15th Triennial World Congress, Barcelona, Spain

DESIGN OF ACTIVE SUSPENSION CONTROL USING SINGULAR PERTURBATION THEORY

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The presence of fast and slow modes in vehicle suspension systems, based on a half car model, is utilized in the design of active suspension control using singular perturbation theory. This strategy is based on the slow-fast control design. The suspension system performance is optimised with respect to ride comfort, road holding and suspension rattle space as expressed by the mean-square-values of body acceleration (including effects of heave and pitch), tire deflections and front and rear suspension travels. The method of design in this study is based on LQG feedback control combined with singular perturbation theory, and at the end a composite LQG controller has been proposed. Numerical simulations in the time domain evaluate the performance of the active suspension system. In spite of the simplified structure of the composite model, simulation results indicate that its performance is comparable to that of the full-state feedback design.

Keywords: Active Suspension System, Optimal Control, Singular Perturbation Theory, LQG, Kalman filter

1. INTRODUCTION

The design of vehicle suspension system is a compromise among several conflicting requirements. On one hand, for ride comfort, the suspension system must isolate vehicle body from the road inputs, whereas for maintaining good vehicle handling and road holding the tyres should continuously touch the road surface and simultaneously the system has to bear the weight of the vehicle. The passive suspension systems of the past have shown all their potentialities and won't promise to bear unexpected miracles. Because of this the giant car manufactures world-wide have focused their attention on active systems . Often, in the initial stage of the design, a quarter car model with a single wheel is devised (E.W. Kamen, B.S. Heck, 1997). Now by considering an active suspension system and with due fact that different criterion are required, amongst other methods, the linear quadratic gaussian (LQG) is a sound method as an optimal control technique.

The singular perturbation theory as a separate science and a mathematical discussion appeared in the study of fluid dynamics in 1904. Applying this theory for the analysis of integrated systems gives the designer extraordinary measuring potentials. Dividing the system into smaller subsystems attains this objective. However, in order to apply the singular perturbation theory to control systems, the system shall bare the two time scale properties, i.e., the system contains two batches of completely separate poles. Some modes in the system will contribute only to the initial stage of the response (transient response) and will eventually die out (assuming stable fast modes). Others continue throughout the entire time history (Nematollahzadeh S. M. 1998). Evidently, the dynamics of a vehicles suspension system is comprised of 2 separate sets of fluctuating modes. One set of modes are natural frequencies of vehicle mass and body springs comprising the slow dynamics (denoted by λ_s), and another set is the national frequencies of tires accounting for fast dynamics (denoted by λ_f). Without loss of generality, the ratio of $|\lambda_s|_{\max} / |\lambda_f|_{\min}$ is taken as the approximate value of the singular perturbation parameter ε given by

$$\varepsilon \cong |\lambda_s|_{\max} / |\lambda_f|_{\min} \qquad (1)$$

This clearly shows that the vehicle suspension system possesses a two-time-scale property, with the body modes as the slow, and the wheel hop modes as the fast modes (Salman M.A. ,Lee A.Y. and Boustany N.M. 1990). In the present study, the full order suspension system is considered first. By defining the appropriate cost function and using the LQG design method, the problem can be solved; next the singular perturbation technique is applied for solving the same problem. The theory has been applied to solve many of the control problems (Calise A.J. 1976), two strategies of the singular perturbation technique can be used. The first strategy is to design a reduced order controller and is using only the slow modes of the vehicle suspension and neglects the fast modes present (assuming stability of the fast modes). This method is the simplest form of controller design, although an appropriate response is not always achieved, and isn't recommended in this case. In the second strategy, the controller is produced by composition of a slow-controller and fast one, which has two advantages: First the design process becomes simpler due to order reduction of the subsystems, second, by exploiting the time scale property, it should be possible to perform multi-stage sampling entailing lower pressure upon the digital controller.
In section 2 the half car model, the related differential equations and the appropriate cost function are presented. In the third part, the LQG problem is solved using standard singular perturbation theory. Its important to note that using a four-wheel vehicle model would render the same characteristics as the half car model (assuming independent suspension systems are employed). In section 4 simulation results are presented. Finally, in section 5 concluding remarks are given.

