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ADAPTIVE CONTROL ARCHITECTURE OF BRAIN MOTOR CONTROL INCORPORATING SKILL ACQUISITION

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Abstract: This paper proposes an architecture and algorithm of brain motor control based on the internal model control (IMC) incorporating the acquisition of skill explicitly. The meaning of skill is extended from acquisition of internal model, which is a widespread view in computational neuroscience, to acquisition of internal model and prefilter with appropriate bandwidth. This extension is essential in view of the fact that no inverse model exists for usual plants.

Keywords: Brain models; Adaptive control; Skill; Model based control.

1. INTRODUCTION

In brain motor control, how human being can get skills has been a long standing issue. Recently, some computational neuroscientists brought forward the idea that the skill is defined to be an acquisition of internal models of the limbs and the environment (e.g., Kawato & Gomi, 1992). They used the so-called *feedback error learning* method to construct an inverse model. Actually, the inverse model may not exist for usual dynamical systems. A usual technique to circumvent the difficulty is to insert a lowpass prefilter that makes the inverse model realizable. However, there still remains the problem of how to choose the bandwidth of the filter. If the bandwidth is too narrow, it leads to degradation of the performance, and if the bandwidth is too broad while the inverse model is not accurate, it may lead to instability. Therefore, the choice of the bandwidth is really a matter of skill. The skillful control must choose the bandwidth of the prefilter concordantly with the accuracy of inverse model.

In this paper, an architecture of learning controller which incorporates the skill acquisition in the above sense is explored. Our approach can be regarded as an adaptive version of windsurfer approach (Anderson & Kosut 1991). In Section 2, proposed adaptive brain motor control architecture is stated. In Section 3, concrete adaptive control algorithm is given. In Section 4, simulation result is shown and Section 5 gives concluding remarks.

2. BASIC ARCHITECTURE

2.1 *Internal Model Control*

The basic architecture is shown in Fig. 1. This scheme is called *Internal Model Control* (Morari & Zafiriou, 1989). Here $\hat{P}$ denotes a model of the

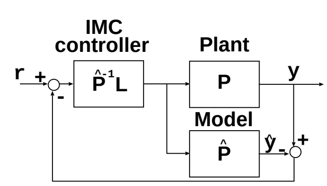

Fig. 1. Architecture of IMC

plant P. L is a prefilter such that controller $\hat{P}^{-1}L$ is a realizable dynamical system. $L(s)$ such as

$$L(s) = \frac{\omega^{d*}}{(s+\omega)^{d*}} \tag{1}$$

is taken in the same way as Morari & Zafiriou (1989), where $d*$ is a positive constant enough to make $\hat{P}^{-1}L$ proper. ω is the only parameter of filter. Roughly speaking, system is robustly stable and bandwidth is narrow when ω is small and vice versa (Morari & Zafiriou, 1989).

In Miall *et al.* (1993), Smith Predictor which is a special case of IMC was employed to explain motor control in the cerebellum. There, interpretation of IMC from the brain motor control point of view was conjectured as follows. Plant corresponds to musculoskeletal system or controlled object. Desired motion r is issued in posterior parietal cortex, passes the IMC controller constructed in the cerebellum and goes to motor cortex by way of ventrolateral thalamic nucleus. In motor cortex, input to musculoskeletal system is produced. Model is acquired in the cerebellum. It produces estimated state of the plant $\hat{y}$. This signal is compared with real output y from peripheral proprioceptors.

2.2 *Adaptive IMC with skill enhancer*

In the above scheme, when model is known or only approximately known, controller acts as a feedback controller. On the other hand, when model is perfect, it works as a feedforward controller. In human skill acquisition process, IMC controller works as feedback at early period of learning and gradually shifts to feedforward as learning progresses. Synaptic plasticity called "Long Term Depression" is conjectured to be a basic substratum of adaptation in the cerebellum (e.g., Kawato & Gomi, 1992).

Refinement of model can be done by minimizing the error between real output y and estimated output $\hat{y}$. Then, IMC controller is constructed by copying parameters of model. But learning of the model only is not enough. Adaptation of the adjustable parameter of prefilter is also important to realize excellent skill enhancement. Skill enhancement is a process of gradually improving control performances. To do so, adaptation of prefilter parameter to decrease control error between y and $L_*(s)r$ is required where $L_*(s) = \frac{\omega_*^{d*}}{(s+\omega_*)^{d*}}$ and ω_* is desired bandwidth. Pathway from inferior olive is considered as conveyer of error signal between reference signal and output (Kawato & Gomi, 1992) or between output and estimated output (Miall & Wolpert, 1996). These signals can be used for adaptation of filter parameter and model.

