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INPUT-OUTPUT DECOUPLING OF NONLINEAR DISCRETE-TIME SYSTEMS BY STATIC OUTPUT FEEDBACK

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Abstract: The input-output decoupling problem for discrete-time nonlinear systems by static output feedback is studied in this paper. The discrete-time nonlinear system is described either by state equations or by high order input-output difference equations (NARMA model). In both cases the necessary and sufficient conditions are given for the solution of the problem. *Copyright © 2002 IFAC*

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1. INTRODUCTION

The input-output decoupling problem of discrete-time nonlinear system by state feedback has been studied extensively, see for instance (Nijmeijer and van der Schaft, 1990) and the references given there. When state is not available for measurement, two approaches are possible: the reconstruction of the state by means of an observer or the use of output feedback. The first solution suffers from the fact that the well-known separation theorem for the combination of a static state feedback with an observer in the linear case is no longer valid in the nonlinear domain. In this contribution we investigate the second solution, i. e. output feedback. We limit ourselves to the case of the static output feedback. The case of the dynamic output feedback is left for the future studies.

The results for nonlinear system described by state equations extend the known results in the continuous-time case (Pothin and Moog, 1998) whereas the results for systems described by NARMA model have no continuous-time analogues.

The paper is organized as follows. In the second section we give a precise problem formulation for input-output decoupling problem if the system is described by state equations as well the necessary and sufficient solvability conditions. In the third section we present the analogous results for the system described by NARMA model. Futhermore, we will give the procedures to obtain the static output feedback if the necessary and sufficient solvability conditions are satisfied. These procedures are given in the sufficiency parts of the proofs and are constructive up to finding the integrating factors and integration of the one-forms. Finally, this contribution ends with some conclusions.

To our best knowledge there exists only a few papers that tackle nonlinear synthesis problems via (static or dynamic) output feedback for discrete-time nonlinear systems (Pothin *et al.*, 2000; Pothin and Moog, 2001). In the first the i/o linearization problem and in the second disturbance decoupling for the single-input single-output sys-

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tem is studied both via static and dynamic feedback.

2. SYSTEMS DESCRIBED BY STATE EQUATIONS

2.1 *Definitions and Problem Statement*

Consider a square invertible discrete-time nonlinear system

$$\begin{aligned} x(t+1) &= f(x(t), u(t)), \quad t = 0, 1, \dots, \\ y(t) &= h(x(t)) \end{aligned} \tag{1}$$

where $x(t) \in I\!R^n, u(t) \in I\!R^m, y(t) \in I\!R^m$, and the maps f and h are supposed to be analytic functions of their arguments. Let $\mathcal{K}$ denote the field of meromorphic functions in a finite number of variables $\{x(t), u(t+k), k \geq 0\}$. The forward-shift operator $\delta : \mathcal{K} \to \mathcal{K}$ is defined by $\delta\phi(x(t), u(t), \dots, u(t+N)) = \phi(f(x(t), u(t)), u(t+1), \dots, u(t+N+1))$. Denote by $\mathcal{E}$ the $\mathcal{K}$-vector space spanned by

$$\Delta\left(\sum_i a_i \mathrm{d}\phi_i\right) \to \sum_i \delta a_i \mathrm{d}\delta\phi_i,$$

where $a_i, \phi_i \in \mathcal{K}$.

Definition 1. The relative degree r_i of the output y_i is defined to be

$$r_i = \min\{k \in I\!N \mid \Delta^k \mathrm{d}y_i(t) \notin \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x(t)\}\}.$$

If such an integer does not exist, set $r_i = \infty$.

Problem Statement: Consider an invertible system of the form (1). Find if possible, a regular static output feedback

$$u(t) = \alpha(y(t), v(t)), \quad \mathrm{rank}_{\bar{\mathcal{K}}}\left[\frac{\partial\alpha(\cdot)}{\partial v(t)}\right] = m \tag{2}$$

such that the closed loop system satisfies

(i) $$\begin{aligned} &\mathrm{d}y_i[t+k] \in \mathrm{span}_{\bar{\mathcal{K}}}\{\mathrm{d}x(t), \\ &\mathrm{d}v_i(t), \dots, \mathrm{d}v_i(t+k-r_i)\}, \\ &\forall i = 1, \dots, m \text{ and } \forall k \geq r_i \end{aligned} \tag{3}$$

(ii) $$\mathrm{d}y_i[t+r_i] \notin \mathrm{span}_{\bar{\mathcal{K}}}\{\mathrm{d}x(t)\} \tag{4}$$

where $\bar{\mathcal{K}}$ is the field of meromorphic functions of $x(t)$, $v(t)$ and a finite number of time shifts of $v(t)$. Note that we have used brackets $[\cdot]$ instead of paranthesis $(\cdot)$ to make a clear distinction between the outputs of the closed-loop, and the open-loop system, respectively. Condition (i) represents noninteraction whereas condition (ii) represents the output controllability.

