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A TOEPLITZ CHARACTERIZATION OF THE STA TICOUTPUT FEEDBACK STABILIZATION PROBLEM FOR LINEAR DISCRETE-TIME SYSTEMS

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Abstract: The static output feedback (SOF) stabilization problem for linear SIMO or MISO discrete-time systems is presented and characterized in terms of the properties of positiv edefinite T oeplitzmatrices. A global minimization problem in a compact sets is introduced whose solution, if any, guarantees closed-loop stabilit y and the fulfillment of an upper bound to a suitable closed loop performance.

1. INTRODUCTION

The static output feedback stabilization problem is one of the most known and still open issues in systems and control, see [1] for a recen t survey. Here we deal with a discrete-time linear system

$$x(k+1) = Ax(k) + Bu(k) \tag{1}$$

$$y(k) = Cx(k) \tag{2}$$

where $x \in I\!R^n$, $u \in I\!R^m$, $y \in I\!R^p$ are the state, input and output vectors, respectively, and A, B, C are matrices with constant real coefficients and appropriate dimensions.

The *static output feedback stabilization* problem for system (1)-(2) consists in finding, if possible, a static control la w described by equations of the form

$$u(k) = Fy(k) \tag{3}$$

such that the closed-loop system is asymptotically stable, *i.e.* the matrix $A + BFC$ has all its eigen values with modulus less than one. If suc h an output feedback does exist, w e say that the system (1)-(2) is *output stabilizable* and that F is a solution of the problem. We make the assumptions that the system is observable and that C has full row rank. In particular, it is possible to define the projection matrix

$$V = I - C'(CC')^{-1}C,$$

whose role will be clear in the sequel.

In this paper w e aim at characterizing the SOF problem as a constrained LMI problem and, at the same time, we propose an algorithm for MISO systems based on the properties of positive Toeplitz matrices.

The paper is organized as follows. In Section 2 w e give some preliminary results, namely tw o necessary and sufficient conditions for the solvabilit y of the SOF stabilization problem for system (1)-(2). In Section 3 we discuss the relations between positive definite Toeplitz matrices, the Yule Walker equations and the SOF stabilization problem. In Section 4 we provide a conceptual algorithm for the solution of the SOF problem *with performance.* Finally, Section 5 contains some concluding remarks.

2. PRELIMINARY RESULTS

In this section some preliminary and general results on the SOF problem for discrete-time systems are collected. The first result is the discrete-time counterpart of the results in [2], [3] and gives a necessary and sufficient condition for SOF stabilizability.

Theorem 1. The system (1)-(2) is output feedback stabilizable if and only if there exists a symmetric positive semi-definite matrix $P \in I\!R^{n\times n}$ and a matrix $G \in I\!R^{m\times n}$ such that

$$\begin{aligned} P &= A'PA + C'C + G'G \\ &\quad - A'PB(I + B'PB)^{-1}B'PA \qquad (4) \\ 0 &= V(P - A'PA)V \qquad (5) \end{aligned}$$

Proof. If F is a stabilizig gain, then take the unique positive definite solution of the Lyapunov equation

$$P = (A + BFC)'P(A + BFC) + C'C + C'F'FC$$

The thesis follows by recognizing that the same solution P solves (4) with

$$G = (I + B'PB)^{1/2}FC + (I + B'PB)^{-1/2}B'PA$$

Notice that, in view of the assumptions, there is no loss of generality in considering only strictly positive definite solutions P of (4).

Vice-versa, if P and G satisfy (4), (5), then take an orthogonal matrix T such that

$$T'GV = (I + B'PB)^{-1/2}B'PAV$$

and define

$$\begin{aligned} F &= (I + B'PB)^{-1/2}T'GC'(CC')^{-1} \\ &\quad - (I + B'PB)^{-1}B'PAC'(CC')^{-1} \end{aligned}$$

Then, it is a fact of cumbersome computation to show that P satisfies

$$P = (A + BFC)'P(A + BFC) + C'C + C'F'FC$$

so that the stability of $A + BFC$ comes from the well known Lyapunov Lemma. ∎

The proof of the previous result allows us to provide the parametrization of all stabilizing SOF gains as follows.

