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ON THE DISCRETE-TIME H^∞ FIXED-LAG SMOOTHING*

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Abstract: This paper deals with the discrete-time H^∞ fixed-lag smoothing problem. Conventionally, this problem is solved by reducing it to a standard H^∞ filtering problem for a *higher order* system that includes not only the actual system dynamics but also the delay caused by the smoothing lag. As the smoothing lag gets larger, such an approach may suffer from computational problems, especially due to the fact that a high dimensional Discrete Algebraic Riccati Equation (DARE) is to be solved. To overcome this disadvantage, in this paper, a new solution to this problem is derived in terms of *one* DARE of the same dimensions as the actual system dynamics.

Keywords: Smoothing filters, H^∞ optimization, Discrete - time systems

1 INTRODUCTION

In this paper, the problem of the discrete-time fixed-lag smoothing over an infinite horizon is studied. The problem is to estimate, on the basis of measurements available up to time k, a linear combination of the system states at time $k-r$, for a given $r > 0$, so called the *smoothing lag*. This problem appears in many signal processing and communication applications, where a certain delay in the estimate is allowed in order to achieve more accurate results.

Various solutions to this problem are available in the literature, both in the context of the H^2 and the H^∞ optimizations. Yet, while the solution and the properties of the H^2 - optimal smoothers are well understood (Anderson and Moore, 1979), the case of designing smoothers to achieve an estimation error less than a required level (i.e., the H^∞ optimization) is still under investigation. Various methods have been used to find a solution to this latter problem: Grimble (1991, 1996) solved the problem by using the polynomial H^2 embedding approach, Theodor and Shaked (1994) dealt with the time-varying finite-horizon case using game theory methods, while in (Zhang *et al.*, 2000) the Krein space polynomial approach has been used. The main idea in all these works is to incorporate the delay resulting from the smoothing lag into the transfer function of the process. This reduces the smoothing problem to a filtering one which, in turn, can be solved using known methods. However, in all these approaches, the dimension of the equivalent filtering problem to be solved increases rapidly as the smoothing lag gets larger. This may lead to computational burden and may induce errors in the computation of the smoother. In addition, the structure of the problem is lost and the affect of the lag on the final solution is not clear.

In this respect, (Colaneri *et al.*, 1998) have developed a solution to the H^∞ discrete-time fixed-lag smoothing problem without making any fictitious augmentation of the state space dimension, by using the J-spectral factorization approach. This solution, however, requires three discrete algebraic Riccati equations (DARE's) and two iterative procedures that eliminate the non-minimum phase and the infinite zeros from the J-spectral factor. Consequently, it may suffer the same computational problems as the afore mentioned solutions.

In this paper a different approach is used. It

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is also based on the solution to the H^∞ filtering problem. However, the solution is given in terms of one H^∞ DARE of the same dimension as the original process dynamics — the same H^∞ DARE that has to be solved in the filtering problem. By using this solution, the amount of computations required to solve this problem is considerably reduced and more accurate solutions can be obtained. Furthermore, the effect of the smoothing lag on the solution can be traced back.

The paper is organized as follows. In Section 2 some preliminary results on the discrete algebraic Riccati equation are collected. In Section 3 the discrete-time H^∞ fixed-lag smoothing problem is formulated and solved. Finally, some concluding remarks are presented in Section 4.

The notation throughout the paper is fairly standard. M' means the transpose of a matrix M. $\sigma(M)$ is the spectrum of a square matrix M. As usual, $\mathbb{D}$ denotes the open unit disk and $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. The compact block notation $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ denotes transfer functions in the z domain in terms of their state-space realization.

2 PRELIMINARIES

This section assembles the mathematical background required in the sequel. First, we review some basic notions and notations concerning the discrete algebraic Riccati equation (DARE). Then, an important relation is established between the stabilizing solution to the H^∞ DARE associated with the system $\begin{bmatrix} \bar{\mathcal{G}}_1(z) \\ \bar{\mathcal{G}}_2(z) \end{bmatrix}$ and that of the H^∞ DARE associated with $\begin{bmatrix} z^{-1}\bar{\mathcal{G}}_1(z) \\ \bar{\mathcal{G}}_2(z) \end{bmatrix}$. This relation is a crucial step in the derivation of the new solution to the discrete-time H^∞ fixed-lag smoothing problem.