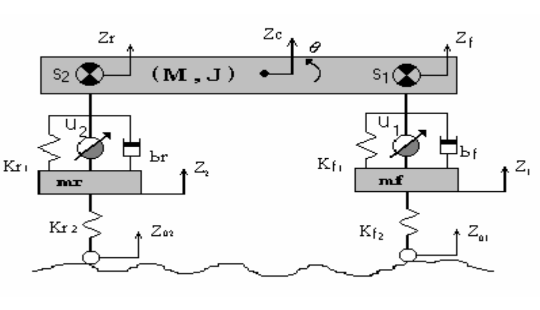

Fig. 1. Half car model with active suspension system.

2. ACTIVE SUSPENSION SYSTEM OF A HALF CAR MODEL

The half car active suspension system is shown in Figure1 (Hac A., Youn I. 1993).
Defining the states, controls and disturbances of the system, respectively by

$$X = (x_1 \quad x_2 \quad \dots \quad x_8)^T \qquad (2a)$$

$$U = \begin{pmatrix} u_1 & u_2 \end{pmatrix}^T \qquad (2b)$$

$$w = \begin{pmatrix} w_1 & w_2 \end{pmatrix}^T \qquad (2c)$$

where

$$\begin{aligned} x_1 &= Z_c + a\theta - Z_1 & x_2 &= \dot{Z}_c + a\dot{\theta} \\ x_3 &= Z_c - b\theta - Z_2 & x_4 &= \dot{Z}_c - b\dot{\theta} \\ x_5 &= Z_1 - Z_{01} & x_6 &= \dot{Z}_1 \qquad (3a) \\ x_7 &= Z_2 - Z_{02} & x_8 &= \dot{Z}_2 \end{aligned}$$

$$w_1(t) = \dot{Z}_{01}(t) \qquad w_2(t) = \dot{Z}_{02}(t) \qquad (3b)$$

then the outputs will be

$$y_1 = x_1 + x_5 \qquad y_2 = x_3 + x_7 \qquad (3c)$$

and U is the control signal. The differential equation representing the state space model is

$$\dot{X} = AX + BU + Lw$$
$$Y = CX + DU + v \qquad (4)$$

Where *A, B, C, D* and *L* are constant matrices and are given in Appendix. *w* and *v* are unbiased process noise and measurement noise respectively with covariance

$E\{ww'\} = W$, $E\{vv'\} = V$, $E\{wv'\} = 0$ (5)

To define the cost function we must take into account the passenger comfort, acceleration in passenger compartment which should be kept as low as possible, and to achieve a good vehicle handling, tire deflection rate proportional to tire strength, should be low, rendering to maximum contact with road. Putting all the above objectives together and considering the random nature of disturbances, the performance measure is defined (Hac A., Youn I. 1993)

$$J = E\{\frac{1}{2}\int_0^\infty \left(\begin{pmatrix} \ddot{Z}_c \\ \ddot{\theta} \end{pmatrix}^T \begin{pmatrix} p_0 & 0 \\ 0 & p_1 \end{pmatrix} \begin{pmatrix} \ddot{Z}_c \\ \ddot{\theta} \end{pmatrix} + \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}^T \begin{pmatrix} p_2 & 0 \\ 0 & p_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} + \begin{pmatrix} x_5 \\ x_7 \end{pmatrix}^T \begin{pmatrix} p_4 & 0 \\ 0 & p_5 \end{pmatrix} \begin{pmatrix} x_5 \\ x_7 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}^T \begin{pmatrix} p_6 & 0 \\ 0 & p_7 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right) dt\}$$

The above criteria contains some weighting factors which are constant and are specified by the designer depending on vehicle movement conditions such as speed, road quality, vibration inputs, etc. After substituting for the accelerations, Z_c and θ, the quadratic performance index is expressed in terms of state and input vectors

$$J = E\{\frac{1}{2}\int_0^\infty \left(X^T Q X + X^T N U + U^T R U \right) dt\} \qquad (7)$$

where the constant matrixes Q, N, R are given in (Nematollahzadeh S.M. 1999).

