_{t_i}, V(t_i) X_{t_i}) \Bigg\}. \tag{3.16}$$

As well known, denoting by V^o the solution of (3.5) for $L = L^o$ (L^o given by (3.2)) it results: $V^o(t) = G(t)$, with G solution of the Riccati equation (3.3). Moreover, for any V solution of (3.5):

$$V(t) - G(t) \geq 0, \ \forall t \in [t_i, t_f], \ \forall L \in \mathbb{R}^{p \times n}. \tag{3.17}$$

The theorem is proven as soon as it is shown that $\forall t \in [t_i, t_f]$, $J(L) - J(L^o) \geq 0$, $\forall L \in \mathbb{R}^{p \times n}$. From (3.16) one has:

$$J(L) - J(L^o)$$
$$= \frac{1}{2} \mathbf{E} \Bigg\{ \int_{t_i}^{t_f} \Big\{ \left[st(V(t)) \right]^T \tilde{\Phi}(t - t_i) \Phi_L(t, t_i) {X_i}^{[2]}$$
$$- \left[st(G(t)) \right]^T \tilde{\Phi}(t - t_i) \Phi_{L^o}(t, t_i) {X_i}^{[2]} \Big\} dt \Bigg\}$$
$$+ \frac{1}{2} \mathbf{E} \Big\{ (X_{t_i}, (V(t_i) - G(t_i)) X_{t_i}) \Big\}.$$

Because of (3.17) the last term in the previous expression is non negative all over the control interval; as far as the first term is concerned it is possible to show that, for any $t \in [t_i, t_f]$ one has

$$\arg\min_{L(\cdot)} \Big\{ \left[st(V(t)) \right]^T \tilde{\Phi}(t - t_i) \Phi_L(t, t_i) {X_i}^{[2]} \Big\}$$
$$= -R(\cdot)^{-1} H(\cdot)^T G(\cdot), \tag{3.18}$$

with G solution of the Riccati equation (3.3), from which we get the required non negativeness even for the first term. As a matter of fact, let $z_L(t)$ solution of the (deterministic) system:

$$\dot{z}_L(t) = (A(t) + H(t)L(t)) z_L(t), \qquad z_L(t_i) = X_i. \tag{3.19}$$

Then, using formulas (B.1) and (A.2), one has:

$$\frac{d}{dt}z_L^{[2]}(t) = \left(\frac{d}{dz}z^{[2]}\right)_{z=z_L(t)} \dot{z}_L(t)$$
$$= U_n^2(I_n \otimes z_L(t))\left[(A(t)+H(t)L(t))z_L(t)\otimes 1\right]$$
$$= U_n^2\left[((A(t)+H(t)L(t))z_L(t))\otimes z_L(t)\right]$$
$$= U_n^2\left[(A(t)+H(t)L(t))\otimes I_n\right] z_L^{[2]}(t),$$

with initial condition: $z_L^{[2]}(t_i) = X_i^{[2]}$. Hence, recalling the definition of the above used semigroup Φ_L it results:

$$z_L^{[2]}(t) = \Phi_L(t,t_i){X_i}^{[2]}. \tag{3.20}$$

Since $\Lambda \geq 0$, and hence $\Lambda \exp\{\Lambda I_{n^2}(t-t_i)\} \geq 0$, $\forall t \in I$:

$$\arg\min_{L(\cdot)} \left[st(V)\right]^T \tilde{\Phi}(t-t_i)\Phi_L(t,t_i){X_i}^{[2]}$$
$$= \arg\min_{L(\cdot)} \left[st(V)\right]^T \Phi_L(t,t_i){X_i}^{[2]},$$

and, using (3.20) and (A.4),

$$\left[st(V(t))\right]^T \Phi_L(t,t_i){X_i}^{[2]} = \left[st(V(t))\right]^T z_L^{[2]}(t)$$
$$= \left(z_L(t)\ ,\ V(t)z_L(t)\right). \tag{3.21}$$