Let $J = J'$ be a square matrix of the same dimension as the input dimension of the discrete-time LTI plant $\bar{\mathcal{G}}(z) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ and associate with $\bar{\mathcal{G}}$ the following operator:

$$R_{\bar{\mathcal{G}}}(Y) \doteq AYA' - Y + BJB' - L\big(CYC' + DJD'\big)L',$$

where

$$L \doteq -\big(AYC' + BJD'\big)\big(CYC' + DJD'\big)^{-1}.$$

The operator $R_{\bar{\mathcal{G}}}(Y)$ is said to be the discrete-time Riccati operator and the equation

$$R_{\bar{\mathcal{G}}}(Y) = 0 \tag{1}$$

is the well known filtering DARE, and finding its stabilizing solution is a crucial step in solving various discrete-time control and filtering problems, such as H^2 ($J = I$) and H^∞ ($J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$) optimizations (Chen and Francis, 1995; Zhou *et al.*, 1995). This equation is extensively investigated in the literature (Lancaster and Rodman, 1995) and the necessary and sufficient conditions for the existence of its stabilizing solution are well understood (Ionescu and Weiss, 1992).

The matrix $Y \in \mathbb{C}^n$ is said to be a solution to DARE (1) if $CYC' + DJD'$ is invertible and Y satisfies (1). It is known (Van Dooren, 1981) that the solutions to DARE (1) are strongly related to the following generalized eigenvalue problem:

$$\Delta - \lambda\Lambda \doteq \begin{bmatrix} A' & 0 & C' \\ -BJB' & I & -BJD' \\ DJB' & 0 & DJD' \end{bmatrix} - \lambda \begin{bmatrix} I & 0 & 0 \\ 0 & A & 0 \\ 0 & -C & 0 \end{bmatrix},$$

and can be computed directly from the deflating subspaces of $\{\Lambda, \Delta\}$. Indeed, DARE (1) has solutions only if this pencil is regular. In this case, in order to compute all the Y's satisfying (1), one just has to find vector bases of the form $\begin{bmatrix} I & Y' & L \end{bmatrix}'$ to all the deflating subspaces of $\{\Lambda, \Delta\}$ having the rank equal to the dimension of the main coefficient A. Then, any of those Y's is a solution to DARE (1).

If $Y = Y' \in \mathbb{R}^n$, where $n \doteq \dim A$, is a solution to DARE (1) and, in addition, $A_L \doteq A + LC$ is asymptotically stable (i.e., all the eigenvalues are inside the unit disk), then Y is said to be a stabilizing solution to DARE (1) and L is its associated stabilizing matrix gain. It was proved in (Ionescu and Weiss, 1992) that if the DARE (1) has a stabilizing solution, then it is unique. The computation of the stabilizing solution to the DARE (1) is based on the fact that $\{\Lambda, \Delta\}$ is an extended symplectic pencil (ESP). That is, it satisfies the following three conditions:

1. $\det(\Delta - \lambda\Lambda) \not\equiv 0$,
2. If $\lambda \notin \{0, \infty\}$ is an eigenvalue of $\Delta - \lambda\Lambda$ of multiplicity r, then so is $1/\lambda$,
3. If 0 is an eigenvalue of Δ of multiplicity r, then it is an eigenvalue of Λ of multiplicity $r + m$, where m is the input dimension of $\bar{\mathcal{G}}$.

An ESP is said to be *dichotomic* if it has no eigenvalues on the unit circle. If $\{\Lambda, \Delta\}$ is dichotomic, it has n eigenvalues inside $\mathbb{D}$ and n eigenvalues outside it. Consider the n-dimensional deflating subspace $\mathcal{Y}_{\mathbb{D}}\{\Lambda, \Delta\}$ corresponding to the eigenvalues inside $\mathbb{D}$. Clearly,

$$\mathcal{Y}_{\mathbb{D}}\{\Lambda, \Delta\} = \mathrm{Im} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix},$$

where $y_1, y_2 \in \mathbb{R}^{n \times n}$, $y_3 \in \mathbb{R}^{m \times n}$, and

$$\Delta \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \Lambda \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} A_{st}, \quad \sigma(A_{st}) \in \mathbb{D}. \tag{2}$$