3- STANDARD LQG DESIGN

The Kalman filter gain matrix K is given by: (Maciejowski J.M. 1989)

$$K = P_e C^T V^{-1} \tag{8}$$

where P_e satisfies the algebraic Riccati equation

$$AP_e + P_e A^T - P_e C^T V^{-1} C P_e + LWL^T = 0 \tag{9}$$

and $P_e = P_e^{\,T} \geq 0$.

The optimal state-feedback gain matrix G is given by

$$G = -R^{-1} B^T P_c \tag{10}$$

where P_c satisfies the following algebraic Riccati equation (MATLAB. 1999)

$$A^T P_c + P_c A - (P_c B + N/2) R^{-1} (B^T P_c + N^T/2) + Q = 0$$

And $P_c = P_c^{\,T} \geq 0$.

Now it is quite clear that the optimal control of the system (4) with the cost function (7) will take the form

$$U = G\hat{X} \tag{12}$$

where the estimated states are obtained from Kalman filter.

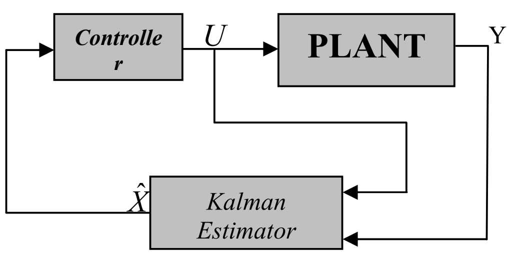

Fig.2. LQG Controller block diagram**.**

4-LQG DESIGN USING SINGULAR PERTURBATION THEORY

Figure 1 is a schematic diagram of the vehicle's active suspension system showing clearly its two-time-scale behaviour, i.e. slow dynamic behaviour of system includes changes in the overhead front and rear tire suspension system and in vertical speed of the main mass at the wheels of system. In contrast, the fast dynamics include change of vertical form in tires and the rate of such changes called unsprung mass. In order to use two-time scale property of the system, Eq.(4) is partitioned as

$$\dot{X}_1 = A_{11}X_1 + A_{12}X_2 + B_1U + L_1w \tag{13a}$$

$$\dot{X}_2 = A_{21}X_1 + A_{22}X_2 + B_2U + L_2w \tag{13b}$$

$$Y = C_1X_1 + C_2X_2 + v \tag{13c}$$

where

$$X_1 = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}^T, X_2 = \begin{pmatrix} x_5 & x_6 & x_7 & x_8 \end{pmatrix}^T$$

and the constant system matrices are [1]:

$$A_{11} = A(1{:}4,1{:}4) \qquad A_{12} = A(1{:}4,5{:}8)$$

$$A_{21} = A(5:8,1:4) \qquad A_{22} = A(5{:}8,5{:}8) \tag{15a}$$

$$B_1 = B(1{:}4,1{:}2) \qquad B_2 = B(5{:}8,1{:}2) \tag{15b}$$

$$C_1 = C(1:2,1:4) \qquad C_2 = C(1:2,5:8) \tag{15c}$$

$$L_1 = L(1{:}4,1{:}2) \qquad L_2 = L(5{:}8,1{:}2) \tag{15d}$$

The standard singular perturbation form of (13) is then given by

$$\dot{X}_1 = A_{11}X_1 + A_{12}X_2 + B_1U + L_1w \tag{16a}$$

$$\varepsilon\dot{X}_2 = \hat{A}_{21}X_1 + \hat{A}_{22}X_2 + \hat{B}_2U + \hat{L}_2w \tag{16b}$$