2.3 *Relation to conventional control architectures*

Anderson & Kosut (1991) proposed an iterative control system synthesis method imitating the process through which human learns how to windsurf. They proposed a method of gradually increasing ω as identification proceeds using IMC scheme. Some work has been done about adaptive IMC. Specifically, in Datta & Ochoa (1998), stability analysis was done and convergence of the estimation error was proved. But in these works, adjustable parameter of IMC filter was fixed. In Papadoulis & Svoronos (1987), adaptive control architecture like IMC was proposed and adaptive law of filter parameter based on the size of estimation error $y - \hat{y}$ was proposed.

In this paper, brain adaptive control architecture is proposed based on adaptive IMC in Datta & Ochoa (1998). In their scheme, model is identified using estimation error $y - \hat{y}$ and construct a controller using estimated parameter. But from control point of view, estimation only is not enough. A mechanism to reduce control error $y - L_*(s)r$ is required and the filter parameter ω will take care of it. In the following, first, methods of estimation of model and construction of the time varying controller in Datta (1998) is shown. Secondly, an adaptive law of filter parameter ω (Nagai & Kimura, 2001) is given.

3. ADAPTIVE ALGORITHM

3.1 *Adaptive IMC*

Controlled system is represented as

$$y = \frac{B(s)}{A(s)}[u] \tag{2}$$

where y is output of the plant, u input of the plant, $A(s)$ a monic stable polynomial with order d, and $B(s)$ a stable polynomial with order n. Assume that d and n are known and plant is strictly proper, that is $n < d$.

For estimation, (2) is transformed into linear form with respect to the parameters as in Datta (1998). Introduce monic stable polynomial $Q_m(s)$ with order d and transform (2) into

$$y = \frac{Q_m(s) - A(s)}{Q_m(s)}[y] + \frac{B(s)}{Q_m(s)}[u] \tag{3}$$

(3) can be represented as

$$y = v^T x \tag{4}$$

where $v = [v_1^T, v_2^T]^T$. v_1 and v_2 are coefficients of polynomial $Q_m(s) - A(s)$ and $B(s)$, respectively, and $x = [x_1^T, x_2^T]^T, x_1 = (p_{d-1}(s)/Q_m(s))[y]$, $x_2 = (p_n(s)/Q_m(s))[u]$, where

$$p_{d-1}(s) = [s^{d-1}, s^{d-2}, \cdots, 1]^T, \tag{5}$$
$$p_n(s) = [s^n, s^{n-1}, \cdots, 1]^T \tag{6}$$
$$Q_m(s) - A(s) = v_1^T p_{d-1}(s) \tag{7}$$
$$B(s) = v_2^T p_n(s) \tag{8}$$

Model for (4) and parameter estimation law are as follows (Datta, 1998)

$$\hat{y} = \hat{v}^T x \tag{9}$$
$$\dot{\hat{v}} = Pro[kex], \hat{v}(0) \in D \tag{10}$$
$$e = \frac{y - \hat{y}}{n_m^2} \tag{11}$$
$$n_m^2 = 1 + f_m \tag{12}$$
$$\dot{f}_m = -\eta_m f_m + u^2 + y^2, f_m(0) = 0 \tag{13}$$

where k is an adaptive gain and positive constant, $\eta_m > 0$ is a constant such that $\frac{1}{Q_m(s)}$ is analytic in $Re[s] \geq -\frac{\eta_m}{2}$, and $Pro[\cdot]$ is a projection operation defined as follows.