A dichotomic ESP is said to be *disconjugate* if the matrix y_1 is non singular. If $\{\Lambda, \Delta\}$ is disconjugate, it is possible to set $Y = y_2 y_1^{-1}$ and $L' = y_3 y_1^{-1}$. It can be proved (Van Dooren, 1981) that the DARE (1) has a unique stabilizing solution if and only if the ESP $\{\Lambda, \Delta\}$ is disconjugate. Moreover, in this case, Y is its unique stabilizing solution and L is its associated stabilizing matrix gain.

The following Lemma is crucial for the reasoning to follow. It assumes that the H^∞ DARE's associated with the systems

$$\begin{bmatrix} \bar{\mathcal{G}}_1(z) \\ \bar{\mathcal{G}}_2(z) \end{bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline C_1 & D_1 \\ C_2 & D_2 \end{array}\right] \tag{3}$$

and

$$\begin{bmatrix} z^{-1}\bar{\mathcal{G}}_1(z) \\ \bar{\mathcal{G}}_2(z) \end{bmatrix} = \left[\begin{array}{cc|c} A & 0 & B \\ C_1 & 0 & D_1 \\ \hline 0 & I & 0 \\ C_2 & 0 & D_2 \end{array}\right] \tag{4}$$

have stabilizing solutions and establishes a relation between them.

Lemma 1. *Let* $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}' & P_{22} \end{bmatrix}$ *be the stabilizing solution to the* H^∞ *DARE associated with* (4):

$$P = \begin{bmatrix} A & 0 \\ C_1 & 0 \end{bmatrix} P \begin{bmatrix} A' & C_1' \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B \\ D_1 \end{bmatrix} [B' \; D_1'] - L_P R_P L_P', \tag{5}$$

with

$$L_P = \begin{bmatrix} L_{11} & L_{21} \\ L_{12} & L_{22} \end{bmatrix} \doteq -\left(\begin{bmatrix} A & 0 \\ C_1 & 0 \end{bmatrix} P \begin{bmatrix} 0 & C_2' \\ I & 0 \end{bmatrix} + \begin{bmatrix} B \\ D_1 \end{bmatrix} [0 \; D_2'] \right) R_P^{-1}$$

and

$$R_P = \begin{bmatrix} 0 \\ D_2 \end{bmatrix} [0 \; D_2'] + \begin{bmatrix} 0 & I \\ C_2 & 0 \end{bmatrix} P \begin{bmatrix} 0 & C_2' \\ I & 0 \end{bmatrix} - \gamma^2 \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

such that

$$|\gamma^2 I - P_{22}| \neq 0, \tag{6a}$$

and

$$|D_2 D_2' + C_2 Y C_2'| \neq 0, \tag{6b}$$

where $Y \doteq P_{11} + P_{12}(\gamma^2 I - P_{22})^{-1} P_{12}'$. *Then,*

$$\begin{aligned} P = & \begin{bmatrix} A \\ C_1 \end{bmatrix} Y [A' \; C_1'] + \begin{bmatrix} B \\ D_1 \end{bmatrix} [B' \; D_1'] \\ & - \left(\begin{bmatrix} B \\ D_1 \end{bmatrix} D_2' + \begin{bmatrix} A \\ C_1 \end{bmatrix} Y C_2' \right) \\ & \times (D_2 D_2' + C_2 Y C_2')^{-1} \\ & \times \left(\begin{bmatrix} B \\ D_1 \end{bmatrix} D_2' + \begin{bmatrix} A \\ C_1 \end{bmatrix} Y C_2' \right)', \end{aligned}$$

$$L_{21} = -(AYC_2' + BD_2')(C_2 Y C_2' + D_2 D_2')^{-1},$$
$$L_{22} = (C_1 Y C_2' + D_1 D_2')(C_2 Y C_2' + D_2 D_2')^{-1},$$

and Y *is the the stabilizing solution to the* H^∞ *DARE associated with* (3):

$$Y = AYA' + BB' - L_Y R_Y L_Y', \tag{7a}$$

with

$$L_Y = -\left[AYC_1' + BD_1' \quad AYC_2' + BD_2' \right] R_Y^{-1} \tag{7b}$$

and

$$R_Y = \begin{bmatrix} D_1 D_1' + C_1 Y C_1' - \gamma^2 I & D_1 D_2' + C_1 Y C_2' \\ D_2 D_1' + C_2 Y C_1' & D_2 D_2' + C_2 Y C_2' \end{bmatrix}. \tag{7c}$$