where

$$\hat{A}_{21} = \varepsilon A_{21} \qquad \hat{A}_{22} = \varepsilon A_{22} \tag{17a}$$

$$\hat{B}_2 = \varepsilon B_2 \qquad \hat{L}_2 = \varepsilon L_2 \qquad ,L_1 = 0 \tag{17b}$$

and ε is the singular perturbation parameter given by (1). Applying the standard singular perturbation approach to decouple the slow-fast subsystems, the slow and the fast dynamics is obtained (Kokotovic P.V. 1986) .

4.1 The slow subsystem

$$\dot{X}_{1s} = A_0X_{1s} + B_0U + L_0w$$

$$Y_s = C_0X_{1s} + D_0U + S_0w + v$$

$$X_{1s}(0) = X_1^{\,0} \tag{18}$$

where subscript s represents the slow part of the vectors, and

$$A_0 = A_{11} - A_{12}\hat{A}_{22}^{\,-1}\hat{A}_{21}$$

$$B_0 = B_1 - \hat{A}_{12}\hat{A}_{22}^{\,-1}\hat{B}_2 \tag{19}$$

$$C_0 = B_1 - C_2\hat{A}_{22}^{\,-1}\hat{A}_{21} \qquad D_0 = -C_2\hat{A}_{22}^{\,-1}\hat{B}_2$$

$$L_0 = -A_{12}\hat{A}_{22}^{\,-1}\hat{L}_2 \qquad S_0 = -C_2\hat{A}_{22}^{\,-1}\hat{L}_2$$

The problem of the slow subsystem controller can be solved by finding the control signal U_s in such a way as to minimise J_s (slow part of cost function)

$$J_s = E\{\frac{1}{2}\int_0^\infty (X_{1s}^{\,T}Q_1X_{1s} + X_{1s}^{\,T}N_1U_s + U_s^{\,T}RU_s)dt\}$$

where

$$Q_1 = Q(1:4,1:4) \; Q_2 = Q(5:8,5:8) \tag{21a}$$

$$N_1 = N(1:4,:) \qquad N_2 = N(5:8,:) \tag{21b}$$

The control signal for the slow subsystem is

$$U_s = G_s\hat{X}_{1s} \tag{22}$$

where $\hat{X}_{1s}$ is the optimal estimated slow states (X_{1s}), provided by the slow Kalman filter

$$\dot{\hat{X}}_{1s} = A_0\hat{X}_{1s} + B_0U_s + K_s[Y - D_0U_s - C_0\hat{X}_{1S}]$$

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[1] Please note that due to space limitation the matrices are given in MATLAB format.

The filter gain K_s is

$$K_s = (P_{es}C_0^T + L_0S_0^T)V^{-1} \qquad (24)$$

and $P_{es} = P_{es}^T \geq 0$ is the stabilizing solution of the slow algebraic Riccati equation

$$(A_0 - L_0S_0^TV_0^{-1}C_0)P_{es} + P_{es}(A_0 - L_0S_0^TV_0^{-1}C_0)^T$$
$$+ L_0(I - S_0^TV_0^{-1}S_0)L_0^T - P_{es}C_0^TV_0^{-1}C_0P_{es} = 0$$

where

$$V_0 = V + S_0S_0^T \qquad (26)$$

4.2 The fast subsystem

$$\varepsilon\dot{X}_{2f} = \hat{A}_{22}X_{2f} + \hat{B}_2U_f + \hat{L}_2w$$
$$Y_f = C_2X_{2f} + v$$
$$X_{2f}(0) = X_2^0 + \hat{A}_{22}^{-1}\hat{A}_{21}X_1^0 \qquad (27)$$