Parameter Projection (Pomet & Praly 1992)
Assume that C^2 function $R_v(\hat{v})$ satisfies the following conditions,
(1)For any real $a \in [0, 1]$, the set $\{\hat{v} | R_v(\hat{v}) \leq a\}$ is convex.
(2)For any $\hat{v}$ such that $R_v(\hat{v}) \in [0, 1]$, $\frac{\partial R_v}{\partial \hat{v}} \neq 0$.
(3)Parameter of true plant v meets $R_v(v) \leq 0$.
(4)$R_v(\hat{v}(0)) \leq 1$.
Under these assumptions, projection is defined as

$$Pro[f_m] = \begin{cases} f_m & \text{if } R_v(\hat{v}) \leq 0, \text{ or if } R_v(\hat{v}) \geq 0 \\ & \text{and } (\frac{\partial R_v}{\partial \hat{v}})^T f_m \leq 0 \\ f_m - \frac{R_v(\hat{v}) \frac{\partial R_v}{\partial \hat{v}} (\frac{\partial R_v}{\partial \hat{v}})^T}{||\frac{\partial R_v}{\partial \hat{v}}||^2} f_m & \text{otherwise} \end{cases} \tag{14}$$

where $f_m = kex$. Using projection, estimated parameter $\hat{v}$ is assured to be in $D = \{\hat{v} | R_v(\hat{v}) \leq 1\}$ for all $t \geq 0$. D is chosen so that coefficient of the highest order of estimated numerator polynomial does not pass 0, estimated numerator and denominator polynomials are stable.

Time varying controller is constructed in the same way as (Datta & Ochoa, 1998; Datta, 1998). At first, introduce description $\hat{A}(s, t)$, which shows a polynomial or a transfer function derived when coefficients of time invariant polynomial or transfer function A(s) are replaced by time variable ones and time is fixed. Estimated model of the plant for fixed t is represented using estimated parameters $\hat{v}_1, \hat{v}_2$ as

$$\hat{P}(s, t) = \frac{\hat{B}(s, t)}{\hat{A}(s, t)} \tag{15}$$

where $\hat{B}(s, t) = \hat{v}_2^T(t) p_n(s), \hat{A}(s, t) = Q_m(s) - \hat{v}_1^T(t) p_{d-1}(s)$. For fixed t, time varying prefilter is

$$\hat{L}(s, t) = \frac{\omega(t)^{d*}}{(s + \omega(t))^{d*}} \tag{16}$$

where $d* = d - l$ so that the order of $\hat{P}^{-1}(s, t)\hat{L}(s, t)$ equals to d. Then IMC controller for fixed t is expressed as

$$\hat{P}(s, t)^{-1} \hat{L}(s, t) \tag{17}$$

Control input u can be represented as

$$u = \hat{P}(s, t)^{-1} \hat{L}(s, t)[r - (y - \hat{y})] \tag{18}$$
$$= \frac{\frac{\omega(t)^{d*} \hat{A}(s,t)}{b_0(t)}}{\frac{\hat{B}(s,t)(s+\omega(t))^{d*}}{b_0(t)}} [r - en_m^2] \tag{19}$$

for fixed t, where $b_0(t)$ is the coefficient of the highest order of $\hat{B}(s, t)$ and equation (11) was used in transformation. This can be transformed into

$$u = \frac{Q_c(s) - \frac{\hat{B}(s,t)(s+\omega(t))^{d*}}{b_0(t)}}{Q_c(s)} [u] + \frac{\frac{\omega(t)^{d*} \hat{A}(s,t)}{b_0(t)}}{Q_c(s)} [r - en_m^2] \tag{20}$$

where $Q_c(s)$ is monic stable polynomial whose order is d. Transform (20) into linear form with respect to time varying coefficients

$$u = l_1^T(t) \frac{p_{d-1}(s)}{Q_c(s)} [u] + l_2^T(t) \frac{p_d(s)}{Q_c(s)} [r - en_m^2] \tag{21}$$

where $l_1(t)$ and $l_2(t)$ are time varying parameters which contain estimated model parameter $\hat{v}(t)$ and a filter parameter $\omega(t)$. Input is generated changing these parameters by adaptive laws.

3.2 *Adaptive law of prefilter parameter*

A small initial value $\omega(0)$ is taken to make input small. Because at early period of learning, large model uncertainty is expected. This is same as Anderson & Kosut (1991). In their research, increasing ω intends to attain desirable bandwidth through extending bandwidth of closed loop step by step. Here, adaptive law of ω is considered. For skill enhancement, an adaptive law which gradually decreases the following value function proposed in Nagai & Kimura (2001) is used.

$$C_z = \frac{1}{2n_z^2} (y - L_*(s)[r])^2 \tag{22}$$

where $L_*(s)$ was defined in Section 2.2 and n_z is a normalizing signal often used to assure signals in adaptive system to be bounded (e.g., Datta, 1998) and also used in model parameter estimator. Normalizing signal n_z is given by

$$n_z^2(t) = \alpha_z + f_z(t) + gy^2(t) \tag{23}$$

$$\frac{df_z(t)}{dt} = -\eta_z f_z(t) + r^2(t) + y^2(t) + (\frac{\partial y}{\partial \omega})^2, f_z(0) = 0 \tag{24}$$

where α_z and g are positive constants, η_z a positive constant such that $\frac{1}{Q_c(s)}$ and $L_*(s)$ analytic in $Re(s) \geq -\frac{\eta_z}{2}$.