Corollary 2.1. Consider the system (1)-(2). The family of all output feedback gains F such that the matrix $A + BFC$ is stable is given by

$$\begin{aligned} F &= (I + B'PB)^{-1/2}T'GC'(CC')^{-1} \\ &\quad - (I + B'PB)^{-1}B'PAC'(CC')^{-1} \end{aligned}$$

where $P = P' \geq 0$ and G solve (4) and (5) and T is any orthogonal matrix satisfying

$$T'GV = (I + B'PB)^{-1/2}B'PAV$$

It is worth pointing out that it is possible to remove the unknown G in Theorem 1 by replacing the Riccati equality with a Riccati inequality. This fact is formalized in the following result.

Theorem 2. The system (1)-(2) is output feedback stabilizable if and only if there exists a symmetric positive semi-definite matrix $P \in I\!R^{n\times n}$ such that

$$\begin{aligned} P &\geq A'PA + C'C - A'PB(I + B'PB)^{-1}B'PA \qquad (6) \\ 0 &= V(P - A'PA)V \qquad (7) \end{aligned}$$

Proof. If F is a stabilizing gain, then the result follows from the proof of Theorem 1. Vice-versa, assume that $P > 0$ satisfies (6), (7). If the rank of

$$S_P = P - A'PA + C'C - A'PB(I + B'PB)^{-1}B'PA$$

is not greater than m, then the proof follows again from Theorem 1 by taking $G \in I\!R^{m\times n}$ such that $G'G = S_P$. Otherwise, let $\tilde{B} = [B\ 0]$ where the zero elements are such that the number of columns $\tilde{m}$ of $\tilde{B}$ equals the rank of S. Notice that replacing B with $\tilde{B}$ does not affect inequality (6). Now, taking $\tilde{G} \in I\!R^{\tilde{m}\times n}$ such that $\tilde{G}'\tilde{G} = S_P$, it is possible to select

$$\begin{aligned} \tilde{F} &= (I + \tilde{B}'P\tilde{B})^{-1/2}T'\tilde{G}C'(CC')^{-1} \\ &\quad - (I + \tilde{B}'P\tilde{B})^{-1}\tilde{B}'PAC'(CC')^{-1} \end{aligned}$$

where T is any orthogonal matrix satisfying

$$T'\tilde{G}V = (I + \tilde{B}'P\tilde{B})^{-1/2}\tilde{B}'PAV.$$

As proved in Theorem 1, such a gain $\tilde{F}$ is such that $A + \tilde{B}\tilde{F}C$ is stable. Letting F denote the first m rows of $\tilde{F}$, the conclusion is drawn that $A + BFC$ is stable as well, so concluding the proof. ∎

Interestingly, the inequality (6) can be cast as an LMI formulation in the unknown $X = P^{-1}$ as follows

$$0 \leq \begin{bmatrix} X & XA' & XC' \\ AX & X+BB' & 0 \\ CX & 0 & I \end{bmatrix} \tag{8}$$

Hence, in principle, the SOF problem is an LMI one (equations (8) and (7)) along with the nonlinear coupling condition $PX = I$.

The main problem concerning the search of a feasible solution P generating a stabilizing gain F is that such a solution can be, generally speaking, arbitrarily large (in some norm to be defined). However, P can be given a system-theoretical interpretation in terms of a closed-loop norm that one is interested in keeping less than a prescribed value for better performance. To see this fact, consider the system

$$x(t+1) = Ax(t) + Bu(t) + \bar{B}w(t) \tag{9}$$

$$y(t) = Cx(t) \tag{10}$$

$$z(t) = \begin{bmatrix} y \\ u \end{bmatrix} \tag{11}$$

where w is the disturbance. Assume that one wants to find a SOF gain F such that the H_2 norm $\|T_{zw}\|_2$, of the closed loop system, with disturbance input w and performance output z is less than a prescribed positive value. It is easy to see that

$$\|T_{zw}\|_2^2 = \mathrm{trace}(\bar{B}'P\bar{B})$$

where P satisfies

$$P = (A+BFC)'P(A+BFC) + C'C + C'F'FC.$$

Theorem 3. Consider the system (9)-(11). There exists F such that $A + BFC$ is stable and $\|T_{zw}\|_2 \leq \gamma$ if and only if there exists a symmetric positive semi-definite matrix $P \in I\!R^{n\times n}$ and a matrix $G \in I\!R^{m\times n}$ such that