Proof. The proof of this Lemma is based on relation (2), written explicitly for the DARE (5), i.e.:

$$\begin{bmatrix} A' & C_1' & 0 & 0 & 0 & C_2' \\ 0 & 0 & 0 & 0 & I & 0 \\ -BB' & -BD_1' & I & 0 & 0 & -BD_2' \\ -D_1 B' & -D_1 D_1' & 0 & I & 0 & -D_1 D_2' \\ 0 & 0 & 0 & 0 & -\gamma^2 I & 0 \\ D_2 B' & D_2 D_1' & 0 & 0 & 0 & D_2 D_2' \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \\ P_{11} & P_{12} \\ P_{12}' & P_{22} \\ L_{11}' & L_{12}' \\ L_{21}' & L_{22}' \end{bmatrix}$$
$$= \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & A & 0 & 0 & 0 \\ 0 & 0 & C_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I & 0 & 0 \\ 0 & 0 & -C_2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \\ P_{11} & P_{12} \\ P_{12}' & P_{22} \\ L_{11}' & L_{12}' \\ L_{21}' & L_{22}' \end{bmatrix} \begin{bmatrix} A_{st}^{11} & A_{st}^{12} \\ A_{st}^{21} & A_{st}^{22} \end{bmatrix}. \tag{8}$$

From the first two block rows of this equation we find that

$$A_{st} = \begin{bmatrix} A' + C_2' L_{21}' & C_1' + C_2' L_{22}' \\ L_{11}' & L_{12}' \end{bmatrix}.$$

By using row 5 of (8) and by assuming that (6a) holds true, we get

$$L_{11}' = (\gamma^2 I - P_{22})^{-1} P_{12}' (A' + C_2' L_{21}')$$

and

$$L_{12}' = (\gamma^2 I - P_{22})^{-1} P_{12}' (C_1' + C_2' L_{22}').$$

Hence

$$[P_{11} \; P_{12}] A_{st} = Y [A' + C_2' L_{21}' \quad C_1' + C_2' L_{22}'].$$

The formulae of P, L_{21} and L_2 follow then by substituting this relation into the 3rd, 4th, and the last block rows of (8), respectively.

To prove that Y satisfies the H^∞ DARE associated with (3), consider the matrix R_Y in (7). This matrix is invertible since $(D_2 D_2' + C_2 Y C_2')$ and its Schur complement $(P_{22} - \gamma^2 I)$ were assumed to be non-singular. Hence, L_Y exists. Simple matrix manipulations yield that

$$\begin{aligned} L_Y R_Y L_Y' = & P_{12}(P_{22} - \gamma^2 I)^{-1} P_{12}' + (AYC_2' + BD_2') \\ & \times (D_2 D_2' + C_2 Y C_2')^{-1} (AYC_2' + BD_2')'. \end{aligned}$$

Add this relation to the formula of P_{11} in terms of Y to get that Y is a solution to (7). The proof that Y is the *stabilizing* solution to (7) follows from the fact that the ESP's associated with (3) and (4) have the same eigenvalues (only the multiplicity of the eigenvalues $\lambda = 0, \infty$ is different). □

In the next section, this Lemma will be applied recursively in order to solve the discrete-time H^∞ fixed-lag smoothing problem.

3 PROBLEM FORMULATION AND SOLUTION

The purpose of this paper is to solve the following design problem:

SP$_r$: Given $\begin{bmatrix}\bar{\mathcal{G}}_1(z)\\ \bar{\mathcal{G}}_2(z)\end{bmatrix}$, (3), and a positive number γ, determine whether a strictly proper and stable filter $\bar{\mathcal{K}}_r(z)$ which guarantees

$$\|z^{-r}\bar{\mathcal{G}}_1(z) - \bar{\mathcal{K}}(z)\bar{\mathcal{G}}_2(z)\|_\infty < \gamma$$

exists. In this case, find such a filter.

