15th Triennial World Congress, Barcelona, Spain

OPTIMAL SEQUENTIAL CHANGE DETECTION AND ISOLATION

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Abstract: The problem of detecting and isolating abrupt changes in random signals and systems is addressed in the paper. The key results of the recently developed optimal theory of change diagnosis (detection and isolation) are presented. The application of the developed theory to the problem of navigation system integrity monitoring is discussed.

Keywords: Decision theory, random processes, dynamic systems, abrupt change detection isolation, fault detection isolation, safety-critical, navigation systems, inertial reference units.

1. INTRODUCTION

Statistical decision tools for detecting and isolating abrupt changes in the properties of stochastic signals and dynamical systems have numerous applications, from on-line fault diagnosis in complex technical systems to detection of signals with unknown arrival time in geophysics, radar and sonar signal processing. The change detection problem (binary hypothesis case) has received extensive research attention (Shiryayev, 1977; Lorden, 1971; Basseville and Nikiforov, 1993; Lai, 1995; Lai, 1998; Lai, 2001). The goal of this paper is to describe the recently developed optimal theory of change diagnosis (detection and isolation) (Nikiforov, 1995*a*; Nikiforov, 1997; Lai, 2000; Nikiforov, 2000). The readers will be provided with some ideas how these new results can be used in practice to solve on-line fault diagnosis problems, especially in safety-critical applications. Some open problems also will be discussed. The paper is organized as follows. First, the problem is stated, namely, the models and criteria are introduced in section 2. Next, the design of the change diagnosis algorithms and their statistical properties are discussed in section 3. Some practical applications are described in section 4. Some open problems are drawn in section 5. Finally, concluding remarks are given in section 6.

2. MODELS AND CRITERIA

In this section, some statistical models and criteria of optimality are introduced. These models and criteria will be used in the rest of the paper to design the change detection/isolation algorithms and investigate their statistical properties.

2.1 *Models with abrupt changes*

Generic model. Let us assume that there exists a discrete time stochastic system

$$Y_k = \mathcal{F}(X_k, \theta(k), \xi_k, k), \qquad (1)$$

where $Y \in \mathbb{R}^r$ is the measured output, $\theta \in \mathbb{R}^r$ is the parameter of interest, X_k is an unknown vector (typically a state), ξ is a zero-mean noise. This system is observed sequentially, i.e. at time n the observations $Y_1, \ldots, Y_n$ are available. Until the unknown time $k_0 - 1$ the parameter vector is $\theta(k) = \theta_0$ and from k_0 it becomes $\theta(k) = \theta_l$ for some l, $1 \leq l \leq K$. Therefore, (1) is a system with *abrupt changes* where θ_0 describes the normal operation mode of the system $\mathcal{F}$ and θ_l describes the abnormal mode number l. It is assumed that the change time k_0 and number l are unknown and non random [1].

[1] This paper is devoted to the min-max approach, the results on the Bayesian approach can be found in (Malladi and Speyer, 1999; Lai, 2000).

Let $(Y_k)_{k\geq 1}$ be a sequence of observations, which are coming from system (1). The problem is to *detect* and *isolate* the change in θ. In other words, to determine the type of fault (number l) as soon as possible. The change detection/isolation algorithm has to compute a *pair* (N,ν) based on the observations $(Y_k)_{k\geq 1}$, where N is the *alarm time* at which a ν-type change is detected/isolated and ν, $1 \leq \nu \leq K$, is the *final decision*.

Let us discuss now several particular cases of model (1) frequently used in practice.

Independent observations. A finite family of distributions $\mathcal{P} = \{P_i,\ i = 0,\dots,K\}$ with densities $\{f_i,\ i = 0,\dots,K\}$ is considered. In the parametric case, it is assumed that $\mathcal{P} = \{P_\theta,\ \theta \in \Theta\}$, where $\theta \in \mathbb{R}^r$, $\Theta = \bigcup_{i=0}^{K}\{\theta_i\}$ and the density function of this family is denoted by $f_\theta(Y)$. It is also assumed that under P_0, the density function of Y_k is f_0 for every $k \geq 1$ and under $P^l_{k_0}$, the density function is f_0 for $k < k_0$ and is f_l for every $k \geq k_0$, $1 \leq l \leq K$.