$$\begin{aligned} P \geq\ & A'PA + C'C \\ & -A'PB(I+B'PB)^{-1}B'PA \end{aligned} \tag{12}$$

$$0 = V(P - A'PA)V \tag{13}$$

$$\mathrm{trace}(\bar{B}'P\bar{B}) \leq \gamma^2. \tag{14}$$

3. TOEPLITZ MATRICES AND YULE-WALKER EQUATIONS

We are most interested in the constraint (5), which can be exploited in order to give more structure to the problem. To do this, we confine the attention to single output systems and, for simplicity, we make reference to the observability canonical form, *i.e.*

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_2 \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix}, \tag{15}$$

$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{16}$$

It is very easy to recognize that, thanks to the particular structure of A and C, the set of $n \times n$ positive definite matrices satisfying (5) is just the set of $n \times n$ positive definite Toeplitz matrices. For instance, in the case $n = 3$, the constrained inherited from (5) on $P = \{P_{ij}\}$ are

$$P_{11} = P_{22} = P_{33},\ P_{12} = P_{23}$$

so that

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12} & P_{11} & P_{12} \\ P_{13} & P_{12} & P_{11} \end{bmatrix}.$$

The Toeplitz matrices are fully characterized by the first row. The question now arise on the condition under which a Toeplitz matrix is positive definite. This problem arises in several topics of signal processing, time series analysis and image processing and significant efforts have been spent to develop efficient algorithms for the solution of positive definite Toeplitz equations. In prediction theory, this efforts brought to the celebrated Yule Walker equations. Specifically, if $P(n) = \{P_{ij}\}$ is an invertible Toeplitz matrix of dimension n, then it is possible to find a solution of the linear equations (henceforth referred to as YW equations)

$$\begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n-1} \\ P_{12} & P_{11} & \cdots & P_{1n-2} \\ \vdots & \vdots & \ddots & \vdots \\ P_{1n-1} & P_{1n-2} & \cdots & P_{11} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n-1} \end{bmatrix} + \begin{bmatrix} P_{12} \\ P_{13} \\ \vdots \\ P_{1n} \end{bmatrix} = 0.$$

The YW equations define the coefficients α_i of a polynomial, whose stability property is related to the positive definiteness of the Toeplitz matrix P. The proof of the following result easily derives from [4].

Theorem 4. A Toeplitz matrix $P(n) = \{P_{ij}\}$ of dimension n, whose coefficients satisfy the YW equations, is positive definite if and only if the polynomial

$$p(z) = z^{n-1} + \alpha_1 z^{n-2} + \cdots + \alpha_{n-2}z + \alpha_{n-1} \tag{17}$$

is Schur. ∎

Proof. First of all, note that an invertible solution $P(n)$ of the YW equations also satisfies the matrix equation

$$P(n-1) = \tilde{A}'P(n-1)\tilde{A} + \gamma(P(n))e_1'e_1 \tag{18}$$

where $\tilde{A}$ is the companion matrix

$$\tilde{A} = \begin{bmatrix} -\alpha_1 & 1 \; 0 \cdots 0 \\ -\alpha_2 & 0 \; 1 \cdots 0 \\ \vdots & \vdots \; \vdots \; \ddots \; \vdots \\ -\alpha_{n-2} & 0 \; 0 \cdots 1 \\ -\alpha_{n-1} & 0 \; 0 \cdots 0 \end{bmatrix} \tag{19}$$

$$\gamma(P(n)) = P_{11} - \left[P_{12} \; P_{13} \; \cdots \; P_{1n} \right] P(n-1)^{-1} \begin{bmatrix} P_{12} \\ P_{13} \\ \vdots \\ P_{1n} \end{bmatrix} \tag{20}$$

and e_1 is the first column of the $n-1$ dimensional identity matrix.

Suppose now that $P(n)$ is a positive solution of the YW equations, i.e. a positive solution of the Lyapunov-like equation (18). From Lyapunov Lemma and the observability of the pair $(\tilde{A}, e_1')$ it follows that $\tilde{A}$ is stable, i.e. the polynomial $p(z)$ is Schur.