Adaptive law is derived from value function (22) using steepest decent method. Projection operator as in (10) is used to keep $\omega(t)$ in a prescribed region.

$$\begin{aligned}\frac{d\omega}{dt} &= Pro[-k_z \frac{\partial C_z}{\partial \omega}], \\ &\quad \omega(0) \in \{\omega | R_\omega(\omega) \leq 1\} \qquad (25) \\ &= Pro[-\frac{k_z}{n_z^2}(y - L_*(s)[r])\frac{\partial y}{\partial \omega}], \\ &\quad \omega(0) \in \{\omega | R_\omega(\omega) \leq 1\}\end{aligned}$$

where $k_z > 0$ is a constant adaptive gain. ω is a scalar, so projection for ω becomes

$$Pro[f_z] = \begin{cases} f_z & \text{if } R_\omega(\omega) \leq 0, \text{ or if } R_\omega(\omega) \geq 0 \\ \text{and} & \frac{\partial R_\omega}{\partial \omega} \cdot f_z \leq 0 \\ f_z - R_\omega(\omega) f_z & \text{otherwise} \end{cases} \tag{26}$$

where $f_z = -\frac{k_z}{n_z^2}(y - L_*(s)[r])\frac{\partial y}{\partial \omega}$ and assume that for any real $a \in [0,1]$, the set $\{\omega | R_\omega(\omega) \leq a\}$ is convex, $R_\omega(\omega(0)) \leq 1, R_\omega(\omega_*) \leq 1$, $\frac{\partial R_\omega}{\partial \omega} \neq 0$ for any ω such that $R_\omega(\omega) \in [0,1]$, and ω which meets $\{\omega | R_\omega(\omega) \leq 1\}$ is positive.

Adaptive law (25) cannot be used because $\frac{\partial y}{\partial \omega}$ is not available. So, approximate $\frac{\partial y}{\partial \omega}$ as follows in the same way as Jordan & Rumelhart (1992) which proposed inverse model adaptation method making use of forward model.

First, assume that estimated model parameters are identical to real plant (constant) parameters, that is

$$P(s) = \hat{P}(s,t) = \frac{\hat{B}(s,t)}{\hat{A}(s,t)} \tag{27}$$

Then feedback signal $en_m^2 = 0$ and closed loop system becomes feedforward. In (21), ignoring the term $en_m^2 = 0$, input of the plant becomes

$$u = l_1^T(t)\frac{p_{d-1}(s)}{Q_c(s)}[u] + l_2^T(t)\frac{p_d(s)}{Q_c(s)}[r] \tag{28}$$

So output y is calculated to be

$$y = \hat{P}(s,t)[u] = \frac{\hat{B}(s,t)}{\hat{A}(s,t)}[l_1^T(t)\frac{p_{d-1}(s)}{Q_c(s)}[u] + l_2^T(t)\frac{p_d(s)}{Q_c(s)}[r] \tag{29}$$

Here, assume that $\dot{\omega}(t)$ is small enough in calculating right hand side of (29). Then,

$$\begin{aligned} y &= l_1(t)^T \frac{p_{d-1}(s)}{Q_c(s)} \frac{\hat{B}(s,t)}{\hat{A}(s,t)}[u] + \frac{\omega(t)^{d*}}{b_0(t)} \frac{\hat{B}(s,t)}{Q_c(s)}[r] \\ &= l_1(t)^T \frac{p_{d-1}(s)}{Q_c(s)}[y] + \frac{\omega(t)^{d*}}{b_0(t)} \frac{\hat{B}(s,t)}{Q_c(s)}[r] \\ &= h_1^T(t)\frac{p_{d-1}(s)}{Q_c(s)}[y] + h_2^T(t)\frac{p_d(s)}{Q_c(s)}[r] \end{aligned} \tag{30}$$

where the relation $y = \hat{P}(s,t)[u]$ was used, and define as follows. $\hat{B}(s,t) = b_0(t)s^n + b_1(t)s^{n-1} + \cdots + b_n(t)$, $h_1(t) = l_1(t)$, $h_2^T(t) = (\omega(t)^{d*}, \frac{\omega(t)^{d*}b_1(t)}{b_0(t)}, \cdots, \frac{\omega(t)^{d*}b_n(t)}{b_0(t)})$