Static regression model with redundancy. Consider the following regression model with additive changes :

$$Y_k = HX_k + \xi_k + G_l(k,k_0), \qquad (2)$$

where $X_k \in \mathbb{R}^p$ is the unknown non random state vector, ξ_n is a zero mean Gaussian white noise with covariance matrix $R = \sigma^2 I_r$, $\sigma^2 > 0$, H is a full rank matrix of size $r \times p$ with $r > p$ and $G_l(k,k_0)$ is the l-type change occurring at time k_0, namely :

$$G_l(k,k_0) = \begin{cases} 0 & \text{if } k < k_0 \\ G_l & \text{if } k \geq k_0 \end{cases},\quad 1 \leq l \leq K. \qquad (3)$$

Dependent observations. It is assumed that under P_0, the conditional density function of Y_k given $\mathcal{Y}_1^{k-1} = Y_1,\dots,Y_{k-1}$ is $f_0(Y_k|\mathcal{Y}_1^{k-1})$ for every $k \geq 1$ and under $P^l_{k_0}$, the conditional density function is $f_0(Y_k|\mathcal{Y}_1^{k-1})$ for $k < k_0$ and is $f_l(Y_k|\mathcal{Y}_1^{k-1})$ for every $k \geq k_0$, $1 \leq l \leq K$.

State space model with additive changes. The following state space model with additive changes is assumed here

$$X_{k+1} = F_k X_k + \zeta_k + U_l(k,k_0) \qquad (4)$$
$$Y_k = H_k X_k + \xi_k + G_l(k,k_0), \qquad (5)$$

where $U_l(k,k_0)$ is defined exactly as $G_l(k,k_0)$ (see equation (3)), ζ_k and ξ_k are zero mean Gaussian white noises.

2.2 *Criteria of optimality*

Intuitive formulation of the criteria. If the change is detected *after* the change time k_0 (the event $\{N \geq k_0\}$ is true), then the delay for detection/isolation is $\tau = N - k_0 + 1$. If the changes in θ are detected *before* time k_0 or if the final decision is *incorrect* ($\nu \neq l$), then these are *false alarms or false isolations* which we characterize in the following manner.

False alarms. Let the observations $(Y_k)_{k\geq 1}$ come from the normal mode system $\mathcal{F}$. Consider the following sequence of alarm times

$$N_0 = 0 < N_1 < N_2 < \cdots < N_r < \cdots,$$

where N_r is the alarm time of the detection/isolation algorithm which is applied to $Y_{N_{r-1}+1}, Y_{N_{r-1}+2},\dots$. Define the *first* false alarm time $N^{\nu=j}$ of a j-type in this sequence :

$$N^{\nu=j} = \inf_{r\geq 1}\{N_r\ :\ \nu_r = j\},\quad 1 \leq j \leq K,$$

where $\inf\{\emptyset\} = \infty$ as usual. If the consequences of different false alarms are equivalent, then their impact is measured by the time before a false alarm.

False isolations. Let the observations $(Y_k)_{k\geq 1}$ come from the abnormal mode number l. We define a false isolation as the following event $\{\nu \neq l\} \cap \{N \geq k_0\}$. In other words, the change is detected correctly but the isolation step is failed. It is intuitively obvious that the optimality criterion must favor fast detection with few false alarms and few false isolations.

Formal definition of the criteria. Several criteria have been recently developed (Nikiforov, 1995*a*; Nikiforov, 1995*b*; Lai, 2000; Nikiforov, 2000).