Define Ω_i, for each output $y_i (i = 1, \dots, m)$, by:

$$\begin{aligned} \Omega_i = \{&\omega(t) \in \mathcal{X} \mid \forall k \in I\!N \\ &\Delta^k\omega(t) \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x(t), \mathrm{d}y_i(t+r_i), \dots, \\ &\mathrm{d}y_i(t+r_i+k-1)\}\} \end{aligned} \tag{5}$$

where $\mathcal{X} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x(t)\}$. The space Ω_i is instrumental for solving the input-output decoupling problem. The spaces Ω_i may be computed as the limit of the following algorithm:

$$\begin{aligned} \Omega_i^0 &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x(t)\}, \\ \Omega_i^{k+1} &= \{\omega(t) \in \Omega_i^k \mid \Delta\omega(t) \in \Omega_i^k \\ &\quad + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_i(t+r_i)\}\}, \quad k \geq 0. \end{aligned} \tag{6}$$

Lemma 2. The subspaces Ω_i, $i = 1, \dots, m$, are invariant with respect to

(i) the state transformation (state space diffeomorphism) $z(t) = \phi(x(t))$,
(ii) the regular static output feedback $u(t) = \alpha(y(t), v(t))$

2.2 *Main Result*

Theorem 3. System (1) is input-output decoupable by static output feedback if and only if

(i) $r_i < \infty$, $i = 1, \dots, m$
(ii) $\mathrm{rank}_{\mathcal{K}}\left(\dfrac{\partial y_1(t+r_1), \dots, \partial y_m(t+r_m)}{\partial u(t)}\right)^T = m$
(iii) $\mathrm{d}y_i(t+r_i) \in \Omega_i \oplus \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_j(t), \mathrm{d}u(t), (\text{for } j = 1, \dots, m, j \neq i)\}$
(iv) $\mathrm{d}\omega_i(t) \wedge \omega_i(t) \wedge \mathrm{d}y_i(t) = 0$.

where

$$\omega_i(t) \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_j(t), \mathrm{d}u(t),$$

$$(\text{for } j = 1, \dots, m, j \neq i)\},$$

is such that $\mathrm{d}y_i(t+r_i) - \omega_i(t) \in \Omega_i$.

Remark. Conditions (i) and (ii) are known to be necessary and sufficient for the solvability of the i/o decoupling problem via the regular static state feedback (Nijmeijer and van der Schaft, 1990).

Proof. *Necessity.* It is enough to show the necessity of (iii) and (iv). Assume the regular static output feedback (2) decouples system (1). From (2) one obtains

$$v = \alpha^{-1}(y(t), u(t)). \tag{7}$$

For the closed-loop system, (4) is fulfilled and one has

$$\mathrm{d}y_i[t+r_i] \in \bar{\Omega}_i \oplus \mathrm{span}_{\bar{\mathcal{K}}}\{\mathrm{d}v_i(t)\} \tag{8}$$

where $\bar{\Omega}_i$ is defined for the closed loop system similarly to (5). By (7)

$$\mathrm{d}v_i(t) \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y(t), \mathrm{d}u(t)\}, \qquad (9)$$

or equivalently,

$$\mathrm{d}v_i(t) \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_i(t)\} \oplus \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_j(t), \mathrm{d}u(t), \\ (\text{for } j = 1, \ldots, m, j \neq i)\}. \qquad (10)$$

From (8) and (9),

$$\mathrm{d}y_i[t + r_i] \in \bar{\Omega}_i \oplus \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_j(t), \mathrm{d}u(t) \\ (\text{for } j = 1, \ldots, m, j \neq i)\}. \qquad (11)$$

which yields (iii),

$$\mathrm{d}y_i(t + r_i) \in \Omega_i \oplus \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_j(t), \mathrm{d}u(t) \\ (\text{for } j = 1, \ldots, m, j \neq i)\}. \qquad (12)$$

To show necessity of (iv), note that from (7), $\omega_i(t)$ is uniquely defined by

$$\omega_i(t) = \xi \left(\sum_{j=1, j\neq i}^{m} \frac{\partial v_i}{\partial y_j} \mathrm{d}y_j(t) + \sum_{k=1}^{m} \frac{\partial v_i}{\partial u_k} \mathrm{d}u_k(t) \right) \qquad (13)$$

in which $\xi \in \mathcal{K}$.