Consider that (30) as a real output of the plant and calculate partial derivative of each side of (30) with respect to ω. Then, $\frac{\partial y}{\partial \omega}$ is generated as

$$\frac{\partial y}{\partial \omega} = h_1^T(t)\frac{p_{d-1}(s)}{Q_c(s)}[\frac{\partial y}{\partial \omega}] + \frac{\partial h_1^T(t)}{\partial \omega}\frac{p_{d-1}(s)}{Q_c(s)}[y] + \frac{\partial h_2^T(t)}{\partial \omega}\frac{p_d(s)}{Q_c(s)}[r] \tag{31}$$

Using adaptive law (23)-(26) and (31), next Lemma can be shown. Proof is given in Appendix.

Lemma 1 (Nagai & Kimura 2001)
(i)$R_\omega(\omega(t)) \leq 1 \quad \forall t \geq 0$
(ii)$\omega(t) \in L_\infty$
(iii)If adaptive gain k_z is small enough, $|\frac{d\omega}{dt}|$ is bounded by any small positive constant δ_z for all $t \geq 0$.

3.3 *Properties of the adaptive system*

Theorem 1 (Nagai & Kimura, 2001)
Let $r(t)$ be a piecewise continuous uniformly bounded reference input. Consider adaptive control scheme composed of plant (2), parameter estimator (9)-(14), time varying controller (21) and an adaptive law of ω (23)-(26) and (31). If adaptive gain k of model estimator and adaptive gain k_z of ω are sufficiently small, all signals in the adaptive system are uniformly bounded and estimation error $y - \hat{y}$ vanishes asymptotically.

Note that the statement of Theorem 1 is identical to that of Theorem 5.3.1 in Datta (1998) except that in Theorem 1, adaptation of ω and restrictions of adaptive gains k and k_z are imposed.

Proof of Theorem 1 can be done through almost the same way as of Theorem 5.3.1 in Datta (1998) using properties assured in Lemma 1.

4. SIMULATION

Plant is given by

$$P(s) = \frac{3s+1}{s^2+2s+2} \tag{32}$$

Reference input is 0.2Hz square wave whose amplitude is 1. Design parameters are as follows,

Constants in normalizing signal :
$\alpha_z = 1, g = 0.1, \eta_z = 1, \eta_m = 1$
Initial value of ω : $\omega(0) = 0.01$
Ideal value of ω : $\omega_* = 5$
Adaptive gain of ω : $k_z = 0.1$
Adaptive gain of model estimator : $k = 10$
Initial value of model parameters :
$\hat{v}_1(0) = -12, \hat{v}_2(0) = -15, \hat{v}_3(0) = 16, \hat{v}_4(0) = 15$
True value of parameters :
$v_1 = 3, v_2 = 4, v_3 = 3, v_4 = 1$
Polynomial for constructing controller :
$Q_c(s) = s^2 + 5s + 6$
Polynomial for constructing model :
$Q_m(s) = s^2 + 5s + 6$

Results are shown in Figure 2 to Figure 7. Left side figures show each signal at 0∼200(s), while right side ones are at 19900∼ 20000(s). Figure 2 shows the output of the plant. As ω grows gradually in Figure 4, y becomes larger. Tracking error $y - L_*(s)r$ is reduced after adaptation as shown in Figure 7. In this simulation, estimated parameter $\hat{v}$ converged to true value.

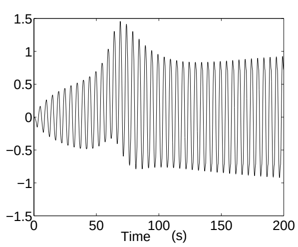

Fig. 2. y : 0-200(s)

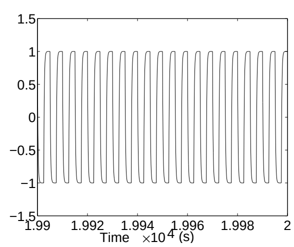

Fig. 3. y : 19900-20000(s)