Worst case detection/isolation delay. The character feature of this approach is a pessimistic estimation of the detection delay and an optimistic estimation of the probability of false isolation. Let $\mathbb{E}^l_{k_0}$ (or $\mathbb{E}_0$) be the expectation with respect to $P^l_{k_0}$ (or P_0). We require that the worst case detection/isolation delay [2] (Nikiforov, 1995*a*) :

$$\overline{\mathbb{E}}^*(N) = \sup_{k_0\geq 1, 1\leq l\leq K} \operatorname{esssup} \mathbb{E}^l_{k_0}\left((N-k_0+1)^+|\mathcal{Y}_1^{k-1}\right) \qquad (6)$$

where $x^+ = \max(0,x)$, should be *as small as possible* for a given minimum γ of the mean times before a false alarm or a false isolation :

$$\mathbb{E}_0\left(\inf_{r\geq 1}\{N_r : \nu_r = j\}\right) \geq \gamma,\ \mathbb{E}^l_1\left(\inf_{r\geq 1}\{N_r : \nu_r = j\}\right) \geq \gamma \qquad (7)$$

for $1 \leq l, j \neq l \leq K$. Next, the isolation's constraint in (7) has been modified to obtain a more practicable performance index, i.e. the probability of false isolation (Nikiforov, 1995*b*)

$$\mathbb{P}^l_1(\nu = j \neq l) \leq \beta \sim \gamma^{-1} \text{ as } \gamma \to \infty. \qquad (8)$$

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[2] Let us assume that $y, \acute{y}, x$ are the random values. We say that the $y = \operatorname{esssup} x$ if : 1) $\mathbf{P}(x \leq y) = 1$; 2) if $\mathbf{P}(x \leq \acute{y}) = 1$ then $\mathbf{P}(y \leq \acute{y}) = 1$, where $\mathbf{P}(A)$ is the probability of the event A.

The above criterion (6) - (8) has been generalized to the case of dependent observations $(Y_k)_{k\geq 1}$ in (Lai, 2000).

Uniformly constrained conditional probability of false isolation. The drawback of criterion (6) - (8) lies in that the probability of false isolation is constrained only if the change time is $k_0 = 1$. Strictly speaking, if $k_0 > 1$ then the probability of false isolation can be greater than β. Simulation shows that it depends on the mutual "geometry" of the hypotheses (Nikiforov, 2000). On the other hand, equation (6) is too pessimistic, at least for some practical problems. To obtain a more tractable criterion, we propose to minimize the maximum mean delay for detection/isolation (Nikiforov, 2000) :

$$\overline{\mathbb{E}}(N) = \sup_{k_0\geq 1, 1\leq l\leq K} \mathbb{E}^l_{k_0}(N - k_0 + 1 \mid N \geq k_0) \quad (9)$$

subject to the constraints

$$\mathbb{E}_0(N) \geq \gamma, \ \sup_{k_0\geq 1} \mathbb{P}^l_{k_0}(\nu = j \neq l | N \geq k_0) \leq \beta, \quad (10)$$

for $1 \leq l, j \neq l \leq K$. Initially (Nikiforov, 2000), the mean time before a false alarm has been defined by using equation (7) but to seek the simplicity a slightly different constraint $\mathbb{E}_0(N) \geq \gamma$ will be used in the rest of the paper.

Uniformly constrained probabilities of false alarm and false isolation within a time window. For some safety-critical applications it is necessary to warrant that the probabilities of false alarm and false isolation within a time window (m_α) are less than a prescribed upper bound. As it has been mentioned in (Lai, 2000), the constraint $\mathbb{E}_0(N) \geq \gamma$ does not necessary imply that the probability of having a false alarm before some specified time is small. To solve this problem, Lai proposes to minimize the mean delay for detection/isolation for every $1 \leq l \leq K$:

$$\mathbb{E}^l_{k_0}(N - k_0 + 1)^+ \quad (11)$$

subject to the constraint

$$\sup_{k\geq 1} \mathbb{P}_0(k \leq N < k + m_\alpha) \leq \alpha m_\alpha, \quad (12)$$

$$\sup_{k_0\geq 1} \mathbb{P}^l_{k_0}(k_0 \leq N < k_0 + m_\alpha \cap \nu \neq l) \leq \alpha m_\alpha, \quad (13)$$

for $1 \leq l \leq K$.