Now, from the deterministic optimal control theory it is well known that

$$J_t(L) \doteq \frac{1}{2}\int_t^{t_f} \left(z_L(\tau)\ ,\ Q + L^TRLz_L(\tau)\right) d\tau$$
$$+\frac{1}{2}\left(z_L(t_f)\ ,\ Fz_L(t_f)\right) = \left(z_L(t)\ ,\ V(t)z_L(t)\right)$$

with V solution of equation (3.5) and the solution of the control problem:

$$\min_L J_t(L),$$
$$\dot{z}_L(t) = (A+HL)z_L(t), \qquad z_L(t_i) = X_i,$$

is given by $L^o = -R^{-1}H^TG$ with G solution of (3.3). Then, taking into account (3.21), (3.18) follows. □

4. CONCLUSIONS

The SLF control problem has been considered as a suitable framework to attack stochastic optimal control problems in a class of nonlinear systems. The case studied has been a finite-horizon optimal control problem with the classical quadratic cost criterion for a controlled system described by a single bilinear stochastic Ito equation. In the control-system theoretical language this means that the system-state is completely observable, and this case can be also referred as the "complete" infomation case. The control has the structure of a linear function of the current state and the SLF control problem consists in finding the linear map minimizing the quadratic cost. Theorem 3.1 provides the solution of this problem. It results that the equations solving the SLF control problem are formally very similar to the ones solving the classical LQG control problem in the complete information case, and hence the simplicity and meaningfulness of the LQG control-scheme results still preserved even in the present sub-optimal case. As in that problem, the solution of the SLF control problem is provided by the control-matrix (equation (3.2)), that depends of another matrix, G, solution of the backward Riccati equation (3.3). Looking at the structure of the solution one can see that it does not depend from the initial condition X_i, and hence, the hypotesis for X_i to be non-random, is not really restrictive (simply, the initial condition can be considered as "unknown"). The SLF control problem in the general case has been here formally stated as the optimization problem (2.6). This will constitute the subject of a future paper.

Appendix A. Kronecker algebra.

Let M and N be matrices of dimension $r \times s$ and $p \times q$ respectively. The Kronecker product $M \otimes N$ is defined as the $(r \cdot p) \times (s \cdot q)$ matrix

$$M \otimes N = \begin{bmatrix} m_{11}N & \dots & m_{1s}N \\ \dots & \dots & \dots \\ m_{r1}N & \dots & m_{rs}N \end{bmatrix},$$

where the m_{ij} are the entries of M. Let M be the $r \times s$ matrix $M = [\, m_1 \quad m_2 \quad \dots \quad m_s \,]$, where m_i denotes i-th column of M, then the stack of M is the $r \cdot s$ vector

$$st(M) = [\, m_1^T \quad m_2^T \quad \dots \quad m_s^T \,]^T. \tag{A.1}$$

Given suitably dimensioned matrices, A, B, C, D, and vectors u, v, the following properties hold (see Bellman (1970), Rodgers (1980)):

$$(A \cdot C) \otimes (B \cdot D) = (A \otimes B) \cdot (C \otimes D), \tag{A.2}$$
$$(A \otimes B)^T = A^T \otimes B^T, \tag{A.3}$$
$$st(A \cdot B \cdot C) = (C^T \otimes A) \cdot st(B), \tag{A.4}$$
$$u \otimes v = st(v \cdot u^T), \tag{A.5}$$
$$tr(A \otimes B) = tr(A) \cdot tr(B), \tag{A.6}$$

The Kronecker power of the matrix M is defined by the recursive rule: $M^{[n]} = M \otimes M^{[n-1]} = M^{[n-1]} \otimes M$, $M^{[0]} = 1$. Although the Kronecker product is not commutative, the following theorem holds (see Rodgers (1980), Carravetta et al. (1996)).