The problem is finding the control signal U_f in such a way as to minimise the following cost function (fast part)

$$J_f = E\{\frac{1}{2}\int_0^\infty (X_{2f}^TQ_2X_{2f} + X_{2f}^TN_2U_f + U_f^TRU_f)\}dt$$

The feedback control signal for fast sub-system is given by

$$U_f = G_f\hat{X}_{2f} \qquad (29)$$

where $\hat{X}_{2f}$ is the optimal estimated fast states (X_{2f}), provided by the fast Kalman filter

$$\varepsilon\dot{\hat{X}}_{2f} = \hat{A}_{22}\hat{X}_{2f} + \hat{B}_2U_f + K_f[Y_f - C_2\hat{X}_{2f}]$$

The filter gain K_f is

$$K_f = P_{ef}C_2^TV^{-1} \qquad (31)$$

and $P_{ef} = P_{ef}^T \geq 0$ is the stabilizing solution of the fast algebraic Riccati equation

$$\hat{A}_{22}P_{ef} + P_{es}\hat{A}_{22}^T + \hat{L}_2W\hat{L}_2^T - P_{ef}C_2^TV^{-1}C_2P_{ef} = 0 \qquad (32)$$

The composite control signal is the sum of the slow and fast control signals

$$U_s + U_f = G_s\hat{X}_{1s} + G_f\hat{X}_{2f} \qquad (33)$$

as shown in Figure 3.

5- APPLICATION TO THE HALF CAR MODEL

The slow-fast controller design presented in section 4.2 is now applied to the half car model of section 2. The nominal parameter values of the model used in the simulation are presented in Table 1. These parameters are specified for a Sedan car model (Hac A., Youn I. 1993). The consonant matrices Q and R of Eq. (7) are attained from the cost function weighting coefficients of Eq. (6)-having values arranged in Table 2.

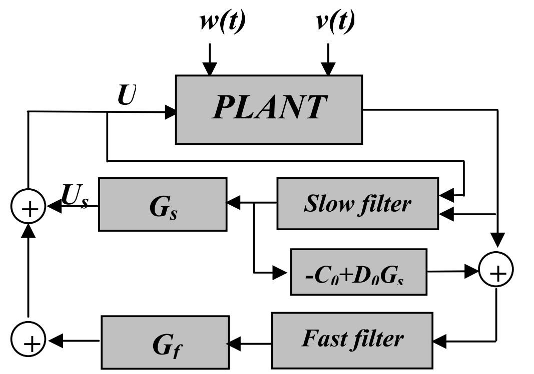

Fig.3-Block diagram representation of the composite stochastic control.

Table 1. Parameter values for a car model.

Index	Value	Index	Value
K_{f1}	10000 N/m	M	500. kg
k_{f2}	100000 N/m	I	910. kg.m2
k_{r1}	10000 N/m	m_1	30 kg
k_{r2}	100000 N/m	m_2	40 kg
b_f	1000 N.s/m	A	1.25 m
b_r	1000 N.s/m	B	1.45 m

Table 2. Coefficient values of the cost function

Index	Value	Index	Value
P0	1	P1	2
P2	900	P3	960
P4	5800	P5	4600
P6	0	P7	0

The weighting coefficients for the cost function depend on the conditions of motion such as the road conditions, and the dynamic speed and to some extend such measures are desirably specified by the designer (Hac A., Youn I. 1993).

In order to observe the performance of each of the above controllers in the presence of external disturbances, a roadway containing a hole along the travelling path has been considered as shown in Figure 4.