3. ALGORITHMS AND THEIR STATISTICAL PROPERTIES

This section is devoted to the design of the change detection/isolation algorithms and investigation of their statistical properties. It is divided into three subsections following the definitions of different criteria of optimality given in subsection 2.2.

3.1 *Worst case detection/isolation delay*

An asymptotic lower bound $n(\gamma)$ for the worst case detection/isolation delay (criterion (6) - (8)) is given by the following equation (Nikiforov, 1995*a*)

$$n(\gamma) \sim \log \gamma / \rho^* \text{ as } \gamma \to \infty, \quad (14)$$

where (when the observations $Y_1, \ldots, Y_n$ are independent) $\rho^* = \min_{1\leq l\leq K} \min_{0\leq j\neq l\leq K} \rho_{l,j}$ and $0 < \rho_{l,j} = \mathbb{E}^l_1 \left(\log f_{\theta_l}(Y_i) / f_{\theta_j}(Y_i)\right) < \infty$. The above equation has been generalized to the case of dependent observations in (Lai, 2000). When the observations are dependent, the definition of the Kullback-Leibler information $\rho_{l,j}$ is much more complicated, the interested reader is referred to (Lai, 2000).

Non recursive algorithm. Unfortunately, the algorithm (N_{nr}, ν_{nr}) which reaches this lower bound ($\overline{\mathbb{E}}^*(N_{nr}; \gamma) \sim n(\gamma) \sim \log \gamma / \rho^*$ as $\gamma \to \infty$) is non recursive (Nikiforov, 1995*a*; Lai, 2000) :

$$N_{nr} = \min_{1\leq l\leq K}\{N_{nr}(l)\}, \nu_{nr} = \arg \min_{1\leq l\leq K}\{N_{nr}(l)\}, \quad (15)$$

where

$$N_{nr}(l) = \inf\left\{n \geq 1 : \max_{1\leq k\leq n} \min_{0\leq j\neq l\leq K} S^n_k(l, j) \geq h\right\},$$

$$S^n_k(l, j) = \sum_{i=k}^{n} \log f_{\theta_l}(Y_i | \mathcal{Y}^{i-1}_1) / f_{\theta_j}(Y_i | \mathcal{Y}^{i-1}_1) \quad (16)$$

is the *log likelihood ratio* (LR) between hypotheses $\mathcal{H}_l : \theta = \theta_l$ and $\mathcal{H}_j : \theta = \theta_j$ and h is a chosen threshold.

It is easy to see that the above algorithm requires, at each n, maximization over all possible change times $k : 1 \leq k \leq n$, so the number of LR computations for $N_{nr}(l)$ at time n grows to infinity with n.

Window-limited algorithm. To reduce the computational complexity, (Lai, 2000) proposes the following window-limited modification of N^l_{nr} :

$$N_{wl}(l) = \inf\left\{n \geq 1 : \max_{n-M_\gamma\leq k\leq n} \min_{0\leq j\neq l\leq K} S^n_k(l, j), \geq h\right\}$$

where $\liminf M_\gamma / \log \gamma > 1/\rho^*$ as $\gamma \to \infty$. Lai has shown that equation (14) still holds for the change detection/isolation algorithm (15) - (16) with this window-limited modification $N_{wl}(l)$.