Sufficiency. When the condition (iii) is fulfilled, one has:

$$\mathrm{d}y_i(t + r_i) = \omega_0(t) + \omega_i(t), \qquad (14)$$

where

$$\begin{aligned} \omega_0(t) &\in \Omega_i \\ \omega_i(t) &\in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_j(t), \mathrm{d}u(t), \\ &\qquad (\text{for } j = 1, \ldots, m, j \neq i)\}. \end{aligned} \qquad (15)$$

From (14) and $dv_i(t) \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y(t), \mathrm{d}u(t)\}$, one deduces:

$$\mathrm{d}y_i(t + r_i) \in \Omega_i + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}v_i(t)\}. \qquad (16)$$

The condition (iv) implies that there exists $\lambda \in \mathcal{K}$ and $\mu \in \mathcal{K}$ such that $\lambda(\omega_i(t) + \mu \mathrm{d}y_i(t))$ is exact. Let,

$$\mathrm{d}v_i(t) = \lambda(\omega_i(t) + \mu \mathrm{d}y_i(t)) \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y(t), \\ \mathrm{d}u(t)\}, \qquad (17)$$

so we get $v_i(t) = \phi_i(\mathrm{d}y(t), \mathrm{d}u(t))$ by integrating the one-form $\lambda(\omega_i(t) + \mu \mathrm{d}y_i(t))$. By (ii), $\phi = (\phi_1, \ldots, \phi_m)$ is an invertible function. Thus, $u(t) = \phi^{-1}(y(t), v(t)) = \alpha(y(t), v(t))$. ■

Example 4. Consider the system

$$\begin{aligned} x_1(t+1) &= x_2^2(t) + u_1(t)x_1(t)x_3(t) \\ x_2(t+1) &= -x_2(t) \\ x_3(t+1) &= x_4(t) + x_3(t)u_2(t) \\ x_4(t+1) &= x_3(t)x_4(t) \\ y_1(t) &= x_1(t) \\ y_2(t) &= x_3(t) \end{aligned} \qquad (18)$$

Compute

$$\begin{aligned} \Omega_1 &= \mathrm{sp}_{\mathcal{K}}\{\mathrm{d}x_1(t), \mathrm{d}x_2(t)\} \\ \Omega_2 &= \mathrm{sp}_{\mathcal{K}}\{\mathrm{d}x_2(t), \mathrm{d}x_3(t), \mathrm{d}x_4(t)\} \end{aligned}$$

and

$$\begin{aligned} \mathrm{d}y_1(t+1) &= 2x_2(t)\mathrm{d}x_2(t) + u_1(t)x_1(t)\mathrm{d}y_2(t) \\ &\quad + u_1(t)x_3(t)\mathrm{d}x_1(t) + x_1(t)x_3(t)\mathrm{d}u_1(t) \\ \mathrm{d}y_2(t+1) &= \mathrm{d}x_4(t) + u_2(t)\mathrm{d}x_3(t) + x_3(t)\mathrm{d}u_2(t). \end{aligned}$$

Then from (iii)

$$\begin{aligned} \omega_1(t) &= u_1(t)x_3(t)\mathrm{d}y_2(t) + x_1(t)x_3(t)\mathrm{d}u_1(t) \\ \omega_2(t) &= x_3(t)\mathrm{d}u_2(t) \end{aligned}$$

One can easily check that (iv) is satisfied and find $\mu_1 = u_1(t)y_2(t)$, $\lambda_1 = 1$, $\mu_2 = u_2(t)$, $\lambda_2 = 1$ so that for $i = 1, 2$ $\lambda_i(\omega_i(t) + \mu_i \mathrm{d}y_i(t)) = \mathrm{d}v_i(t)$ is exact. Integrating $dv_i(t)$, $i = 1, 2$, one obtains

$$\begin{aligned} v_1(t) &= u_1(t)y_1(t)y_2(t) \\ v_2(t) &= y_2(t)u_2(t) \end{aligned}$$

which yields the static output decoupling feedback

$$u_1(t) = \frac{v_1(t)}{y_1(t)y_2(t)} \qquad u_2(t) = \frac{v_2(t)}{y_2(t)}$$

3. SYSTEMS DESCRIBED BY NARMA MODELS

3.1 *Definitions and Problem Statement*

Consider a discrete-time nonlinear system with the same number of inputs and outputs, described by the set of input-output equations

$$\begin{aligned} y_i(t + n_i) = f_i(&y_j(t), \ldots, y_j(t + n_{ij} - 1), \\ &u_k(t), \ldots, u_k(t + s_{ik}), \\ &j, k = 1, \ldots, m) \end{aligned} \qquad (19)$$