Theorem A.1. *For any given pair of matrices* $A \in \mathbb{R}^{r \times s}$, $B \in \mathbb{R}^{n \times m}$:

$$B \otimes A = C_{r,n}^T (A \otimes B) C_{s,m}, \tag{A.7}$$

where the commutation matrix $C_{u,v}$ is the $(u \cdot v) \times (u \cdot v)$ matrix such that its (h,l) entry is given by:

$$\{C_{u,v}\}_{h,l} = \begin{cases} 1, & \text{if } l = (|h-1|_v)u + \left(\left[\frac{h-1}{v}\right]+1\right); \\ 0, & \text{otherwise.} \end{cases} \tag{A.8}$$

Appendix B. The vector Ito formula in the Kronecker formalism

All of the following results, but formula (B.3), can be found in Carravetta et al. (2000). They constitute a powerful machinery that allows the calculation, for a given stochastic process ϕ, of the stochastic differential of the process $\phi^{[h]}$, where $[h]$ is any integer Kronecker power. Let $x \in \mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}^{m \times p}$ twice differentiable. Let the matrix $(d/dx) \otimes F(x) \in \mathbb{R}^{m \times (n \cdot p)}$ be defined as

$$\frac{d}{dx} \otimes F(x) \doteq \left[\frac{\partial F(x)}{\partial x_1} \;\; \dots \;\; \frac{\partial F(x)}{\partial x_n} \right].$$

Lemma B.1. *For any $h \geq 1$, $l > 1$ integers:*

$$\frac{d}{dx} \otimes x^{[h]} = U_n^h (I_n \otimes x^{[h-1]}) \tag{B.1}$$

$$\frac{d}{dx} \otimes \frac{d}{dx} \otimes x^{[l]} = O_n^l (I_{n^2} \otimes x^{[l-2]}), \tag{B.2}$$

where

$$U_n^h \doteq \Big(\sum_{\tau=0}^{h-1} C^T_{n,n^{h-1-\tau}} \otimes I_{n^\tau} \Big),$$

$$O_n^h \doteq \sum_{\tau=0}^{h-1} \sum_{s=0}^{h-2} (C^T_{n,n^{h-1-\tau}} \otimes I_{n^\tau}) \cdot (I_n \otimes C^T_{n,n^{h-2-s}} \otimes I_n),$$

*$C_{\cdot,\cdot}$, are commutation matrices (*Theorem A.1*).*

From Lemma B.1 the following useful equality can be easily derived

$$O_n^2 st\{I_n\} = 2 \cdot st\{I_n\}. \tag{B.3}$$

Indeed, looking at (B.2) it follows that matrix O_n^2 satisfy the relation:

$$\frac{d}{dx} \otimes \frac{d}{dx} \otimes x^{[2]} = O_n^2,$$

and by calculation of the derivatives in the left hand side of the previous equation, (B.3) can be directly verified. □

Theorem B.2. *Let $(X_t, \mathcal{F}_t)$ be a vector continuous semimartingale in $\mathbb{R}^n$ described by the Ito's stochastic differential: $dX_t = d\beta_t + dM_t$, where $(\beta_t, \mathcal{F}_t)$ is an a.s. continuous bounded variation process and $(M_t, \mathcal{F}_t)$ is a square integrable martingale. Let $F : \mathbb{R}^n \to \mathbb{R}^p$, be a continuous function endowed with the first and second derivatives. Then the process $Z_t = F(X_t)$ is a square integrable semimartingale, whose differential is given by*

$$dZ_t = \Big(\frac{d}{dx} \otimes F(x)\Big)_{x=X_t} dX_t + \frac{1}{2}\Big(\frac{d}{dx} \otimes \frac{d}{dx} \otimes F(x)\Big)_{x=X_t} d\langle M\rangle_t^{(2)},$$

with $\langle M\rangle_t^{(2)}$ as in (2.1)

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