3.2 *Uniformly constrained conditional probability of false isolation*

An asymptotic lower bound $n(\gamma, \beta)$ for the maximum mean delay for detection/isolation (criterion (9) - (10)) is given by the following equation

$$n(\gamma, \beta) \sim \max\left\{\log \gamma / \rho^*_{\mathrm{d}}, \log \beta^{-1} / \rho^*_{\mathrm{i}}\right\} \quad (17)$$

as $\min\{\gamma, \beta^{-1}\} \to \infty$, where $\rho^*_{\mathrm{d}} = \min_{1\leq j\leq K} \rho_{j,0}$, and $\rho^*_{\mathrm{i}} = \min_{1\leq l\leq K} \min_{1\leq j\neq l\leq K} \rho_{l,j}$.

Recursive algorithm. The mean number of LR computations for the non recursive change detection/isolation algorithm at a current instant k is $O(\gamma)$, the window-limited modification approximately involves $O(\log\gamma/\rho^*)$ LR computations at every time k. These results are again improved by the following fully recursive algorithm (Nikiforov, 2000) :

$$N_r = \min_{1\le l\le K}\{N_r(l)\}, \nu_r = \arg\min_{1\le l\le K}\{N_r(l)\}, \quad (18)$$

where

$$N_r(l) = \inf\left\{n\ge 1 : \min_{0\le j\ne l\le K}[S_n(l,j) - h_{l,j}] \ge 0\right\},$$

and the recursive decision functions $S_n(l,j)$ are defined by

$$S_n(l,j) = g_n(l,0) - g_n(j,0) \quad (19)$$
$$g_n(l,0) = (g_{n-1}(l,0) + Z_n(l,0))^+ ,$$

with $Z_n(l,0) = \log f_{\theta_l}(Y_n)/f_{\theta_0}(Y_n)$ and $g_0(l,0) = 0$ for every $1\le l\le K$ and $g_n(0,0)\equiv 0$. The thresholds $h_{l,j}$ are chosen by the following formula :

$$h_{l,j} = \begin{cases} h_d & \text{if } \ 1\le l\le K \ \text{ and } j = 0 \\ h_i & \text{if } 1\le j,l\le K \ \text{ and } j\ne l \end{cases}, \quad (20)$$

where h_d is the detection threshold and h_i is the isolation threshold. The statistical properties of this recursive algorithm is given by the following asymptotic equation (Nikiforov, 2000) :

$$\overline{\mathbb{E}}(N_r) \sim n(\gamma,\beta) \sim \max\left\{\log\gamma/\rho^*_\mathrm{d}, \log\beta^{-1}/\rho^*_\mathrm{i}\right\} \quad (21)$$

as $\gamma\to\infty$ $\beta\to 0$ and $\log\gamma \gtrsim \log\beta^{-1}$. The slightly different definition of the mean time before a false alarm in (10) with respect to equation (7) does not change the asymptotic relation between $\overline{\mathbb{E}}(N_r)$, γ and β in equation (21).

The tuning of the thresholds h_d and h_i and the comparison between the nonrecursive (15) - (16) and recursive (18) - (19) algorithms, by using Monte Carlo simulation, can be found in (Nikiforov, 2000).

3.3 *Uniformly constrained probabilities of false alarm and false isolation within a time window*

An asymptotic lower bound for $\mathbb{E}^l_{k_0}(N - k_0 + 1)^+$ (criterion (11) - (13)) is given by the following equation (Lai, 2000) :

$$\mathbb{E}^l_{k_0}(N - k_0 + 1)^+ \ge \mathbb{P}_0(N\ge k_0)|\log\alpha|/(\rho_l + o(1))$$

as $\alpha\to 0$ uniformly in $k_0\ge 1$, for every $1\le l\le K$, where $\rho_l = \min_{0\le j\ne l\le K}\rho_{l,j}$. The time window length m_α satisfies (Lai, 2000) $\liminf m_\alpha/|\log\alpha| > 1/\rho^*$ but $\log m_\alpha = o(\log\alpha)$ as $\alpha\to 0$. The criterion and window limited change detection/isolation algorithm have been extended to the case of composite alternatives. The interested reader is referred to (Lai, 2000).

4. EXAMPLES

The goal of this section is to illustrate how the developed theory is used to solve an important practical problem - navigation systems integrity monitoring.