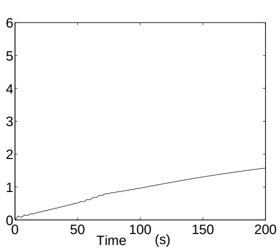

Fig. 4. ω : 0-200(s)

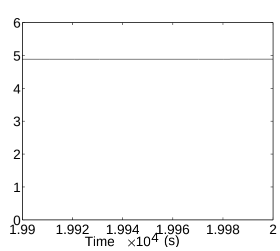

Fig. 5. ω : 19900-20000(s)

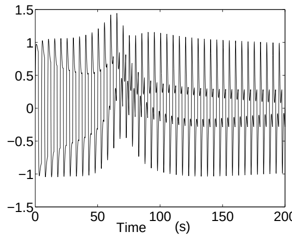

Fig. 6. $y - L_*(s)r$: 0-200(s)

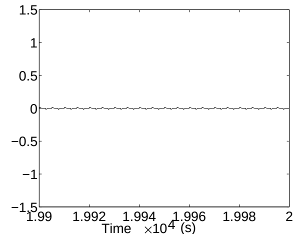

Fig. 7. $y - L_*(s)r$: 19900-20000(s)

5. CONCLUSION

In this paper, an adaptive scheme is proposed as a possible architecture for brain motor control which incorporates skill acquisition. The skill is defined in this paper as an ability to tune a parameter that controls the gain as well as its bandwidth of the adaptive control system. A stability proof has been established. A design simulation has been done, that describes the feasibility of this method. Hopefully, it is desirable to find a physiological support of this method.

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7. APPENDIX

Proof of Lemma 1

In the proof, γ denotes a positive constant. Properties (i) and (ii) of Lemma 1 are guaranteed clearly by projection. Proof of claim (iii) is as follows. First, consider the case

(a) $R_\omega(\omega) \le 0$, or $R_\omega(\omega) \ge 0$ and $\frac{\partial R_\omega}{\partial \omega} \cdot f_z \le 0$.

In this case, from (25)

$$\Big|\frac{d\omega}{dt}\Big| = \frac{k_z}{n_z^2}\Big|y - L_*(s)[r]\Big|\Big|\frac{\partial y}{\partial \omega}\Big| \tag{33}$$

In the following, it will be shown that $\frac{y-L_*(s)[r]}{n_z}$ and $\frac{\frac{\partial y}{\partial \omega}}{n_z}$ are bounded. Then, from (33), $|\frac{d\omega}{dt}|$ can be bounded by any small positive constant δ_z by choosing adaptive gain k_z small enough. At first, $\frac{y-L_*(s)[r]}{n_z}$ is bounded is shown. From triangle inequality

$$\frac{|y - L_*(s)[r]|}{n_z} \le \frac{|y|}{n_z} + \frac{|L_*(s)[r]|}{n_z} \tag{34}$$

The first term of right hand side in (34) is evaluated as

$$\frac{|y|}{n_z} \le \frac{|y|}{\sqrt{\alpha_z + g|y|^2}} \tag{35}$$

where

$$n_z = \Big(\alpha_z + (||r_t||_2^{\eta_z})^2 + (||y_t||_2^{\eta_z})^2 + (||(\frac{\partial y}{\partial \omega})_t||_2^{\eta_z})^2 + gy^2\Big)^{\frac{1}{2}} \tag{36}$$

and exponentially weighted L_2 norm (e.g., Datta, 1998) of w is defined as

$$||w_t||_2^\eta = (\int_0^t e^{-\eta(t-\tau)} w^T(\tau)w(\tau)d\tau)^{\frac{1}{2}} \tag{37}$$

The facts $(||y_t||_2^{\eta_z})^2 \ge 0$, $(||r_t||_2^{\eta_z})^2 \ge 0$ and $(||(\frac{\partial y}{\partial \omega})_t||_2^{\eta_z})^2 \ge 0$ is used in (35). Therefore, $\frac{y}{n_z}$ is bounded. $L_*(s)$ is strictly proper because plant is strictly proper. Using Lemma 2 (shown as bellow),

$$|L_*(s)[r]| \le ||L_*(s)||_2^{\eta_z}||r_t||_2^{\eta_z} \le \gamma||r_t||_2^{\eta_z} \tag{38}$$

So

$$\Big|\frac{L_*(s)[r]}{n_z}\Big| \le \frac{\gamma||r_t||_2^{\eta_z}}{n_z} \le \frac{\gamma||r_t||_2^{\eta_z}}{\sqrt{\alpha_z + (||r_t||_2^{\eta_z})^2}} \tag{39}$$

and it follows that $\frac{L_*(s)[r]}{n_z}$ is bounded. As a result, $\frac{y-L_*(s)[r]}{n_z}$ is bounded.