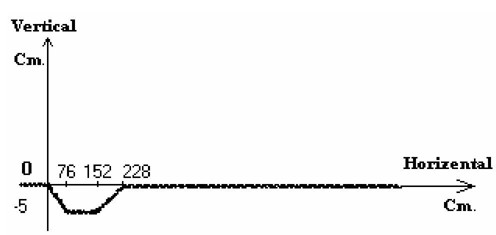

Fig. 4- Road model used for simulations

Such a particular disturbance simulates both the slow and the fast modes in the system. In addition, a white noise signal with the following power spectral density has been selected as a disturbance signal to model the speed vibrations of the road:

$$S_w(\varpi) = \frac{2\alpha\sigma^2}{\varpi^2 + (\alpha v_0)^2} \qquad (34)$$

This can result from passing the white noise $\zeta(t)$ from the low-pass filter

$$\dot{\varpi}(t) + \alpha v_0 \varpi(t) = \zeta(t), \qquad (35)$$

Where the noise signal measures at $2\sigma\alpha v_0$ with σ and α being the road consonants and v_0 the dynamic speed. For an asphalt road these constant coefficients contain the following values (Hac A. 1992)

$$\sigma^2 = 9 * 10^{-6} m^2 \qquad \alpha = 0.15 m^{-1} \qquad (36)$$

The vehicle speed set to v_0 = *20 m/sec*. The time response of outputs such as the amount of variation in the suspension system above the front wheel, the amount of flexure of the front wheel, the vertical acceleration of the car body and the amount of angular changes in the car body are considered for both approaches. In each of these simulations, the outputs are equivalent to the distance of the front section of car body from the road surface and the same distance at the rear car section. These outputs are important in LQG controller design, since in such a design, by using the optimal estimator, the measurable output must provide a good and proper estimate for the modes (X_1, X_2). The combination of this filter together with the gains obtained from the optimal regulator produces the LQG controller. To measure these two outputs, sensors denoted by S_1 and S_2 (see Fig.1) are attached to the front and the rear of the car which allow the estimator (Kalman filter) to measure distances from the road surface.

Computing the controller gain matrix (12) for the standard LQG design, we get

$$G = \begin{pmatrix} 7777 & 1530 & 1410 & 300 & -2110 & -480 & 2610 & -100 \\ 20 & 0 & 300310 & 7060 & -200 & -0 & 53070 & -2770 \end{pmatrix}$$

Applying this controller to the full order system, and computing the cost function (7), we obtain

$$J_{full} = 166.94 \qquad (38)$$

In order to apply the singular perturbation theory, first, lets check the two-time-scale property of the system. Figure 5 shows the Eigen-spectrum of the system. As it can be seen clearly in the Figure, the two sets of Eigen values are well separated. The ratio defined by (1) is computed

$$|\lambda_s|_{\max} / |\lambda_f|_{\min} \cong 0.14 \qquad (39)$$

The calculations of the controller gains matrices using the singular perturbation theory for both the slow and the fast subsystems result in the following

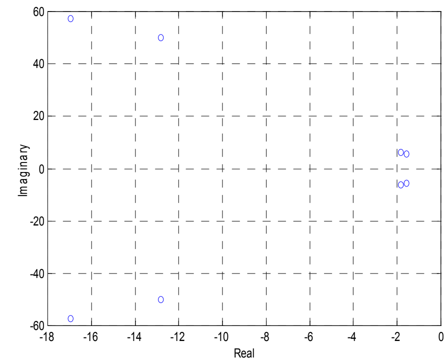

Fig. 5 Eigen spectrum of the model

$$G_1 = \begin{pmatrix} 8054 & 1540 & 15227 & 493 \\ 5 & 0 & 307866 & 11162 \end{pmatrix} \qquad (40a)$$

$$G_2 = \begin{pmatrix} -3678 & -490 & 72 & -1 \\ -317 & -6 & -2499 & -483 \end{pmatrix} \qquad (40b)$$

and the composite cost function measured by the sum of the slow and the fast functions (20) and (28), is computed to be

$$J_{comp.} = 168.30 \qquad (41)$$

Design of the controller using the singular perturbation theory entails a %0.81 cost increase that is negligible, and clearly justifies the utilization of this approach.