4.1 *Navigation systems integrity monitoring*

For many safety-critical applications, a major problem of the existing navigation systems consists in its lack of integrity. The goal of the integrity monitoring (as it is defined by the International Civil Aviation Organization (ICAO)) is to detect and isolate faults so that they can be removed from the navigation solution before they sufficiently contaminate the output. Let us start our discussion with two types of navigation systems : the Strapdown Inertial Reference Unit (SIRU) and the Global Positioning System (GPS).

The model of SIRU with a degradation. Conventional redundant SIRU incorporates $r\ge 5$ single degree-of-freedom sensors (laser giros or accelerometers) (Sturza, 1988; Nikiforov *et al.*, 1993). We assume that r skewed axis inertial sensors are equally spaced on a cone with half-angle $\alpha = 54.736\,\mathrm{deg}$. A simplified measurement model of SIRU is defined by the following static regression model with redundancy :

$$Y_k = H\mathcal{A}_k + \zeta_k,\ \zeta_k = \zeta_{k-1} + \xi_k + G_l(k,k_0), \quad (23)$$

where $\mathcal{A}_k\in\mathbb{R}^3$ is a non random unknown state vector (say, acceleration), $Y_k\in\mathbb{R}^r$ is a vector of measurements, ζ_k is the accelerometer biases modelled as random walks, $\xi_k\in\mathbb{R}^r$ is a Gaussian white noise with zero mean and covariance $R = \sigma^2 I_r$, $\sigma^2 > 0$, $H = (h_{ij})$ is a matrix of size $r\times 3$, $h_{i1} = \cos\beta_i$, $h_{i2} = \sin\beta_i\sin\alpha$, $h_{i3} = -\cos\alpha$, $\beta_i = 360(i-1)/r\,\mathrm{deg}$, $1\le i\le r$ and $G_l(k,k_0)$ is an additional bias occurring at time k_0 in an l-th accelerometer's error, $1\le l\le r$. It is easy to see that equation (23) can be reduced to equation (2) by using the first difference

$$\nabla Y_k = H\nabla\mathcal{A}_k + \xi_k + G_l(k,k_0),\ \nabla(.)_k \overset{\mathrm{def}}{=} (.)_k - (.)_{k-1}.$$

The model of GPS with a degradation. The GPS navigation solution is based upon accurate measuring the distance (*range*) from r visible satellites with known locations $X_i = (x_i,y_i,z_i)^T$, $1\le i\le r$, to a user (vehicle) at $X_u = (x_u,y_u,z_u)^T$. The distance from the i-th satellite to the user is defined as $d_i = \|X_i - X_u\|$. The *pseudo-range* (i.e. measure of the distance) r_i from the i-th satellite to the user can be written as $r_i = d_i + c\,b + \xi_i$, $1\le i\le r$, where $b\in\mathbb{R}$ is a user clock bias, $c\simeq 2.9979\cdot 10^8\mathrm{m/s}$ is the speed of light and ξ_i is an additive pseudo-range error at the user's position. Let us introduce the following vectors: $R = (r_1,\dots,r_r)^T$ and $X = (X_u^T, b)^T$. By linearizing the pseudo-range equation with respect to the state vector X around the working point X_0, we get the measurement equation

$$Y = R - R_0 \simeq Hx + \xi,\ \ x = X - X_0,$$

where $R_0 = (r_{1_0}, \ldots, r_{r_0})^T$, $r_{i_0} = \|X_i - X_{u_0}\| + c\, b_0$, $\xi = (\xi_1, \ldots, \xi_n)^T$, $H = \frac{\partial R}{\partial X}\big|_{X=X_0}$ is the Jacobian matrix of size $r \times 4$ and $\xi \in \mathbb{R}^r$ is a Gaussian white noise with zero mean and covariance $R = \sigma^2 I_r$, $\sigma^2 > 0$. The degradation of GPS channels is represented by an additional biases in the pseudo-ranges (Nikiforov, 1996) :