$i = 1, \ldots, m$, where the maps f_i, $i = 1, \ldots, m$ are supposed to be analytic functions of their arguments. We assume that the system is strictly proper i. e. that $s_{ik} < n_i$, for $i, k = 1, \ldots, m$. Moreover, we assume *system (19) to be in canonical form*, which means that and $n_{ij} < \min(n_i, n_j)$ and $n_1 + \ldots + n_m := n$ is the order of the system. The latter implies that whenever (19) admits a Kalmanian realization, the indices n_i, associated to each output y_i, $i = 1, \ldots, m$, are the observability indices of any observable state-space realization of order n. Moreover, being in canonical form also means that system (19) generically satisfies the condition

$$\frac{\partial f_i(\cdot)}{\partial(y(t), u(t))} \not\equiv 0. \qquad (20)$$

The form (19) is an extension of the echelon canonical matrix fraction description introduced in (Popov, 1969) for linear systems.

Given a NARMA model (19), it is always possible to transform it into an *extended* state-space realization. Specifically, this representation is obtained from equations (19) by taking $z(t)$ as the following state vector, involving both past outputs and past inputs:

$$\begin{aligned} z^T(t) = & (y_j(t), \ldots, y_j(t+n_j-1), \\ & u_k(t), \ldots, u_k(t+s), j, k = 1, \ldots, m) \end{aligned}$$

where $s = \max\{s_{ik}, i, k = 1, \ldots, m\}$. Note that this extended state vector is of dimension $n_1 + \ldots + n_m + m(s+1)$ and the extended state-space model may be written down directely from the input-output model (19) as:

$$\begin{aligned} z_{i,1}(t+1) &= z_{i,2}(t) \\ &\vdots \\ z_{i,n_i-1}(t+1) &= z_{i,n_i}(t) \\ z_{i,n_i}(t+1) &= f_i(z(t)), \quad i = 1, \ldots, m \\ z_{m+j,1}(t+1) &= z_{m+j,2}(t) \\ &\vdots \\ z_{m+j,s}(t+1) &= z_{m+j,s+1}(t) \\ z_{m+j,s+1}(t+1) &= w_j(t), \quad j = 1, \ldots, m. \end{aligned} \qquad (21)$$

The disadvantage of the extended state-space realization is that it is nonminimal, and therefore either non-controllable or non-observable or both. In (Kotta, 1998) the necessary and sufficient conditions were given when the input-output equations of the form (19), or equivalently (21), can be transformed into the observable and controllable state-space form. Since every NARMA model cannot be described in the minimal state space form, it is worth to study the input-output decoupling problem for the system described by the input-output model(19).

Let $\mathcal{K}_e$ denote the field of meromorphic functions in a finite number of variables $\{z(t), w(t+k), k \geq 0\}$. The forward-shift operator $\delta_e : \mathcal{K}_e \to \mathcal{K}_e$ is defined by $\delta_e \phi(z(t), w(t), \ldots, w(t+N)) = \phi(f_e(z(t), w(t)), w(t+1), \ldots, w(t+N+1))$, where f_e denotes the state transition map of the extended system (21). Denote by $\mathcal{E}_e$ the $\mathcal{K}_e$-vector space spanned by $\{\mathrm{d}\phi \mid \phi \in \mathcal{K}_e\}$. The operator δ_e induces a forward-shift operator $\Delta_e : \mathcal{E}_e \to \mathcal{E}_e$ by

$$\Delta_e \left(\sum_i a_i \mathrm{d}\phi_i \right) \to \sum_i \delta_e a_i \mathrm{d}\delta_e \phi_i,$$

where $a_i, \phi_i \in \mathcal{K}_e$.

Definition 5. The relative degree r_i of the output y_i is defined to be (for equations in the canonical form)

$$r_i = n_i - \max\{s_{ik}, k = 1, \ldots, m\} := n_i - s_i$$

If all $s_{ik} = 0$, then set $r_i = \infty$.