Note that **SP**$_0$ is the standard a-priori filtering problem for system (3). Its solution is well-known (Hassibi *et al.*, 1999):

Theorem 1. *Assume that*

(A1): The pair $(A,\, C_2)$ *is detectable;*

(A2): The matrix $\begin{bmatrix} A - e^{j\theta}I & B \\ C_2 & D_2\end{bmatrix}$ *is right invertible* $\forall\theta \in [0, 2\pi)$.

Then, there exists a filter $\bar{\mathcal{K}}_0(z)$ *which solves* **SP**$_0$ *if and only if the DARE* (7) *has a stabilizing solution* $Y \geq 0$ *such that*

$$D_1D_1' + C_1YC_1' < \gamma^2 I. \tag{9}$$

If so, one possible solution to **SP**$_0$ *is*

$$\bar{\mathcal{K}}_0(z) = \left[\begin{array}{c|c} A + L_2C_2 & -L_2 \\ \hline C_1 & 0\end{array}\right],$$

where $L_Y \doteq \begin{bmatrix} L_1 & L_2\end{bmatrix}$ *is the stabilizing matrix gain associated with* Y.

Clearly, if the backward shift operator z^{-r} is absorbed into the transfer matrix $\bar{\mathcal{G}}_1(z)$, **SP**$_r$ can be solved by using Theorem 1, since **SP**$_r$ is just the standard a-priori filtering problem for the system

$$\begin{bmatrix} z^{-r}\bar{\mathcal{G}}_1 \\ \bar{\mathcal{G}}_2\end{bmatrix} \doteq \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C}_1 & \tilde{D}_1 \\ \tilde{C}_2 & \tilde{D}_2\end{array}\right] = \left[\begin{array}{cccccc|c} A & 0 & \dots & \dots & 0 & & B \\ C_1 & 0 & \dots & \dots & 0 & & D_1 \\ 0 & I & \ddots & & \vdots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & & \vdots \\ 0 & \dots & 0 & I & 0 & & 0 \\ \hline 0 & 0 & \dots & 0 & I & & 0 \\ C_2 & 0 & \dots & \dots & 0 & & D_2 \end{array}\right] \begin{matrix} \\ \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} (r-1)\text{ rows} \\ \\ \\ \end{matrix}. \tag{10}$$

Such an approach, however, may suffer from computational problems as the smoothing lag r gets larger, especially due to the fact that a higher dimensional DARE is to be solved.

Our goal is to find the solution to **SP**$_r$ in terms of the parameters of $\bar{\mathcal{G}}_1(z)$ and $\bar{\mathcal{G}}_2(z)$. To this end, we assume that the following difference Riccati equation admits a solution:

$$\begin{aligned} P_{j+1} &= AP_jA' + BB' - (AP_jC_2' + BD_2') \\ &\quad \times (C_2P_jC_2' + D_2D_2')^{-1} \\ &\quad\quad \times (AP_jC_2' + BD_2')', \end{aligned} \tag{11a}$$

where $j = 1, \dots, r$ and where $P_1 = Y$ is the stabilizing solution to the H^∞ DARE (7). The following expressions are also required in the sequel:

$$\begin{aligned} R_{j+1} &= AP_jC_1' + BD_1' - (AP_jC_2' + BD_2') \\ &\quad \times (C_2P_jC_2' + D_2D_2')^{-1} \\ &\quad\quad \times (C_1P_jC_2' + D_1D_2')', \end{aligned} \tag{11b}$$

$$\begin{aligned} H_j &= -(AP_jC_2' + BD_2') \\ &\quad \times (C_2P_jC_2' + D_2D_2')^{-1}. \end{aligned} \tag{11c}$$

We are now in the position to state the main result of this paper.