$$Y_k = Hx_k + \xi_k + G_l(k, k_0). \tag{24}$$

Let us assume that at least six satellites are visible. As it follows from equations (23) and (24), the optimal estimate of the user's fix x_k (or the vehicle's jerk $\nabla\mathcal{A}_k$) is given by the least squares (LS) algorithm [3] :

$$\widehat{x}_k = (H^T H)^{-1} H^T Y_k. \tag{25}$$

As it follows from equations (23), (24) and (25), a fault G_l, affecting the sensors (or channels), implies an additional error $\mathbb{E}(\widehat{x}_k - x_k) = (H^T H)^{-1} H^T G_l$ in the vector $\widehat{x}_n$ (or $\widehat{\nabla\mathcal{A}_k}$) which contaminates the output of the navigation system (Nikiforov *et al.*, 1993; Nikiforov, 1996).

4.2 *Integrity monitoring algorithms*

Change detection/isolation with nuisance parameters. The characteristic feature of the regression model with redundancy (2) ((23) or (24)) is the fact that the vector X_k ($\mathcal{A}_k$ or x_k) is unknown. This type of statistical problem is usually called a detection with *nuisance parameters*. Let us define the following hypotheses :

$$\mathcal{H}_l = \{Y \sim \mathcal{N}(HX + G_l, \sigma^2 I_n),\ X \in \mathbb{R}^m\}, \tag{26}$$
$$\mathcal{H}_j = \{Y \sim \mathcal{N}(HX + G_j, \sigma^2 I_n),\ X \in \mathbb{R}^m\}, \tag{27}$$

where G_l, G_j ($0 \le l, j \ne l \le K$) are the *informative* parameters, and X is the *nuisance* parameter. We are interested in detecting a change from 0 to G_l, while considering X as an *unknown* parameter of model (2). It follows from section 3 that the lower bound for the detection/isolation delay is a monotone decreasing function of the Kullback-Leibler information. Therefore, the design of the *minimax algorithm* (in the sense of minimizing the detection/isolation delay) consists of finding a pair of the *least favorable values* X^l and X^j for which the Kullback-Leibler information $\rho_{l,j} = \rho(X^l, X^j)$ is minimum, and in computing the LR

$$S_k^k(l, j) = \log f_{G_l}(Y_k; X^l)/f_{G_j}(Y_k; X^j)$$

of the optimal algorithm for these values. The Kullback-Leibler information $\rho_{l,j}$ is given by $\rho_{l,j}(x) = \frac{1}{2\sigma^2}\|Hx + G_{l,j}\|^2$, where $x = X^l - X^j$ and $G_{l,j} = G_l - G_j$. Therefore, we minimize $\rho_{l,j}(x)$ with respect to x (the interested reader is referred to (Nikiforov, 1995*a*)). The LR for hypotheses (26) - (27) under the least favorable value x^* is given by

$$S_k^k(x^*; l, j) = \frac{1}{\sigma^2} G_{l,j}^T \Pi Y_k - \frac{1}{2\sigma^2} G_{l,j}^T \Pi G_{l,j}, \tag{28}$$

where $\Pi = I - H(H^T H)^{-1} H^T$. It is of interest to note that the above LR is a function of the *parity vector* ε_k of the analytical redundancy approach (Nikiforov, 1995*a*). This parity vector ε_k is the transformation of the measurements Y_k into a set of $r - p$ ($p = 3; 4$) linearly independent variables by projection onto the left null space of the matrix H.

Therefore, to apply the results of section 3 to the GPS (or SIRU) integrity monitoring, it is sufficient to replace the LR in the definition of $N_{nr}(l)$ ($N_{wl}(l)$ or $N_r(l)$) by the LR $S_k^k(x^*; l, j)$ given by (28). A comprehensive comparison (analytical and numerical) of the proposed sequential algorithm with the so-called"snapshot" (based on the last observation Y_k) and fixed size sample approaches can be found in (Nikiforov, 1996; Nikiforov, 1997). The significant superiority of the sequential algorithm has been shown.