The simulation results of both design approaches are compared graphically. Figures 6 through 9 show the time response of the front suspension, front tire deflection, vertical acceleration, and pitch rate respectively. As shown in the figures, the trajectories are almost identical justifying, once again, the utilisation of singular perturbation approach.

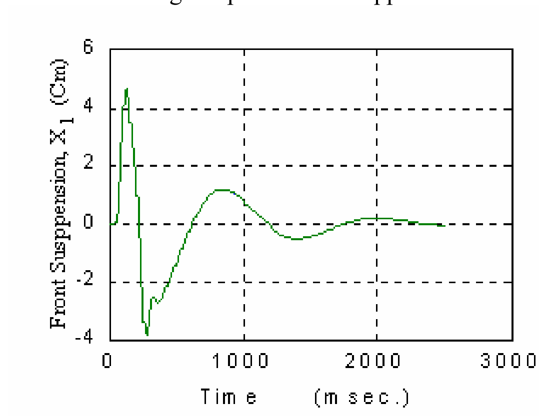

Fig.6- Size of the suspension system above the front wheel. (-) full states and, (…) composite control design.

Of course, in the simple problem of half car model presented here, use of such a strategy may not give a much outstanding vision, yet, in an expanded problem with further dimensions, better and greater advantages are discernible.

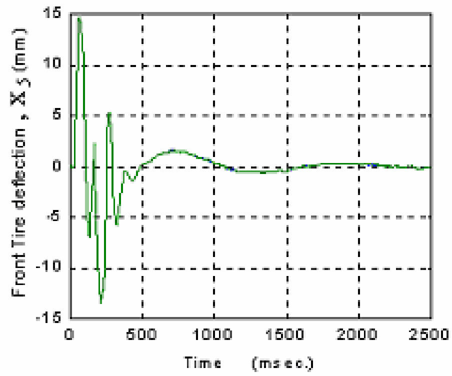

Fig.7- Size of the deformation on the front wheel
(-)full states and, (..) composite control design.

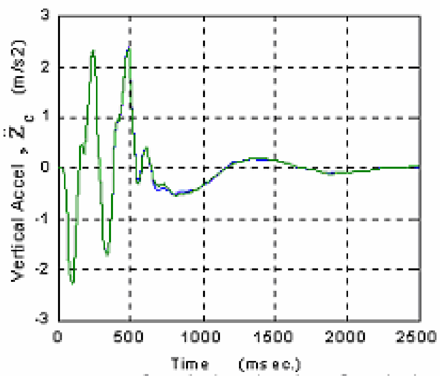

Fig.8- Amount of vertical accleration of car body.
(-) full states and, (..) composite control design.

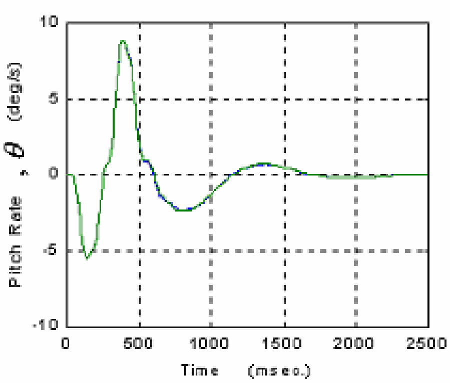

Fig.9-Limits of the pitch rate of the body.
(-)full states and, (..) composite control design.

6- CONCLUSION

Controller design for active suspension system has been presented. Singular perturbation theory was utilized to design LQG controllers. Although half-car model has been used, the eight-order dynamic system enjoys the decoupling into two forth-order subsystems. This, in turn, results in solving lower-order Riccati equations to obtain the estimator gains, as well as the controller gain matrices. Simulation results were shown to be very close approximation to the full-order design. Computation time savings, parallel processing are of importance, and are enhanced by the techniques presented here. Extension of the approach to the full-car model will certainly magnify the benefits of singular perturbation theory. That is the subject we are looking into, and hope to report in the future.

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