Problem statement. Consider a nonlinear system of the form (19). Find, if possible, a regular static output feedback

$$\begin{aligned} & u(t) = \alpha(y(t), v(t)), \\ & \mathrm{rank}_{\mathcal{K}_e} \frac{\partial \alpha(y(t), v(t))}{\partial v(t)} = m \end{aligned} \qquad (22)$$

such that the closed-loop system satisfies

(i) $\mathrm{d}y_i[t+k] \in \mathrm{span}_{\mathcal{K}_e}\{\mathrm{d}y_i[t], \ldots, \mathrm{d}y_i[t+n_i-1], \mathrm{d}v_i(t), \ldots, \mathrm{d}v_i(t+s_i), \ldots, \mathrm{d}v_i(t+k-r_i)\}$ (23)

$$\forall i = 1, \ldots, m \text{ and } \forall k \geq n_i$$

(ii) $\mathrm{d}y_i[t+n_i] \notin \mathrm{span}_{\mathcal{K}_e}\{\mathrm{d}y_i[t], \ldots, \mathrm{d}y_i[t+n_i-1]\}$. (24)

Condition (i) represents noninteraction whereas (ii) represents the output controllability.

3.2 *Main result*

Theorem 6. System (19) is input-output decouplable by static output feedback if and only if

(i) $r_i < \infty$, $i = 1, \ldots, m$

(ii) $\mathrm{rank}_{\mathcal{K}_e} \dfrac{\partial y_i(t+n_i)}{\partial u_j(t+s_i)} = m$

(iii) $\mathrm{d}y_i(t+n_i) \in \mathrm{span}_{\mathcal{K}_e}\{\omega_i, \Delta^{-1}\omega_i, \ldots, \Delta^{-\sigma_i}\omega_i\} \oplus \mathrm{span}_{\mathcal{K}_e}\{\mathrm{d}y_i(t), \ldots, \mathrm{d}y_i(t+n_i-1)\}$

(iv) $\mathrm{d}\omega_i(t) \wedge \omega_i(t) \wedge \mathrm{d}y_i(t+\sigma_i) = 0$

where

$$\omega_i = \sum_{j \neq i} \frac{\partial f_i(\cdot)}{\partial y_j(t+\sigma_i)} \mathrm{d}y_j(t+\sigma_i) + \sum_{k=1}^{m} \frac{\partial f_i(\cdot)}{\partial u_k(t+\sigma_i)} \mathrm{d}u_k(t+\sigma_i).$$

and $\sigma_i = \max\{k \in I\!N$ such that

$$\frac{\partial f_i(\cdot)}{\partial(y_j(t+k), j \neq i, u(t+k))} \not\equiv 0\}$$

Proof. Sufficiency. Condition (iii) guarantees that by differentiating (19) we should get

$$\begin{aligned} \mathrm{d}y_i(t+n_i) &= \sum_{k=0}^{n_i-1} a_k \mathrm{d}y_i(t+k) \\ &+ \sum_{r=0}^{\sigma_i} b_{\sigma_i - r}\Delta^{-r}\omega_i(t+\sigma_i) \end{aligned} \tag{25}$$

Obviously, $\sigma_i \geq s_i$. Conditions (i) and (iii) yield $\omega_i \notin \mathrm{sp}\{\mathrm{d}y(t+\sigma_i)\}$ and $\sigma_i = s_i$. Condition (iv) will imply that there exist $\lambda \in \mathcal{K}_e$ and $\mu \in \mathcal{K}_e$ such that $\lambda(\omega_i(t+s_i) + \mu \mathrm{d}y_i(t+s_i))$ is exact. Let

$$\mathrm{d}\phi_i(t+s_i) = \lambda(\omega_i(t+s_i) + \mu\mathrm{d}y_i(t+s_i)) \in \mathrm{span}_{\mathcal{K}_e}\{\mathrm{d}y(t+s_i), \mathrm{d}u(t+s_i)\},$$

so we get $v_i(t) \triangleq \phi_i(y(t), u(t))$ by integrating the one-form $\lambda(\omega_i(t+s_i) + \mu\mathrm{d}y_i(t+s_i))$ and shifting the result backwards s_i steps.