Next, it will be shown that $\frac{\frac{\partial y}{\partial \omega}}{n_z}$ is bounded. In (31), elements of $h_1(t)$, $\frac{\partial h_1^T(t)}{\partial \omega}$, $\frac{\partial h_2^T(t)}{\partial \omega}$ are bounded because they are composed of constants, $\frac{1}{b_0(t)}$, $b_1(t)$, $\cdots, b_n(t)$ and $\omega(t)$ which are bounded and $b_0(t) \ne 0$ for all $t \ge 0$ by projection. Define $h_1^T(t) = (h_{1,1}(t), \cdots, h_{1,d}(t))$ and $h_2^T(t) = (h_{2,1}(t), \cdots, h_{2,d+1}(t))$, then

$$\begin{aligned}\Big|\frac{\partial y}{\partial \omega}\Big| \le \Big|&h_{1,1}(t)\frac{s^{d-1}}{Q_c(s)}[\frac{\partial y}{\partial \omega}] + \cdots + h_{1,d}(t)\frac{1}{Q_c(s)}[\frac{\partial y}{\partial \omega}]\Big| \\ &+ \Big|\frac{\partial h_{1,1}(t)}{\partial \omega}\frac{s^{d-1}}{Q_c(s)}[y] + \cdots + \frac{\partial h_{1,d}(t)}{\partial \omega}\frac{1}{Q_c(s)}[y]\Big| \\ &+ \Big|\frac{\partial h_{2,1}(t)}{\partial \omega}\frac{s^d}{Q_c(s)}[r] + \cdots \\ &+ \frac{\partial h_{2,d+1}(t)}{\partial \omega}\frac{1}{Q_c(s)}[r]\Big|\end{aligned} \tag{40}$$

Here, using Lemma 2,

$$\Big|\frac{\partial y}{\partial \omega}\Big| \le \gamma + \gamma||y_t||_2^{\eta_z} + \gamma||(\frac{\partial y}{\partial \omega})_t||_2^{\eta_z} \tag{41}$$

$\frac{|\frac{\partial y}{\partial \omega}|}{n_z}$ meets the following inequality

$$\frac{|\frac{\partial y}{\partial \omega}|}{n_z} \le \frac{\gamma + \gamma||y_t||_2^{\eta_z} + \gamma||(\frac{\partial y}{\partial \omega})_t||_2^{\eta_z}}{n_z} \tag{42}$$

So, $\frac{\frac{\partial y}{\partial \omega}}{n_z}$ is bounded.

After all, $\frac{y-L_*(s)[r]}{n_z}$ and $\frac{\frac{\partial y}{\partial \omega}}{n_z}$ are bounded.

(b)In the case of otherwise.

In this case, $R_\omega \ge 0$. And projection keeps ω in $\{\omega | R_\omega(\omega) \le 1\}$. Moreover, by an assumption, $R_\omega(\omega(0)) \le 1$. As a result, $0 \le R_\omega(\omega) \le 1$. So, it follows that $|\frac{d\omega}{dt}| = |f_z - R_\omega(\omega)f_z| = |f_z||1 - R_\omega(\omega)| \le |f_z|$ where $f_z \equiv -\frac{k_z}{n_z^2}(y - L_*(s)[r])\frac{\partial y}{\partial \omega}$. So in the same way as the case (a), claim (iii) of Lemma 1 can be proved. This completes the proof.

Lemma 2 (Datta, 1998)

Consider the following linear system,

$$y = T(s)[u] \tag{43}$$

where, $T(s)$ is a rational function of s and assume that $u \in L_{2e}$ and $T(s)$ is analytic in $Re(s) \ge -\frac{\eta}{2}$. If $T(s)$ is strictly proper, then

$$|y(t)| \le ||T(s)||_2^\eta ||u_t||_2^\eta \tag{44}$$

where

$$||T(s)||_2^\eta \equiv \frac{1}{2\pi}\Big\{\int_{-\infty}^{\infty}\Big|H(j\omega - \frac{\eta}{2})\Big|^2 d\omega\Big\}^{\frac{1}{2}} \tag{45}$$