Conversely, assume that $p(z)$ is Schur. Moreover, note that any solution $P(n)$ of equation (18) is such that $P(n-1)$ is Toeplitz and positive definite, by stability of $\tilde{A}$. Now, fix a positive coefficient $\bar{\gamma}$ and consider the Lyapunov equation

$$P(n-1) = \tilde{A}' P(n-1) \tilde{A} + \bar{\gamma} e_1' e_1$$

and find the unique positive definite solution $P(n-1)$. According to the YW equations, take

$$P_{1n} = - \left[P_{1n-1} \; P_{1n-2} \; \cdots \; P_{11} \right] \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n-1} \end{bmatrix}$$

and note that, with this selection of P_{1n}, $\gamma(P(n)) = \bar{\gamma}$. Hence, $P(n)$ is a positive definite solution of the YW equations. ■

The theorem above establishes a bijective correspondence between the set of positive $n \times n$ matrices satisfying the constraint (5) and the set of stable Schur polynomials of degree $n-1$. Interestingly, the inverse of $P(n)$ is the Hermite matrix associated with the polynomial

$$\pi(z) = zp(z)$$

Precisely,

$$B(z, z^*) = \frac{\pi(z)\pi(z^*) - z^{2k}\tilde{\pi}(z^{-1})\tilde{\pi}(z^{-1*})}{zz^* - 1} =$$

$$\beta \left[1 \; z^* \; \cdots \; z^{*n-1} \right] P^{-1} \begin{bmatrix} 1 \\ z \\ \vdots \\ z^{n-1} \end{bmatrix}$$

where β is a positive scalar. The Bezoutian $B(z, z^*)$ and the Hermitian matrix P^{-1} are related by the property that $\pi(z)$ is positive definite if and only if $P > 0$, which, as easily seen, coincides with the result given in Theorem 4, see also [5]. This correspondence and further details on generalized Bezoutians can be found in [6].

Remark 3.1. Notice that in the YW equations there are $n-1$ parameters of the polynomial $p(z)$ and n parameter of the Toeplitz matrix $P(n)$. If the coefficients α_i are given and $P(n)$ is a solution of the YW equations, then it is easy to see that also $\delta P(n)$ is solution, for any scalar δ. Hence $P(n)$ is well defined up to a generic constant real number. ■

If λ is an eigenvalue of $\tilde{A}$ (root of $p(z)$), and x the corresponding eigenvector, then from (18) it follows

$$(1 - |\lambda|^2) x' P(n-1) x = x_1^2 \gamma(P(n)) \tag{21}$$

where x_1 is the first entry of the vector x. Notice that, thanks to the very structure of $\tilde{A}$, x_1 cannot be zero. Notice also that the two conditions $\gamma(P(n)) > 0$ and $P(n-1) > 0$ are necessary and sufficient for $P(n) > 0$. Now, assume that

$$\epsilon I \leq P(n) \leq kI \tag{22}$$

From (21) we have

$$(1 - |\lambda|^2) = \frac{x_1^2 \gamma(P(n))}{x' P(n-1) x} \geq \frac{x_1^2}{x'x} \frac{\epsilon}{k}$$

Moreover,

$$\begin{aligned} x_2 &= (\lambda + \alpha_1) x_1 \\ x_{i+1} &= \lambda x_i + \alpha_i x_1, \; i = 2, 3, \cdots, n-2. \end{aligned}$$

Hence, if r is an upper bound for a generic coefficient of a Schur polynomial, it follows that

$$\frac{x_1^2}{x'x} \geq \left[\sum_{i=0}^{n-2} (1 + ir)^2 \right]^{-1}$$

so that the maximum modulus root of the polynomial $p(z)$ satisfies

$$|\lambda_{max}| < SM_1 := 1 - \left[\sum_{i=0}^{n-2} (1 + ir)^2 \right]^{-1} \frac{\epsilon}{k} \tag{23}$$

An upper bound for the coefficients of a $n-1$-order Schur polynomial is easily verified to be

$$r = \max_i \binom{n-1}{i} \quad (24)$$

A different way to work out a stability margin for a discrete-time system is to resort to the results of [7] starting from the equation (18), seen as a Lyapunov equation once the positive scalar $\gamma(P(n))$ is given. Hence, from [7] it follows

$$|\lambda_{max}| < 1 - \frac{\gamma(P(n))^2}{2^{2n-3}P_{11}}$$

so that

$$|\lambda_{max}| < SM_2 := 1 - \frac{\epsilon^2}{2^{2n-3}k} \quad (25)$$

Finally, it is possible to define the stability margin

$$SM = min\{SM_1,\ SM_2\}. \quad (26)$$

The following result summarizes the arguments discussed so far.