Theorem 2. *Assume that (A1), (A2) hold true and that recursion* (11) *admits a solution. Then, there exists a filter* $\bar{\mathcal{K}}_r(z)$ *which solves* **SP**$_r$ *if and only if DARE* (7) *has a stabilizing solution* Y *such that the matrix* $A + H_rC_2$ *is Schur and such that* $M_r < \gamma^2 I$, *where*

$$\begin{aligned} M_0 &= D_1D_1' + C_1YC_1', \\ M_1 &= M_0 - (D_1D_2' + C_1YC_2') \\ &\quad (D_2D_2' + C_2YC_2')^{-1}(D_1D_2' + C_1YC_2')', \\ M_2 &= M_1 - R_2'C_2'(D_2D_2' + C_2P_2C_2')^{-1}C_2R_2, \\ M_j &= M_{j-1} - Q_j'(D_2D_2' + C_2P_jC_2')^{-1}Q_j,\ j > 2, \end{aligned}$$

and where

$$Q_j \doteq C_2 \prod_{i=2}^{j-1}(A + H_iC_2)\, R_2\,.$$

If so, one possible solution to **SP**$_r$ *is*

$$\bar{\mathcal{K}}_r(z) = -z^{-r}\left[\begin{array}{c|c} A + L_{2,1}C_2 & L_{2,1} \\ \hline C_1 & 0\end{array}\right] - \sum_{j=1}^{r} z^{-j}L_{2,r+2-j} - \left[\begin{array}{c|c} A + L_{2,1}C_2 & L_{2,1} \\ \hline C_2 & I\end{array}\right], \tag{12}$$

where

$$L_{2,1} = H_r, \tag{13a}$$

$$\begin{aligned} L_{2,2} &= -(C_1P_rC_2' + D_1D_2') \\ &\quad \times (C_2P_rC_2' + D_2D_2')^{-1}, \end{aligned} \tag{13b}$$

$$L_{2,j} = -S_j'(C_2P_rC_2' + D_2D_2')^{-1}, \tag{13c}$$

and where

$$S_j \doteq C_2 \prod_{i=3}^{j-1} (A + H_i C_2)\, R_{r+3-j}, \quad j = 3, \ldots, r+1.$$

Proof. The first part of the proof is based on the afore mentioned fact that $\mathbf{SP}_r$ can be treated as a priori filtering problem for the system (10). Thus, we start by applying step-by-step Theorem 1 to the system (10) and we simplify the result as much as possible.

First, note that assumptions *(A1)* and *(A2)* in Theorem 1 (the detectability and the absence of unit circle zeros for $\bar{\mathcal{G}}_2(z)$) are not affected by r since $\bar{\mathcal{G}}_2(z)$ is common to all $\mathbf{SP}_r$. Then, assume for the moment that the H^∞ DARE associated with (10) has a stabilizing solution $\tilde{Y}$, denote its stabilizing matrix gain by $\tilde{L} \doteq [\tilde{L}_1\ \tilde{L}_2]$ and let $\tilde{L}_2$ be of the form $\tilde{L}_2 = [L'_{2,1}\ L'_{2,2} \ldots L'_{2,r+1}]'$. The condition $\tilde{Y} \geq 0$ is equivalent to the requirement $\sigma(\tilde{A} + \tilde{L}_2\tilde{C}_2) \in \mathbb{D}$ (see (Hassibi *et al.*, 1999) for a proof). This in turn, is equivalent to $\sigma(A + L_{2,1}C_2) \in \mathbb{D}$, due to the particular forms of $\tilde{A}$ and $\tilde{C}_2$. It will be shown in the sequel that $L_{2,1} = H_r$. Hence, the condition $\tilde{Y} \geq 0$ can be replaced with the requirement $\sigma(A + H_r C_2) \in \mathbb{D}$ which is much easier to check. Condition (9) can also be simplified. By using the particular forms of $\tilde{D}_1$ and $\tilde{C}_1$ it is found that this condition is equivalent to $\tilde{Y}_{r+1,r+1} < \gamma^2 I$, where $\tilde{Y}_{r+1,r+1}$ is the last matrix block of $\tilde{Y}$. It will be shown in the sequel that $\tilde{Y}_{r+1,r+1}$ is equal to M_r. Finally, we simplify the expression of $\bar{\mathcal{K}}_r(z)$. According to Theorem 1

$$\bar{\mathcal{K}}_r(z) = \left[\begin{array}{c|c} \tilde{A} + \tilde{L}_2\tilde{C}_2 & -\tilde{L}_2 \\ \hline \tilde{C}_1 & 0 \end{array}\right].$$

By substituting the particular forms of $\tilde{A}$, $\tilde{C}_1$, $\tilde{C}_2$ and the partition of $\tilde{L}_2$ into this formula, (12) follows.