Now, we can rewrite (25) as

$$\mathrm{d}y_i(t+n_i) = \sum_{k=0}^{n_i-1} a_k^{'} \mathrm{d}y_i(t+k) + \sum_{r=0}^{\sigma_i} b_{\sigma_i-r}\Delta^{-r}\mathrm{d}\phi_i(y(t+\sigma_i), u(t+\sigma_i)).$$

By (ii), the decoupling matrix of the system (19) has a full rank which will imply the invertibility of $\phi = (\phi_1, \ldots, \phi_m)^T$. Thus

$$u(t) = \phi^{-1}(y(t), v(t)) \triangleq \alpha(y(t), u(t))$$

which yields the closed loop system

$$\begin{aligned} y_i[t+n_i] &= F_i(y_i[t], \ldots, y_i[t+n_i-1], \\ &\quad v_i(t), \ldots, v_i(t+s_i)) \end{aligned} \tag{26}$$

Necessity. Assume the regular static output feecback (22) decouples system (19). From (22) one obtains

$$v(t) = \alpha^{-1}(y(t), u(t)). \tag{27}$$

For the closed-loop system, (i) and (ii) are fulfilled and one has

$$\begin{aligned} \mathrm{d}y_i[t+n_i] &= \sum_{k=0}^{n_i-1} \alpha_i \mathrm{d}y_i[t+k] \\ &+ \sum_{r=0}^{s_i} \beta_r \mathrm{d}v_i(t+r) \end{aligned} \tag{28}$$

By (27)

$$dv_i(t) \in \mathrm{span}_{\mathcal{K}_e}\{\mathrm{d}y(t), \mathrm{d}u(t)\}$$

and so one can rewrite (28) as follows

$$\begin{aligned} \mathrm{d}y_i(t+n_i) &= \sum_{k=0}^{n_i-1} \alpha_i^{'} dy_i(t+k) \\ &+ \sum_{r=0}^{s_i} \beta_{s_i-r}\left[\sum_{i\neq j} \frac{\partial v_i}{\partial y_j} dy_j(t+r)\right. \\ &+ \left.\frac{\partial v_i}{\partial u}\mathrm{d}u(t+r)\right] \end{aligned} \tag{29}$$

Now, comparing (29) and (19) necessarily, in the open-loop system

$$\omega_i = \beta_0\left[\sum_{j\neq i}\frac{\partial v_i}{\partial y_j}\mathrm{d}y_j(t+s_i) + \frac{\partial v_i}{\partial u}\mathrm{d}u(t+s_i)\right] \tag{30}$$

Thus (iii) is fulfilled, and from the structure of (30) it is obvious that

$$\mathrm{d}\omega_i \wedge \omega_i \wedge \mathrm{d}y_i(t+s_i) = 0.$$

So, also (iv) is fulfilled.

Example 7. Consider the i/o difference equations

$$\begin{aligned} y_1(t+2) &= y_1(t+1) \\ &+ u_1(t+1)y_1(t+1)y_2(t+1) \\ &- u_1(t)y_1(t)y_2(t) \\ y_2(t+2) &= y_2(t+1)y_2(t) \\ &+ y_2(t+1)u_2(t+1) - y_2^2(t)u_2(t) \end{aligned} \tag{31}$$

Compute

$$\begin{aligned} \omega_1(t+1) &= u_1(t+1)y_1(t+1)\mathrm{d}y_2(t+1) \\ &+ y_1(t+1)y_2(t+1)\mathrm{d}u_1(t+1) \\ \omega_2(t+1) &= y_2(t+1)\mathrm{d}u_2(t+1) \end{aligned} \tag{32}$$

The conditions of Theorem 6 are satisfied for (31) and (32) and integrating the one forms

$$\begin{aligned} &\omega_1(t+1) + u_1(t+1)y_2(t+1)\mathrm{d}y_1(t+1) \\ &\omega_2(t+1) + u_2(t+1)\mathrm{d}y_2(t+1) \end{aligned}$$

and shifting the result backwards we get

$$\begin{aligned} v_1(t) &= u_1(t)y_1(t)y_2(t) \\ v_2(t) &= y_2(t)u_2(t). \end{aligned}$$

Note that the classical state space realization of (31) is (18) and that the i/o decoupling output feedbacks coincide for (31) and (18).

4. CONCLUSIONS

A disadvantage of the known solution to the i/o decoupling problem is that one has to measure the whole state in general, but in many industrial applications this is impossible. Therefore, it is natural to pose the problem of what can be achieved, if the control law may depend only on the outputs. The necessary and sufficient solvability conditions are given both in cases if the system is described by state or i/o equations.

An open problem is to extend the solution to the case of dynamic output feedback. Further research is also required to treat the i/o decoupling problem by measured output feedback when the measured outputs are different from the controlled outputs. Moreover, if the system is realizable in the state space form it would be interesting to theoretically prove that the results in Section 2 and 3 coincide.

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