Theorem 5. Consider the Schur polynomial $p(z)$ and let $P(n)$ be a positive definite solution of the YW equations. Then, the maximum modulus root of the polynomial is bounded by SM, where SM is given by (26).

4. ALGORITHM FOR MISO SYSTEMS

The considerations in the previous sections can be used to devise a constructive algorithm for the computation of a stabilizing static output feedback (if any). Also, the closed-loop performance will be taken into account, in the sense that a positive number k is pre-selected which is an upper bound of the norm of P which has to satisfies (7). Recall that this bound can be interpreted, at the light of the result in Theorem 3, as an upper bound of the closed-loop norm attained by a stabilizing static output feedback control law. We refer to this problem as the SOF problem with closed-loop performance degree k.

The algorithm makes use of the following two sets

$$S_1 = (-SM,\ SM),$$
$$S_2 = \{(a,b) \in I\!R^2 \mid z^2 + \frac{a}{SM}z + \frac{b}{SM^2} \text{ is Schur}\}$$

and of the number k whose meaning has been defined above.

The algorithm is based on the choice of n scalar parameters $x_1^\star$ to $x_n^\star$, belonging to $S_1 \times S_2^{\frac{n-2}{2}} \times I\!R^+$, if n is even and to $S_2^{\frac{n-1}{2}} \times I\!R^+$, if n is odd. Notice that the set S_2 is indeed a triangle. The rationale underlying the choice is as follows. With the numbers $x_1, x_2, \cdots, x_{n-1}$ it is possible to work out a $n-1$ order polynomial $p(z)$ with zeros belonging to the disk of radius SM. Once this polynomial is found, one can solve the YW equations so as to find a positive definite matrix $\bar{P}(n)$ with norm equal to one (recall Remark (3.1)). Then, it is possible to select the last coefficient $x_n \leq k$ so that $P = x_n P(n)$ has norm less than k. Notice that P still solves the YW equations (again recall Remark (3.1)). Of course, in general, P does not solve (6), so that a global minimization procedure can be defined. To this end, consider the vector

$$x = (x_1,\ x_2\ \cdots,\ x_{n-1})$$

and the set

$$\Xi = S_1 \times S_2^\nu \times [0,\ k]$$

with $\nu = \frac{n-2}{2}$ if n is even and $\nu = \frac{n-1}{2}$ if n is odd. Now, the problem consists in solving

$$\min_{x \in \Xi}\ \lambda_{max}(-S_P)$$

where

$$S_P = P - A'PA + C'C - A'PB(I + B'PB)^{-1}B'PA$$

Notice that the searching set Ξ is compact. Once the above (global) minimum is achieved there are two possibilities. If this minimum is non-positive then the algorithm terminates, *i.e.* the numbers $x_1^\star$ to $x_n^\star$ of the minimum argument of the objective function can be used to construct a matrix $P = P(n) > 0$ such that conditions (4) and (5) hold. From this matrix P it is possible to construct a stabilizing feedback. If this minimum is positive then the SOF problem with closed-loop performance degree k does not have a solution. Of course, it is well possible that the problem has a solution with a larger value of k.

5. CONCLUSIONS

The SOF stabilization problem for linear discrete-time systems has been discussed. It is shown that the problem can be given a characterization in terms of two LMIs with a bilinear constraint. In the case of SIMO or MISO systems, it is also shown that there is a connection between the solvability of the SOF stabilization problem, the positive definite Toeplitz matrices, and the Yule Walker equations. Finally, using these connections, an algorithm to solve the SOF stabilization problem with performance has been given.

Further studies are in progress to extend the results in Section 3 to MIMO systems, and to write explicit algorithms solving the (global) optimization problem disscussed in Section 4.

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