In order to complete the proof, we only have

1. to find a simpler form to the requirement that the H^∞ DARE associated with (10) has a stabilizing solution, and
2. to calculate $\tilde{Y}_{r+1,r+1}$ and $L_{2,j}$, $j = 1, \ldots, r+1$.

This is accomplished by applying Lemma 1 recursively, r times.

First, consider the case $r = 1$ and note that condition (6b) holds true since we assumed that recursion (11) admits a solution. Also note that the solvability of $\mathbf{SP}_r$ requires that $\tilde{Y}_{2,2} < \gamma^2 I$. Hence, we are interested only in the case where $|\gamma^2 I - \tilde{Y}_{2,2}| \neq 0$. Consequently, Lemma 1 can be applied. Under these assumptions, according to this lemma, the H^∞ DARE associated with (10) has a stabilizing solution $\tilde{Y}$ if and only if so has DARE (7). The expressions for $L_{2,1}$ and $L_{2,2}$ in (13) and the fact that $\tilde{Y}_{2,2} = M_1$ (for the case $r = 1$) follow directly from Lemma 1.

So far we have completed the proof for the case $r = 1$ only. This was done by applying the solution for the case $r = 0$ to the system $\begin{bmatrix} z^{-1}\bar{\mathcal{G}}_1(z) \\ \bar{\mathcal{G}}_2(z) \end{bmatrix}$ and exploiting Lemma 1 to simplify it. By applying the solution for the case $r = 1$ to the same system, it is possible to find a solution to the case $r = 2$. By following the same reasoning as before, we find that Lemma 1 can be used to simplify this solution in the same manner. By continuing this procedure several times, the solution to the general case presented in this theorem is proved by induction. □

Remark. It is worth mentioning that P_r is the first matrix block of $\tilde{Y}$. Since $\tilde{Y} = \tilde{Y}' \geq 0$ is a necessary condition for the solvability of $\mathbf{SP}_r$, so is $P_r \geq 0$.

Remark. Note that the value of M_r is monotonically non increasing as a function of the smoothing lag r. Thus, as expected, the smoother has the potential to achieve a better performance level γ than the a priori filter, and the larger is the smoothing lag, a better estimate can be obtained.

Remark. Also note that the solution presented in Theorem 2 is based on the assumption that the difference Riccati equation (11) admits a solution. This condition is not necessary for the solvability of $\mathbf{SP}_r$ and represents a small disadvantage of the solution presented in this paper. It is known, (Hassibi *et al.*, 1999), that there exist situations in which the difference Riccati equation (11) presents a finite escape point. In those cases there exist a certain value of r for which the H^∞ DARE associated with the system (10) does not have a stabilizing solution. Clearly, for this certain value of the smoothing lag, $\mathbf{SP}_r$ has no solution. However, for larger values of r $\mathbf{SP}_r$ might have a solution and Theorem 2 can not be used to find it. These singular cases can be circumvented by introducing a small change in the required performance level γ. By doing so the difference Riccati equation (11) will admit a solution which, in turn, will enable the use of Theorem 2.

4 CONCLUSIONS

In this paper, the problem of the discrete-time H^∞ fixed-lag smoothing over an infinite horizon has been studied. In contrast to other solutions available in the literature, the solution presented in this paper is based on *one* DARE of the *same dimensions* as the original system dynamics. Consequently, the amount of computations required to solve this problem is considerably reduced and more accurate solutions can be obtained.

It is worth mentioning that the solution to the discrete-time H^∞ smoothing problem is still more complicated and less transparent than the corresponding continuous-time solution in (Mirkin, 2001). We believe that a more elegant discrete-time solution exists as well. The derivation of such a solution is the subject of the current research.

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