5. UNEXPECTED PROBLEMS

Let us discuss here some new aspects of the change detection and isolation. It turns out that these recently identified problems are of key interest for the statistical FDI in safety-critical applications. The navigation system integrity monitoring will be used to illustrate these problems.

5.1 *Reliable detection/isolation*

First, the main goal of the navigation system integrity monitoring is to detect the system degradation when it leads to an unacceptable growth of the output errors. All other faults are of no importance, moreover, their detections can be considered by the user as a false alarm. Second, the "traditional" change detection/isolation methods are formulated as that of the quickest detection/isolation of abrupt changes in the properties of stochastic signals and systems (see section 3). A navigation sensor fault also should be detected quickly but for safety-critical applications the user imposes the constraint on the maximum delay for the detection. Specifically, the ICAO fixes the probability of missed detection within a given *time-to-alarm* τ_a. Therefore, this new optimality criterion requires to minimize the maximum probability of missed detection

$$\bar{\mathbb{P}}(N) = \sup_{k_0 \ge 1} \max_{1 \le l \le K} \mathbb{P}_{k_0}^l (N - k_0 + 1 > \tau_a)$$

subject to the constraint (12) and (13). Moreover, a change detection/isolation algorithm should be adaptive, i.e. it has detect/isolate any faults that will cause a navigation error above a given limit (vertical or

[3] The LS algorithm is presented here to illustrate an impact of a channel (sensor) fault on the navigation accuracy. In fact, the Kalman filter usually processes the outputs of the SIRU (accelerometers and gyros) but the problem of the navigation system contamination still holds in the general framework.

horizontal) for any current GPS satellite constellation. The interested reader is referred to (Younes *et al.*, 1998; Bakhache and Nikiforov, 2000).

5.2 *Multi-sensor navigation systems*

The integration of several navigation subsystems (INS, GPS, Loran-C, baro-altimeter, radio-altimeter,...) is traditionally proposed to improve navigational accuracy, integrity and continuity on vehicles having two (three) INS plus aiding subsystems such as GPS, Loran-C and others. The error model of such a multi-sensor navigation system (with a degradation) is given by equations (4) - (5). The log LR of this state space model can be computed by using the innovation sequence $(e_k)_{k\geq 1}$ of the Kalman filter based upon the nominal state space model (without the terms $U_l(k,k_0)$ and $G_l(k,k_0)$). This random sequence can be modelled as

$$e_k \sim \begin{cases} \mathcal{N}(0,R_k) & \text{if } k<k_0 \\ \mathcal{N}(\eta_l(k,k_0),R_k) & \text{if } k\geq k_0 \end{cases}, \qquad (29)$$

where $\eta_l(k,k_0)=\eta(U_l(k,k_0),G_l(k,k_0)))$ is the dynamic profile of the innovation sequence after the abrupt change number l. Unfortunately, the existence of this profile makes the isolation problem very difficult, especially in the context of the reliable detection/isolation. No results in a mathematically precise sense exist in the literature.

6. CONCLUSION

The paper presents the key results of the recently developed optimal theory of change diagnosis (detection and isolation). Several frequently used models of signals and systems with abrupt changes and three criteria of optimality have been discussed. These criteria have the min-max character, this choice is motivated by the fact that the change time and the behavior of the system's environment are not simply unknown but can be intentionally chosen to maximize their negative impacts on the considered system in safety-critical applications.

The information lower bounds for the detection/isolation delay are given for these criteria. The change detection/isolation algorithms that asymptotically reach these bounds have been discussed.

The application of the developed theory to the problem of navigation systems (SIRU and GPS) integrity monitoring has been discussed. In the course of this application, some open problems have been recently identified. These new crucially important problems have been briefly presented in the paper.

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